2

Spheres and circles

Eudoxus' system of concentric spheres

The Babylonians were concerned with predicting the time at which a particular phenomenon (e.g. a planetary opposition) would occur, since it was the date that was ominous. Their astronomy was thus concerned with the analysis of discrete processes. On the other hand, as we shall see, the Greeks in their astronomy focused on predicting where a celestial body would be at a given time, and they were thus concerned with modelling a continuous process, which naturally leads to the use of geometrical methods. The first person to produce a geometrical model of celestial motions was Eudoxus, who was born in Cnidus on the western coast of Asia Minor in about 400 BC. According to Diogenes Laërtius' *Lives of Philosophers* (written in AD third century) he was taught geometry by Archytas of Tarentum, one of the leading Pythagorean philosophers, and studied with Plato, who was about 30 years older than him. None of Eudoxus' works have survived. Most of the information we have about his system of concentric spheres comes from a brief contemporary account in Aristotle's *Metaphysics* and a more substantial description due to Aristotle's influential commentator Simplicius (AD sixth century).[1]

[1] Simplicius based his work on that of the philosopher Sosigenes (AD second century – not to be confused with the Sosigenes who helped Julius Caesar reform the calendar in the first century BC), and ultimately on the lost history of astronomy by Eudemus, who lived only a generation after Eudoxus. It is important to recognize that Simplicius was writing 900 years after the event and he may well have been putting his own interpretation on Eudoxus' work. By itself, the information supplied by Aristotle and Simplicius does not provide a clear picture of Eudoxus' model, but in a classic paper in 1877, the Italian Giovanni Schiaparelli produced a reconstruction of the Eudoxan system which has remained the accepted version ever since. Recently, however, Yavetz (1998, 2001) has argued that Schiaparelli's interpretation is not the only one that can be put on the original sources which is consistent with what we know about Greek astronomy of the time, and that there is no reason to believe one rather than the other. Here we will stick to the standard interpretation.

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Eudoxus had a considerable influence in many areas of mathematics. In fact, he was the inspiration behind two of the most profound mathematical advances of the fourth century BC. The first of these was the theory of proportion, which formed the basis of much of Euclid's *Elements* (see p. 36), and the second was the method of exhaustion, which was used extensively in the same work and represents the beginnings of the subject now known as the 'integral calculus'. Both of these contributions were of fundamental significance for the development of mathematics, but it would appear that, as far as other scholars from antiquity were concerned, his main claims to fame were his astronomical discoveries,[2] one of which was an accurate estimate for the length of the tropical year, namely 365 days 6 h.

Eudoxus' theory based on concentric spheres marks the beginning of a long development in Greek mathematical astronomy, culminating in the work of Ptolemy about 500 years later.[3] Usually, it is claimed that Eudoxus' theory, described in his treatise *On Speeds*, was a response to a challenge of Plato. Simplicius tells us that:

> Eudoxus of Cnidus was the first of the Greeks to concern himself with hypotheses of this sort, Plato having . . . set it as a problem to all earnest students of this subject to find what are the uniform and ordered movements by the assumption of which the phenomena in relation to the movements of the planets can be saved.[4]

In other words, what uniform circular motions could be used to represent the motions of the planets? Quite how much of the impetus for a geometrical theory of the heavens was due to Plato is open to question, however.[5]

The discovery of the spherical nature of the Earth, together with Pythagorean ideas about uniform circular motions, may well have suggested to Eudoxus that the motion of the stars and planets could be modelled by means of a system of concentric spheres with the Earth at their common centre. He supposed that each of the Sun, Moon and planets occupied its own sphere which was attached via its poles to a larger sphere that rotated about other poles. Third and fourth spheres could be added, each rotating at different rates and about different axes, according to mathematical and observational needs. Apart from the sphere of the fixed stars, Eudoxus used three spheres each for the Sun and Moon and four spheres for each of the planets, making twenty-seven in all. In each case, one

[2] Lasserre (1964), p. 124.
[3] It has been claimed (Landels (1983)) that Eudoxus' model of the Universe represents the first ever example of the technique of mathematical modelling, but while the most obvious mathematical phenomena are modelled quantitatively by Eudoxus' theory, the more subtle effects are only reproduced qualitatively, if at all.
[4] SIMPLICIUS on *De caelo*. Translation from Heath (1932).
[5] See, for example, Goldstein (1980), (1983), Knorr (1990).

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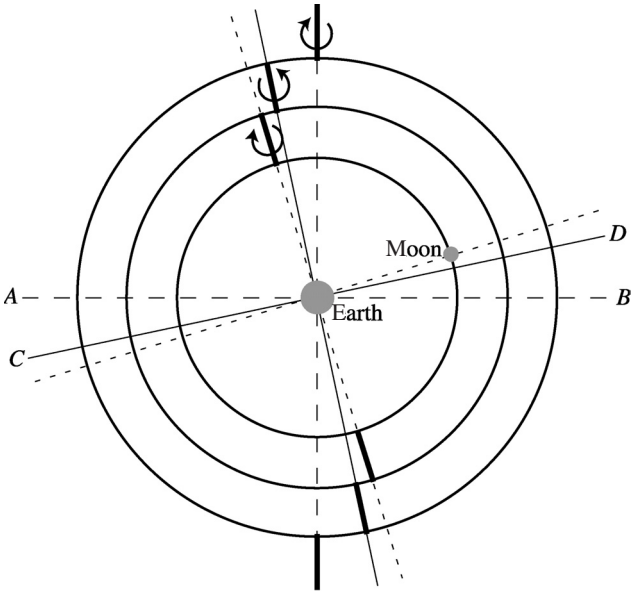

