Quasiregularly Elliptic Manifolds and Cohomology

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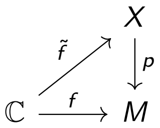
2-Dimensional Case

Let M be a compact Riemann surface and

$$f : \mathbb{C} \to M$$

a nonconstant holomorphic map.

- What type of surface can M be?
  - By the uniformization theorem, the universal cover X of M is $\mathbb{D}$, $\mathbb{C}$ or $\widehat{\mathbb{C}}$.



2-Dimensional Case

$$\begin{array}{ccc} & & X \\ & \overset{\tilde{f}}{\nearrow} & \downarrow p \\ \mathbb{C} & \xrightarrow{f} & M \end{array}$$

- If $X = \mathbb{D}$, then f is constant.
- If $X = \widehat{\mathbb{C}}$, then $M = \widehat{\mathbb{C}}$.
- If $X = \mathbb{C}$, then $M \simeq S^1 \times S^1$.

How can we generalize this to higher dimensions?
conformal $\to$ quasiconformal
holomorphic $\to$ quasiregular

Quasiregular Maps

Let M be a closed, connected, orientable Riemannian manifold.

Defintion

A map $f : \mathbb{R}^n \to M$ is K-*quasiregular* if $f \in W^{1,n}_{\text{loc}}(\mathbb{R}^n)$, f is nonconstant and

$$||Df||^n \leq KJ_f$$

- A homeomorphic K-quasiregular map is K-quasiconformal.
- A 1-quasiregular map in dimension 2 is holomorphic.

Question

What manifolds admit quasiregular maps (quasiregularly elliptic)?

Revisit $\mathbb{C}$

A quasiregular map $f\colon \mathbb{C} \to M$ can always be decomposed

$$f = g \circ \phi$$

where $\phi\colon \mathbb{C} \to \mathbb{C}$ is quasiconformal and $g\colon \mathbb{C} \to M$ is holomorphic (Stoïlow's theorem).

So in dimension 2 the question of quasiregular ellipticity reduces to the holomorphic case.

Higher Dimensions

Proposition (Bonk and Heinonen '01)

If there exists a 1-quasiregular map $f\colon \mathbb{R}^n \to M$, then M is conformally equivalent to a quotient of S^n or T^n.

1-quasiregular is a restrictive condition in higher dimensions.

- This is also true for 1-quasiconformal maps, which are locally Möbius transformations for $n \geq 3$ (Liouville's theorem, Liouville 1850, Gehring '62, Reshetnyak '67).

So instead study K-quasiregular maps for $K \geq 1$.

Fundamental Group

In dimension 2, the fundamental group was the main obstruction for admitting holomorphic maps.

Theorem (Varopoulos)

If M is an n-dimensional Riemannian manifold that is quasiregularly elliptic, then $\pi_1(M)$ has a growth order bounded above by n.

- Proof relies on lifting f to a noncompact universal covering space.
- As in dimension 2, this result is independent of the distortion K.
- Gromov ('81) asked whether there exists a simply connected manifold that is not quasiregularly elliptic.

K-Dependency

The situation is not identical for $K = 1$ and $K > 1$.

Theorem (Rickman '80)

A K-quasiregular map $f : \mathbb{R}^n \to S^n$ can omit at most $C(n, K)$-points.

Theorem (Rickman '85, Drasin and Pankka '15)

For $N \in \mathbb{N}$, there exists a quasiregular map $f : \mathbb{R}^n \to S^n$ that omits N points.

- In higher dimensions, the distortion constant can lead to different results.

K-Dependency

We can look for obstructions in other invariants besides the fundamental group.

Theorem (Bonk and Heinonen '01)

If M is K-quasiregularly elliptic, then

$$\dim H^l(M) \leq C(n,l,K),$$

where $H^l(M)$ is the degree l de Rham cohomology of M.

They conjecture that $C(n,l,K) = \binom{n}{l}$, which is attained since T^n is quasiregularly elliptic.

Dynamic Result

Theorem (Kangasniemi '17)

If M admits a noninjective uniformly quasiregular map, then

$$\dim H^l(M) \leq \binom{n}{l}.$$

- A result by Martin, Volker and Peltonen ('06) gives that M is quasiregularly elliptic.
- Proof uses pointwise orthogonality properties of rescaled differential forms on M.

Main Result

What about the case when M is not assumed to admit a uniformly quasiregular map?

Theorem (P. '18)

If M is quasiregularly elliptic, then

$$\dim H^l(M) \leq \binom{n}{l}$$

- This bound is optimal because T^n is quasiregularly elliptic.

Main Result

Corollary (P. '18)

There exist simply connected manifolds that are not quasiregularly elliptic.