Fig. 2.1. Eudoxus' scheme for the motion of the Moon, according to Simplicius.

of the spheres performed exactly the same function: it rotated once every 24 h and accounted for the daily rotation of the heavens. Thus, the motions of the Sun, Moon and planets with respect to the fixed stars actually were modelled using only nineteen uniform circular motions. It is not clear whether Eudoxus thought of the spheres as being real physical entities, or whether he regarded them as purely mathematical constructions. Indeed, he may not have thought that the distinction was significant.[6]

In Eudoxus' theory for the Moon (see Figure 2.1, in which AB represents the celestial equator and CD the ecliptic), the outermost of the three spheres completes 1 revolution from east to west in 1 day, thus accounting for the diurnal rotation of the Moon. In order to account for the motion of the Moon around the ecliptic, Eudoxus introduced a second sphere attached to the first through an axis passing through the poles of the ecliptic and rotating from west to east. Eudoxus also knew that the Moon did not follow exactly the ecliptic, but deviated a little above and below it, and accounted for this by including a

[6] Wright (1973) has argued that the very nature of the system makes it probable that Eudoxus did think of his spheres as having a physical reality.

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third sphere, the axis of rotation of which makes a small angle with that of the second sphere.

It is not known what speeds Eudoxus chose for his second and third spheres, or what inclination he gave the third sphere relative to the second; Simplicius merely states that the rotation of the third sphere was slow. It seems likely that Eudoxus would have noted that the Moon's variation in latitude amounted to something like $\pm 5^\circ$ and so would have used a value close to 5° for the inclination of the third sphere to the second. For the inclination of the second sphere to the first he would have used his value for the obliquity of the ecliptic and there is reason to believe that this was $1/15$ of a circle, or 24°. This is the angle subtended at the centre by a regular fifteen-sided polygon, the construction of which is given in Euclid's *Elements*, Book IV, Proposition 16, and which, according to Proclus, was included because of its usefulness in astronomy.[7]

Eudoxus' theory for the Sun was very similar to his lunar theory. The axis of the innermost sphere was inclined at a very small angle to the second, thus giving the Sun a deviation from the ecliptic. The ecliptic seems to have been defined vaguely as some great circle passing through the zodiac rather than as the path of the Sun, the definition in terms of the motion of the Sun not being formulated until the second century BC by Hipparchus. The solar theory implied that the Sun moves with constant speed relative to the fixed stars, from which it follows that the seasons are all of equal length – a statement known to be false, since in 432 BC two astronomers from Athens – Meton and Euctemon – measured the times between summer solstice and autumn equinox, autumn equinox and winter solstice, winter solstice and spring equinox, and spring equinox and summer solstice, as 90, 90, 92 and 93 days, respectively. Eudoxus' theory also assumed that the heavenly bodies were each at a fixed distance from the Earth, and so was incapable of modelling the observed changes in the apparent diameters of the Sun and Moon. This is most significant for the Moon,

[7] As to the speeds of the second and third spheres, there is a certain amount of disagreement. According to Dicks (1970), the period of rotation of the second sphere would have been the synodic month and the slow rotation of the third sphere in the opposite direction would not have affected the speed of the Moon very much, but would have accounted for the Moon's movement in latitude and the fact that its greatest deviations occur at points which shift steadily westwards. However, many have taken the view that Simplicius got it wrong and that it was the second sphere that rotated slowly, with a period of over 18 years, and the innermost sphere rotated once in a period of a little over 27 days. This provides a much more accurate representation of the Moon's deviations in latitude and also models the so-called *Saros* period of 223 synodic months ($\approx$ 18 years), which was used by the Babylonians to investigate the periodic recurrence of various lunar phenomena, including eclipses. This view, which originated in the nineteenth century and has been repeated many times since, is, according to Dicks, an example of a misleading interpretation of early astronomical thought by attributing to it a sophistication inconsistent with contemporary knowledge. The fact that Eudoxus' system of concentric spheres is poor quantitatively in virtually all other respects lends credence to Dicks' opinion, but others (e.g. Thoren (1971)) do not share his view.

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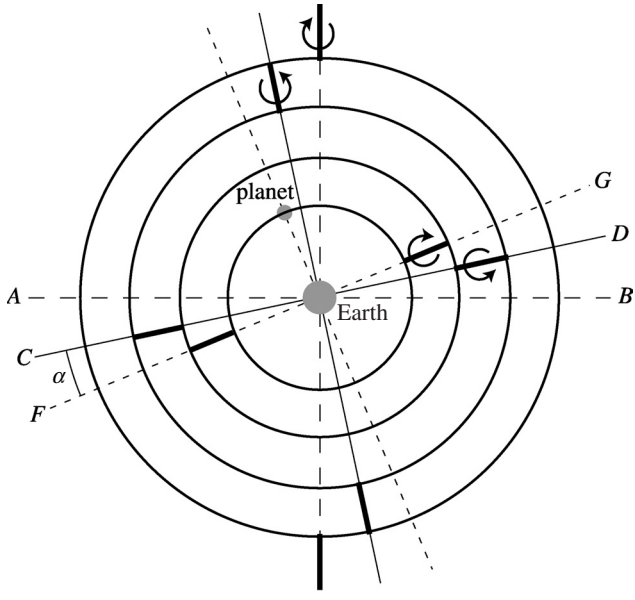

Fig. 2.2. Eudoxus' scheme for the motion of the planets.

the greatest apparent diameter of which is 14 per cent larger than its smallest. Simplicius was well aware that, even judged by the astronomical knowledge of Eudoxus' day, the system of concentric spheres had serious deficiencies.

> Nevertheless the theories of Eudoxus and his followers fail to save the phenomena, and not only those which were first noticed at a later date, but even those which were before known and actually accepted by the authors themselves.[8]

Perhaps the main fascination with Eudoxus' theory is with his method for accounting for the retrograde motion of the planets, which he did by adding an extra sphere (see Figure 2.2).[9] Just as in the case of the Sun and the Moon, the first and second sphere together account for the diurnal rotation and the regular motion around the ecliptic in the appropriate zodiacal period. The axis of the third sphere was in the plane of the equator of the second sphere (i.e. in the ecliptic plane) and the axis of the fourth sphere was inclined at a small angle α to that of the third. Crucially, the third and fourth spheres rotated in

[8] SIMPLICIUS on *De caelo*. Translation from Heath (1932).

[9] Since Simplicius lived nearly 1000 years after Eudoxus, it is quite possible that the phenomena he attempted to describe using Eudoxus' scheme, such as retrograde motion, were not those that Eudoxus originally had in mind (see Goldstein (1997)). However, Yavetz (1998) claims that, given the nature of the Eudoxan model, the most likely explanation is that retrograde motion was the object of the design.

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opposite directions but at the same rate of 1 revolution per synodic period of the planet.

The motion of the third and fourth spheres together combine to generate a figure-of-eight-type curve, which Eudoxus termed a *hippopede*, named after the device used to tether a horse by its feet.

> ... the fourth sphere, which turns about the poles of the inclined circle carrying the planet and rotates in the opposite sense to the third, i.e. form east to west, but in the same period, will prevent any considerable divergence (on the part of the planet) from the zodiac circle, and will cause the planet to describe about this same zodiac circle the curve called by Eudoxus the *hippopede* ...[10]

When superimposed on the regular motion induced by the outermost two spheres, the hippopede has the effect of producing small deviations in latitude and occasional periods of retrograde motion, as required. This curve was only of limited success in modelling the observed planetary motion, but is an early example of the study of curves in space and its construction demonstrates a great deal of mathematical skill.[11]

With modern mathematical techniques, visualization of Eudoxus' hippopede is much simpler than it was for the ancients. If we introduce orthogonal unit vectors $\{\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1\}$, where $\mathbf{k}_1$ is along the axis of rotation of the fourth sphere (FG in Figure 2.2) and $\mathbf{i}_1$ and $\mathbf{j}_1$ lie in its equatorial plane and rotate with it, then the position of the planet is given, for all times, by the vector $\mathbf{i}_1$. Next, we introduce a second set of vectors $\{\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2\}$ which are the same as the first except that the vectors $\mathbf{i}_2$ and $\mathbf{j}_2$ do not rotate with the sphere. Hence, the vectors $\mathbf{i}_2, \mathbf{j}_2$ are related to $\mathbf{i}_1, \mathbf{j}_1$ via a 2×2 rotation matrix, and if the angular rotation rate of the fourth sphere is $-\Omega$ we have:

$$\begin{pmatrix} \mathbf{i}_2 \\ \mathbf{j}_2 \\ \mathbf{k}_2 \end{pmatrix} = \begin{pmatrix} \cos \Omega t & \sin \Omega t & 0 \\ -\sin \Omega t & \cos \Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{pmatrix} = A_1 \begin{pmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{pmatrix},$$

for example. Finally, we introduce an orthogonal set of vectors fixed in the third sphere with $\mathbf{k}_3$ along its axis of rotation (CD in Figure 2.2) and, hence, in the ecliptic plane, which makes an angle α with $\mathbf{k}_1$ and $\mathbf{k}_2$. Since the angular speed

[10] Simplicius on *De caelo*. Translation from Heath (1932).

[11] The question as to how Eudoxus devised his construction of the hippopede has been the subject of much research and speculation (see, for example, Neugebauer (1953) and Riddell (1979)). Aaboe (1974) claims that Eudoxus' model could have been meant only as a qualitative description of the planets, because if one enters in the appropriate periods for Venus or Mars there is simply no way of producing retrograde motion. Yavetz (1998) disputes this for the case of Mars.

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Fig. 2.3. An example of a planetary path in Eudoxus' theory.

of this sphere is Ω, we have:

$$\begin{pmatrix} \mathbf{i}_3 \\ \mathbf{j}_3 \\ \mathbf{k}_3 \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\Omega t & -\sin\Omega t & \sin\alpha\cos\Omega t \\ \cos\alpha\sin\Omega t & \cos\Omega t & \sin\alpha\sin\Omega t \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathbf{i}_2 \\ \mathbf{j}_2 \\ \mathbf{k}_2 \end{pmatrix} = A_2 \begin{pmatrix} \mathbf{i}_2 \\ \mathbf{j}_2 \\ \mathbf{k}_2 \end{pmatrix},$$

for example. It follows that the position of the planet in terms of $\{\mathbf{i}_3, \mathbf{j}_3, \mathbf{k}_3\}$ is:

$$A_2 A_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos^2\Omega t + \sin^2\Omega t \\ (\cos\alpha - 1)\cos\Omega t\sin\Omega t \\ -\sin\alpha\cos\Omega t \end{pmatrix}.$$

This, then, is the parametric equation of the hippopede. For the purpose of illustrating the type of planetary paths that can result from Eudoxus' model, we can superimpose a uniform motion in the $\mathbf{k}_3$ direction, corresponding to the motion around the ecliptic, and obtain various curves like that shown in Figure 2.3, in which the angle α was taken as 5°. The curve shows how Eudoxus' construction leads to paths that deviate by a small amount above and below the ecliptic, and are predominantly in one direction, but with regular periods of retrograde motion.