- For example, $M = \#^m(S^2 \times S^2)$ for $m \geq 4$.

Theorem (Rickman '06)

$(S^2 \times S^2)\#(S^2 \times S^2)$ *is quasiregularly elliptic.*

Outline of the Proof

- Using f, pull back Poincaré pairs on M.
- Rescale the forms in $\mathbb{R}^n$ to get a collection of differential forms on $B(0,1)$.
- The rescaled forms are then pointwise orthogonal, which says that the number of forms should be bounded above by $\dim \bigwedge^l \mathbb{R}^n = \binom{n}{l}$.
    - This uses a weak reverse Hölder inequality for Jacobians of quasiregular maps into manifolds with nontrivial cohomology.

Rescaling Procedure I

In the proof of the Bonk and Heinonen result the authors use a rescaling procedure on the map $f\colon \mathbb{R}^n \to M$.

- This gives that f is uniformly Hölder continuous.

Instead of rescaling the map f, rescale the pullbacks of differential forms.

- If $\alpha \in \Omega^l(M)$, let $\eta = f^*\alpha$. Let $B_j = B(a_j, r_j) \subset \mathbb{R}^n$ be a sequence of balls such that

$$A(B_j) := \int_{B_j} J_f \to \infty.$$

Define

$$\eta_j = \frac{1}{A(B_j)^{1/p}} T_j^*(\eta)$$

where $T_j(x) = a_j + r_j x$, $p = n/l$.

Rescaling Procedure II

$$\eta_j = \frac{1}{A(B_j)^{1/p}} T_j^*(\eta)$$

Rescaling functions in the Rickman-Picard theorem context was used in a paper by Eremenko and Lewis '91.

- They rescale $\mathcal{A}$-harmonic functions of the form $\log |f|$ to get functions on $B(0,1)$.
- The new functions satisfy strong pointwise estimates.

Rescaling forms on M was used by Kangasniemi in the dynamical counterpart of the theorem.

- The rescaled forms satisfy pointwise orthogonality.

Rescaling Procedure III

$$\begin{aligned}
\int_{B(0,1)} |\eta_j|^p &= \frac{1}{A(B_j)} \int_{B_j} |f^*\alpha|^{n/l} \\
&\leq \frac{||\alpha||_\infty}{A(B_j)} \int_{B_j} ||Df||^n \\
&\leq \frac{K||\alpha||_\infty}{A(B_j)} \int_{B_j} J_f \\
&= K||\alpha||_\infty
\end{aligned}$$

So η_j is uniformly bounded in $L^p(B(0,1))$ and has a convergent subsequence in the weak topology.

Example of Rescaling I

$M = T^2$ and $f(z) = e^z$.
$H^1(T^2) \cong \{adz + bd\bar{z} : a, b \in \mathbb{R}\}$.

$$\eta = f^* dz = e^z dz$$

In the rescaling, need B_j such that

- $A(B_j) \to \infty$
- $A(B_j) \leq CA(\frac{1}{2}B_j)$

where

$$A(B) = \int_B J_f$$

$B_j = B(j, 1)$ works. On $B(0, 1)$,

$$\eta_j = \frac{1}{A(B_j)^{1/2}} e^{j+z} dz \sim \frac{e^j}{e^j} e^z dz$$

Example of Rescaling II

In the limit,

$$\eta_j \to e^z dz$$

Similarly, for $\theta = f^* d\bar{z}$,

$$\theta_j \to e^{\bar{z}} d\bar{z}$$

Let ω on $B(0,1)$ be a differential form such that

$$\omega \wedge e^z dz = \omega \wedge e^{\bar{z}} d\bar{z} = 0$$

Conclude that $\omega = 0$ on $B(0,1)$.

Orthogonality

If $k = \dim H^l(M)$, then let $(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$ be Poincaré pairs on M.

$$\int_M \alpha_a \wedge \beta_b = \delta_{ab}$$

So, if $\eta_a = f^* \alpha_a$ and $\theta_b = f^* \beta_b$, then in the rescaling

$$\tilde{\eta}_a \wedge \tilde{\theta}_b = 0$$

$a \neq b$, for almost every $x \in B(0,1)$.
So $Z_a = \{x \in B(0,1) : \eta_a(x) = 0\}$ cover $B(0,1)$.