As a quantitative predictive tool, Eudoxus' system of concentric spheres would not have been much use and, since it was developed before the Greeks came into contact with the accurate arithmetic astronomy of the Babylonians, it is reasonable to think that quantitative prediction was not its purpose. However, as the goals of Greek astronomy evolved, other astronomers attempted to modify the scheme so that it better represented the observed phenomena. Callippus of Cyzicus, who was a pupil of a pupil of Eudoxus, added seven more spheres to Eudoxus' scheme.

> Callippus set down the same arrangement of spheres as did Eudoxus, and gave the same number as he did for Jupiter and Saturn, but for the sun and moon he thought there were two spheres still to be added if one were going to account for the appearances, and one for each of the remaining planets.[12]

[12] ARISTOTLE *Metaphysics*, Book $\Lambda 8$, 1073b. Translation from Aristotle (1999).

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The two extra spheres for the Sun were introduced to accommodate the variation in the length of the seasons – summer, autumn, winter and spring – which were measured by Callippus, around 330 BC, as 92, 89, 90 and 94 days, respectively, each being accurate to within 1 day.

The details of Eudoxus' theory are not known with any certainty, but we do know that the scheme exerted a profound influence over the development of astronomical thought. It was not as good at predicting the future positions of heavenly bodies as the arithmetical schemes of the Babylonians, but it was far more influential. This was because it demonstrated the power of geometrical techniques, in that superpositions of simple uniform rotations could be used to model extremely complex behaviour, and because it (as modified by Callippus) was adopted by the giant of Greek scientific philosophy – Aristotle – whose teachings dominated intellectual thought for the next 2000 years. As a scientific theory, Eudoxus' system of concentric spheres is best described as *ad hoc*. It explains the phenomena only in as much as they are built into the model. It does not predict or explain any independent result, and is untestable because the model's intrinsic parameters simply can be modified whenever observations fail to fit in with predictions.[13]

Aristotle

Aristotle was a pupil of Plato, but the two men differed significantly in their approach to the understanding of the natural world. Plato was an idealist who focused on mathematics as the underlying reality, whereas Aristotle took a more pragmatic approach and emphasized physical aspects (e.g. cause and effect); he searched for reasons why things were as they were. Aristotle's modification of Eudoxus' system (he described his cosmology in the work *On the Heavens*, or *De caelo* in Latin) was not undertaken from the point of view of a mathematical astronomer, but as part of his bold attempt to unify all the separate branches of natural philosophy. As mentioned above, it is unclear whether Eudoxus regarded the spheres in his system as mathematical constructions or material entities, but in Aristotle's version the spheres were regarded as real physical objects, motion being imparted to the planets by the divine and eternal 'ether' (a word derived from the Greek for 'runs always') which filled the spheres. This insistence on the physical reality of the spheres led to the addition of many more spheres to

[13] The status of Eudoxus' homocentric spheres as a scientific theory is discussed and compared with later models in O'Neil (1969).

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Callippus' model, twenty-two in all, so as to undo the motion of each planet before beginning the set of spheres associated with the next planet.

Aristotle wrote on many other subjects (e.g. physics, chemistry and biology) and attempted, with a considerable degree of success, to organize all knowledge into a unified whole. As a result, it became very difficult to criticize one aspect of Aristotle's teachings without criticizing it all. This led to a degree of stability and is one of the reasons why Aristotelian cosmology had such a pervasive influence. Central to Aristotle's teaching was the idea of a stationary Earth at the centre of the Universe:

> . . . the natural movement, both of parts of the earth and of the earth as a whole, is toward the centre of the universe – this is why the earth now actually lies at the centre . . . It happens that the same point is the centre both of the earth and of the universe. . . . It is clear from these considerations that the earth neither moves nor lies away from the centre.[14]

These conclusions followed from Aristotle's terrestrial physics in which natural motion was toward or away from the centre of the universe, depending on the 'heaviness' of the material concerned. When thought of in terms of the physics of the time, ideas like a moving Earth were indeed ridiculous, even if they could be used accurately to model celestial phenomena. A new terrestrial physics would be needed before astronomers could make the leap to a planetary Earth.

Aristotle made a complete distinction between the sublunary world (in which he included comets and the Milky Way!) and heavenly objects, beginning with the Moon and extending to the stars. The physics of these regions were totally different. In the former (described in his *Meteorology*) everything was explained in terms of the four elements earth, water, air, and fire. The central Earth was surrounded by water (the oceans) which itself was surrounded by a shell of air (the atmosphere). Between the atmosphere and the Moon there was a shell of fire. These regions represented the natural habitat of each element and thus explained why fire travelled upwards but a stone fell toward the earth. Within the sublunary world everything, including life itself, was imperfect and changing continually.

On the other hand, the celestial regions were ordered perfectly and never changed. This supralunary world was formed out of a fifth element, the ether (or in medieval terminology 'quintessence') and uniform circular motions were considered natural for heavenly bodies, straight line motion being natural

[14] Aristotle *On the Heavens*, II. Translation from Heath (1932).