Equidistribution of f

Since f is quasiregular, it satisfies equidistribution properties as a map into M.
Let $\alpha \in \Omega^l(M), \beta \in \Omega^{n-l}(M)$,

$$\lim_{j\to\infty}\left|\frac{1}{\int_{B_j}(\psi\circ T_j)J_f}\int_{B_j}(\psi\circ T_j)f^*(\alpha\wedge\beta)-1\right|=0$$

when

$$\int_M \alpha\wedge\beta = 1$$

and $\psi \in C_c^\infty(B(0,1))$.

$$\int_{B(0,1)}\psi\tilde{\eta}_a\wedge\tilde{\theta}_a \sim \lim_{j\to\infty}\int_{B_j}(\psi\circ T_j)J_f$$

J_f is Nonzero

By the orthogonality condition, the sets $Z_a = \{x \in B(0,1) : \eta_a(x) = 0\}$ cover $B(0,1)$. So on B_j,

$$\int_{T_j(Z_a)} J_f \geq CA(B_j)$$

for one of the η_a. But

$$\frac{1}{A(B_j)} \int_{T_i(Z_a)} J_f \sim \int_{T_j(Z_a)} \eta_{a,j} \wedge \theta_{a,j} \to 0.$$

This gives a contradiction, which implies that $\dim H^l(M) \leq \binom{n}{l}$.

Reverse Hölder Inequality I

In the argument above actually need to use a reverse Hölder inequality for J_f.

Theorem (Bojarski and Iwaniec '83)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a K-quasiregular map. Then $f \in W^{1,nq}_{loc}(\mathbb{R}^n)$ for $1 < q \leq Q(n, K)$, where $Q(n, K)$ depends only on n and K. If $B \subset \mathbb{R}^n$ is a ball, then

$$\left(\int_{\frac{1}{2}B} J_f^q \right)^{1/q} \leq C(n, q, K) \frac{1}{|B|^{1/q'}} \int_B J_f \tag{1}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Crucially, $C(n, q, K)$ is independent of f and B.

- This theorem does not directly apply since $f : \mathbb{R}^n \to M$. If $H^l(M) = 0$ for $1 \leq l \leq n-1$, then the theorem does not necessarily hold.

Reverse Hölder Inequality II

In our case there is an l so that $H^l(M) \neq 0$.

Proposition

Let M be a closed Riemannian manifold and let $f : \mathbb{R}^n \to M$ be K-quasiregular. If there exists an integer l with $1 \leq l \leq n-1$ such that $H^l(M) \neq 0$, then the Jacobian of f satisfies the weak reverse Hölder inequality,

$$\frac{1}{|\frac{1}{2}B|} \int_{\frac{1}{2}B} J_f \leq C(n, M, K) \left(\frac{1}{|B|} \int_B J_f^{n/(n+1)} \right)^{(n+1)/n},$$

where $B \subset \mathbb{R}^n$ is an arbitrary ball.

- Once the proposition is shown, then the reverse Hölder inequality for an exponent $b > 1$ follows from Gehring's lemma.

Proof of the Proposition

Since $H^l(M) \neq 0$, there exists as before a Poincaré pair, α and β, so that

$$\int_M \alpha \wedge \beta = 1.$$

Through this,

$$\int_B J_f \sim \int_B f^*\alpha \wedge f^*\beta.$$

And $d\alpha = 0$, so on B, $f^*\alpha = du$.
If $\psi \in C_c^\infty(B)$, then

$$\begin{aligned} \left| \int_B \psi du \wedge f^*\beta \right| &= \left| \int_B d\psi \wedge u \wedge f^*\beta \right| \\ &\leq ||d\psi||_\infty ||u||_s ||f^*\beta||_t \end{aligned}$$

$$\int_B \psi f^*(\alpha \wedge \beta) \leq \|d\psi\|_\infty \|u\|_s \|f^*\beta\|_t$$

For a suitable ψ,

$$\|d\psi\|_\infty \leq \frac{1}{|B|^{1/n}}.$$

The Poincaré-Sobolev inequality for differential forms (Iwaniec and Lutoborski '93) gives

$$\|u\|_s \leq \|f^*\alpha\|_{s^*}$$

$$|f^*\alpha| \leq CJ_f^{l/n} \quad \text{and} \quad |f^*\beta| \leq CJ_f^{(n-l)/n}$$

Choosing exponents correctly gives the reverse Hölder inequality for J_f.

Further questions

- What about the case when M is not compact?
  - For $n = 2$, $M \simeq \mathbb{C}$ or $S^1 \times \mathbb{R}$.
  - For $n > 2$, the answer must depend on K by the Rickman-Picard theorem.
- Does there exist a quasiregularly elliptic manifold where the quasiregular map does not factor through the torus?
  - If $\#^3 S^2 \times S^2$ is quasiregularly elliptic, then the map cannot factor through the torus (Pankka and Souto '12).
- Suppose $\dim H^l(M) = \binom{n}{l}$, what does this imply about M?
  - For $l = 1$, there must exist a covering map $p\colon T^n \to M$ (Luisto and Pankka '16).

Thank you!