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on Earth. Aristotle believed that all the heavenly bodies where spherical; such a shape was in his view 'appropriate' for heavenly bodies.[15] The belief that the Moon was spherical is attributed to the Pythagoreans, who argued that the shape of the phases of the Moon are those that would be observed if the Moon were spherical and illuminated on one side by the Sun. Aristotle argued that if one heavenly body was spherical, they all must be. He also supported the idea that the Earth was a sphere based on a number of observations. Among these were the facts that during a lunar eclipse the shadow cast by the Earth on the Moon is consistent with a spherical Earth, and that when one travels north or south the stars which are visible above the horizon change, indicating that the angle between the observer's horizon and the celestial equator has changed. A spherical Earth is consistent also with Aristotle's terrestrial physics in which the matter making up the Earth is subject to a constant tendency to move toward its centre.

Aristotle was aware that the apparent daily motion of the stars could equally well be accounted for by a rotation of the Earth about its own axis, as suggested by his contemporary Heraclides, but he rejected this idea on the ground that circular motion was a property of the heavenly bodies only. Thus, for Aristotle the diurnal rotation of the stars was real and he needed to determine its cause. Plato had thought of the heavenly bodies as gods who supply their own movement, but in Aristotle's philosophy every motion required some driving force. To set the celestial mechanism in motion, Aristotle invented the concept of a supreme Prime Mover. The Prime Mover did not actually do anything to move the sphere of the stars, since that would imply motion and, hence, a new cause; instead, the sphere moved due to the desire for perfection that He aroused in it. Aristotle used the motion of the sphere of the stars to prove that the Universe is finite, since, he argued, a line from the Earth of infinite length could not possibly complete 1 revolution in a finite time, whereas the fixed stars rotated once every 24 h. He backed this up by pointing out that a body with a centre cannot be infinite and yet the Universe has a centre, namely the centre of the Earth.

Another Aristotelian principle, closely related to the finiteness of the Universe and very significant in later developments, was the impossibility of a vacuum. This principle, which was supported by a number of simple experiments, followed from Aristotle's theory of motion in which the speed of a body was, other things being equal, inversely proportional to the resistance

[15] The status of the Moon itself in Aristotle's writings is somewhat ambiguous. It was common in Greek intellectual culture to ascribe an Earthly nature to the Moon and when Aristotle refers to the relation between the Moon and the sublunary world such characteristics are often present. When discussing celestial motion, however, the Moon is a prefect, unchanging sphere (see Montgomery (1999)).

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of the medium through which it was moving. In a vacuum there would be no resistance and it would thus take no time at all to travel between two points. Aristotle argued that space and matter could not be considered independently: it was impossible to have one without the other. There was no matter outside the sphere of the fixed stars and, hence, no space; there was nothing. Some Greek philosophers (the Stoics) agreed with Aristotle that there could be no vacuum within the Cosmos, but believed that outside the sphere of the fixed stars there was an infinite void. There were those who took a contrary view however, most notably the atomists, led by Leucippus and Democritus, who believed that the Universe consisted of atoms moving about in an otherwise infinite void.

Aristotle's cosmology certainly was ingenious but, from a technical point of view, turned out to be unsatisfactory as there were a number of observable phenomena it could not hope to reproduce or explain. Within a century of Aristotle's death, mathematicians who had come into contact with the quantitative predictive astronomy of the Babylonians were developing geometrical theories that violated many of Aristotle's fundamental physical principles in order to account for subtle irregularities in the positions of the heavenly bodies. The influence of Aristotle's cosmology did not diminish, however. Following the demise of the Greek civilization, a large quantity of Aristotle's work (unlike that of Plato) was translated and became available to scholars in the Islamic world and then in Europe during the Middle Ages. For more than two millennia Aristotle's conceptually simple view of the Universe reigned supreme, though many of those who adhered to it were unaware of its deficiencies.

Most people believed that the heavenly bodies had great influence over things that happened on Earth, and the study of these influences through astrology was undertaken widely by professional astronomers. Some influences were obvious (e.g. the Sun's position on the ecliptic is tied to the seasons) and it was natural to generalize and assume similar influences for all the stars and planets. Aristotelian cosmology helped to reinforce these ideas, since it provided a plausible mechanism by which the motion of heavenly bodies could affect terrestrial phenomena. The constant friction between the celestial spheres, and ultimately between the sphere of the Moon and the sublunary world, was the primary mechanism for all change on Earth. Much of the desire for accurate planetary tables came from those who wished to use them to cast horoscopes, and these were often relied upon when making important decisions. As a result, astrology was a major driving force behind technical astronomy. Also, since astrology fits in well with an Earth-centred cosmology, but loses much of its ideological force when the Earth is relegated to a planet, it helped to shackle astronomers to geocentrism.

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Developments in mathematics

Eudoxus' concentric spheres encouraged the study of spherical astronomy (sphaerics), the first known works on which were *On the Moving Sphere* and *On Risings and Settings* by Autolycus of Pitane, written between 330 and 300 BC. Autolycus' works, which contain a variety of simple geometrical theorems, seem to have been superseded quickly by Euclid's *Phaenomena*, believed to have been written about 300 BC.[16] From then on, sphaerics became a standard branch of Greek mathematics. Euclid's *Phaenomena* is a geometrical treatment of various problems relating to the risings and settings of stars and, in particular, addresses at length the problem of how to determine the duration of daylight on a given day at a given location.

The *Phaenomena* is, however, only a minor work of Euclid, who was responsible for the most successful textbook ever written: the *Elements*, also written in about 300 BC, and used as the primary source for teaching geometry to schoolchildren until the 1970s.[17] Of Euclid's life we know virtually nothing, but it is assumed that he studied in Athens and then, after the death of Alexander the Great in 323 BC, migrated to Alexandria where he established a school, known as the Museum.

The *Elements* was not the first textbook of elementary Greek mathematics, but clearly it was so superior to the others that it alone has survived. It consists of thirteen books dealing with the basic properties of rectilinear figures and circles, the theory of proportion, the theory of numbers, the classification of incommensurables, and solid geometry. The style of the book, consisting entirely of definitions, axioms, theorems and proofs, determined the nature of all subsequent Greek mathematical works and, indeed, much mathematics that is written today still bears the hallmark of this early Greek paradigm.

The contents of the *Elements* show that in the three centuries since the beginnings of deductive mathematics in Ionia, the Greeks had made great progress in mathematics. Two things are particularly important for our discussion of mathematical astronomy. First, Greek mathematics was almost entirely geometry. The algebraic methods that we find indispensable today were not introduced until AD ninth century, and the idea that algebraic equations could be used to represent curves and surfaces did not become fully understood until the seventeenth century. Second, the only curves that appear in the *Elements* are the circle and the straight line, and these two geometrical objects dominated Greek mathematical thought for 1000 years.

[16] A translation of the *Phaenomena* is given in Berggren and Thomas (1996).

[17] The *Elements* has appeared in more editions than any other book apart from the Bible. The most famous English translation is that of Sir Thomas Heath (Euclid (1956)).

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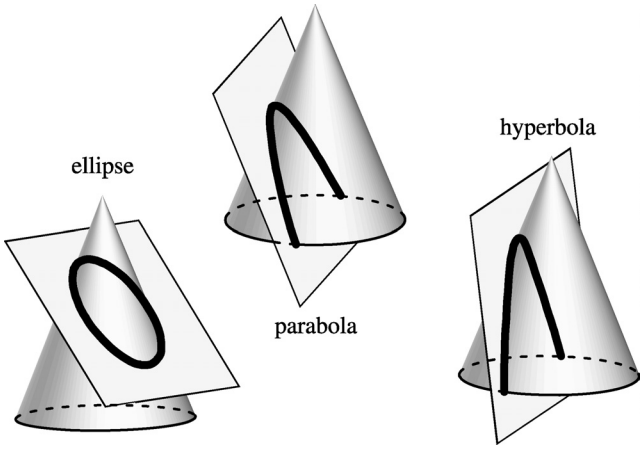

Fig. 2.4. Conic sections.

Other curves were discovered slowly however, and the most important of these were the conic sections – the ellipse, parabola and hyperbola – credited to Menaechmus, a pupil of Eudoxus, in about 350 BC. About 130 years later, Apollonius wrote his treatise on conic sections that is one of the masterpieces of Greek mathematics. Apollonius obtained these curves as the intersections of a plane and a cone, with the three curves being the result of using a different angle of intersection, as shown in Figure 2.4. If the intersecting plane exactly is parallel to one of the generating lines of the cone, we obtain a parabola; otherwise, we get an ellipse or hyperbola, as shown.

Menaechmus, however, used a different construction in which the intersecting plane is always perpendicular to one of the generating lines of the cone, and the different curves are obtained by varying the vertex angle of the cone. It is not known for certain what led to the discovery of the conic sections. One possibility is that they were stumbled upon in the course of attempting to solve the three famous construction problems of antiquity (squaring the circle, duplicating the cube and trisecting the angle). However, it has been suggested[18] that Menaechmus' curious construction – which insists on the perpendicularity of the intersecting plane to the generating lines of the cone – suggests an astronomical heritage. As the Sun moves in a circular arc across the sky, the surface described by the ray that just clips the top of a *gnomon* clearly is conical, and the path mapped out by the end of the *gnomon*'s shadow can thus be thought of as the intersection between a plane (the ground) and this cone. Hence, it will be a

[18] Neugebauer (1948).

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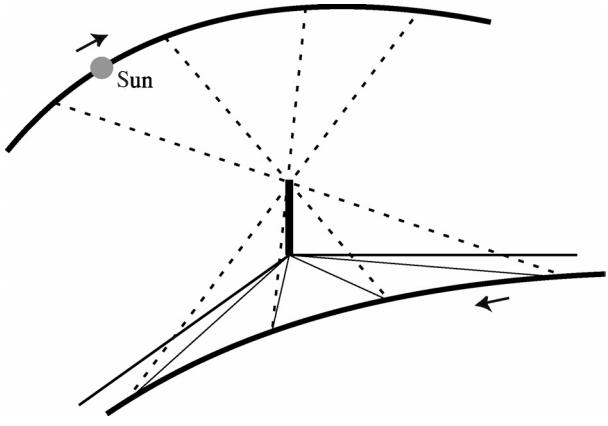

Fig. 2.5. The shadow of a sundial.

conic section and, provided the Sun dips below the horizon during a revolution, it will be a hyperbola (see Figure 2.5). If at some point during the motion of the Sun it is directly overhead, then the *gnomon* will be parallel to one of the generating lines of the cone and perpendicular to the intersecting plane.

The conic sections were recognized by the Greeks as being important geometrical objects, but they were not thought to have any physical significance. The only physically significant curves were the circle – circular motion being natural for heavenly bodies – and the straight line – rectilinear motion being natural for earthly bodies. This mode of thinking did not change until the seventeenth century when Galileo discovered that projectiles follow parabolic paths and Kepler showed that planets have elliptical orbits.

Aristarchus

The predominant Greek view of the Universe had the Earth at the centre. This was Eudoxus' starting point and was fundamental to the whole of Aristotelian physics. An alternative cosmology in which the Sun, Earth and the other celestial bodies revolved around a central fire had been suggested by the Pythagorean Philolaus, and another notable exception to the prevailing viewpoint was that espoused by Aristarchus, from the island of Samos in Asia Minor.[19] Aristarchus

[19] Very little is known about Aristarchus. According to Hipparchus he observed the time of the summer solstice in 281 BC.

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is credited by Archimedes as having postulated that the Earth is not at the centre of the Universe, but that it orbits the Sun. It was Copernicus, in the sixteenth century, who first developed a serious cosmology based on this principle, and so Aristarchus is often referred to as the Copernicus of antiquity. However, the attribution of a fully fledged heliocentric theory to Aristarchus is based actually on extremely scant evidence. Much is made nowadays of the following passage from Archimedes:

> But Aristarchus of Samos brought out a book consisting of certain hypotheses, in which the premises lead to the conclusion that the universe is many times greater than that now so called. His hypotheses are that the fixed stars and the sun remain motionless, that the earth revolves about the sun in the circumference of a circle, the sun lying in the middle of the orbit, and that the sphere of the fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface.[20]

Archimedes' work, of which the above quotation forms a part, is a demonstration of his skill in using the Greek numeration system to manipulate very large numbers. Ostensibly, Archimedes was attempting to calculate the number of grains of sand that the Universe could hold and he concluded that this had to be less than the unimaginably large number, 10^{63}. Clearly, this required a knowledge of the size of the Universe, and Archimedes tried deliberately to overestimate this. He criticized Aristarchus' heliocentric hypothesis because, in order for it to reproduce the observed phenomena; the stars would have to be infinitely far away, contradicting Archimedes' hypothesis that the Universe is finite. The problem is that of stellar parallax: if the Earth is orbiting the Sun, then the observed longitude at which a fixed star is found would vary as the Earth moved around its orbit (see Figure 2.6). Such changes were, however, not observed.

There are many theories, all very speculative, of why the heliocentric theory did not catch on and was superseded so completely by geocentric astronomy. Perhaps the most plausible is the simple fact that the geometrical skill of the Greeks allowed them to devise ingenious constructions that modelled accurately the motions of all the heavenly bodies without having to take the drastic step of removing the Earth from its privileged position at the centre of the Universe.

[20] ARCHIMEDES *Psammites* (*Sand Reckoner*). Translation from Heath (1932). Aristarchus' heliocentric theory is discussed at length in Heath (1913), but Wall (1975) is of the opinion that Heath has made rather too many assumptions and that there is no evidence that Aristarchus ever wrote a treatise on his heliocentric hypothesis. Indeed, it is very likely that all known references to Aristarchus' theory are derived from this one remark of Archimedes.

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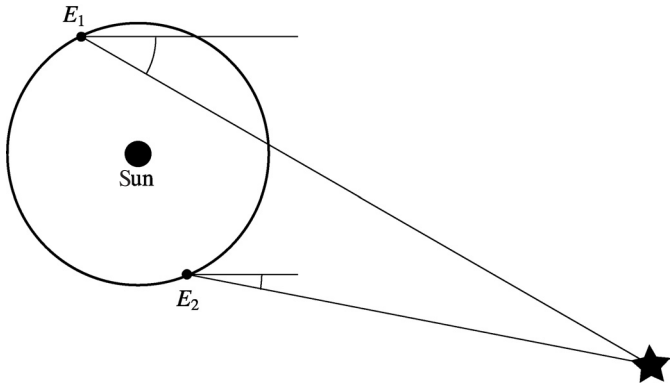

Fig. 2.6. Stellar parallax. E_1 and E_2 are two points on the orbit of the Earth.

The earliest complete astronomical treatise that has survived from ancient Greece is Aristarchus' *On the Sizes and Distances of the Sun and Moon*.[21] The organization of the work, starting with axioms and then making logical deductions from them, is similar to that of the great contemporary works in geometry such as Euclid's *Elements*. The treatise is based on six hypotheses:

(i) The Moon receives its light from the Sun.
(ii) The Earth is positioned at the centre of the sphere in which the Moon moves.
(iii) When the Moon appears to us halved, the great circle which divides the dark and bright portions of the Moon is in the direction of our eye.
(iv) When the Moon appears to us halved, its angular distance from the Sun is 87°. (This value is considerably in error. The true value is about 89° 50′.)
(v) The breadth of the Earth's shadow is that of two Moons. (This figure presumably was based on the length of lunar eclipses. The correct figure is nearer three lunar diameters.)
(vi) The Moon has an angular diameter of 2°. (About 4 times too large.[22])

[21] A translation is given in Heath (1913).
[22] Further on in the work Aristarchus assumed that the angular diameter of the Sun is the same as that of the Moon, i.e. 2°. Archimedes claimed that Aristarchus later discovered the much more accurate value of $1/2°$ for the angular diameter of the Sun, in agreement with his own observations (see Shapiro (1975)). It is likely that Aristarchus' figure of 2° for the Moon's angular diameter was not based on measurements at all, but simply assumed for the purposes of his demonstration (see van Helden (1985), p. 8).

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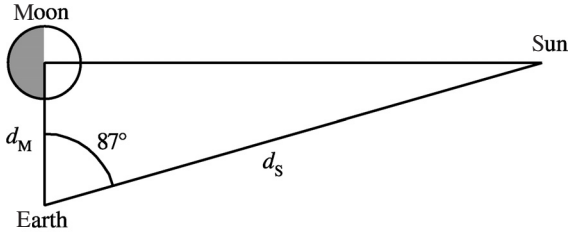

Fig. 2.7. Aristarchus' method for determining the relative distances from the Earth of the Sun and Moon.

Using the first four of these hypotheses, which are illustrated in Figure 2.7, Aristarchus concluded that the ratio of the distance from the Earth to the Sun, d_S, to that of the moon, d_M, satisfies the inequality

$$18 < d_S/d_M < 20.$$

The derivation of the value of d_S/d_M from Figure 2.7 would now be a simple exercise in trigonometry ($d_M/d_S = \cos 87^\circ = \sin 3^\circ$) but this had not yet been invented. The reason for the inequality that results from Aristarchus' calculations has nothing to do with an estimate of the accuracy of the experimental data, but instead represents the accuracy with which Aristarchus could determine what we would now write as $\sin 3^\circ$. Here, we will demonstrate his argument for showing that $d_S/d_M > 18$, which is particularly elegant. In Figure 2.8, E, S and M represent the Earth, Sun and Moon with $\angle ESM = 3^\circ$. The line EA is a continuation of ME such that $|AM| = |MS|$, and SC bisects the angle MSA. The line BC is perpendicular to AS so that $|BC| = |CM|$.

Aristarchus began by noting that

$$\frac{\angle CSM}{\angle ESM} = \frac{22\frac{1}{2}}{3} = \frac{15}{2},$$

from which it follows, using the equivalent of the fact that $\tan\alpha/\tan\beta > \alpha/\beta$ if $\beta < \alpha < 90^\circ$ (a result which was well known in Aristarchus' time; a proof can be found in Euclid's *Optics*), that

$$\frac{|CM|}{|EM|} > \frac{15}{2}. \tag{2.1}$$

Next, we observe that

$$\frac{|CA|^2}{|CM|^2} = \frac{|CA|^2}{|CB|^2} = 2 > \frac{49}{25},$$

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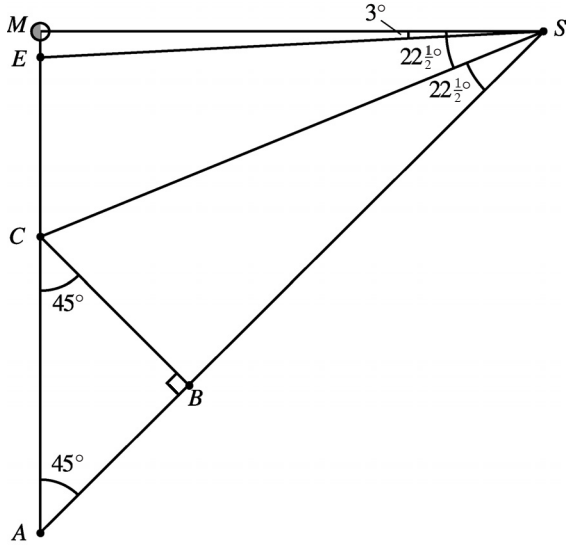

Fig. 2.8. Aristarchus' method for showing that $d_S/d_M > 18$.

and, hence, that $|CA|/|CM| > 7/5$, which in turn implies that

$$\frac{|AM|}{|CM|} = \frac{|CA| + |CM|}{|CM|} > \frac{12}{5}. \tag{2.2}$$

(Note that here Aristarchus has used the well-known Pythagorean approximation to $\sqrt{2}$, namely $7/5$.) Putting Eqns (2.1) and (2.2) together we obtain

$$\frac{|AM|}{|EM|} = \frac{|AM|}{|CM|} \cdot \frac{|CM|}{|EM|} > \frac{12}{5} \cdot \frac{15}{2} = 18.$$

But $|ES| > |MS| = |AM|$, and so we arrive finally at the desired result

$$\frac{d_S}{d_M} = \frac{|ES|}{|EM|} > 18.$$

Having determined the relative distances of the Sun and Moon, Aristarchus immediately could deduce the relative sizes of the Sun and Moon since (from the observed 'fact' that the Moon exactly eclipses the Sun) he assumed that they have the same angular diameter. Hence, the radii of the Sun and Moon, r_S and r_M, for example, satisfy

$$18 < r_S/r_M < 20.$$

Aristarchus' geometry based on the bisected Moon is flawless, but the inaccuracies in his observations mean that his results are not. The correct value

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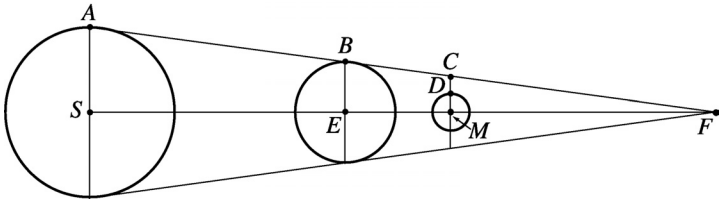

Fig. 2.9. A simplified illustration of Aristarchus' method for determining the relative sizes of the Sun and Moon.

of r_S/r_M is about 400. However, the conclusion that the distance to the Sun is about 19 times greater than the distance to the Moon appears frequently in astronomical writings over the course of the 2000 years following Aristarchus.

Aristarchus then went on to determine the sizes of the Sun and Moon relative to the Earth. His method was fairly complicated, but a simplified version is given below and illustrated in Figure 2.9.[23] The centres of the Sun, Earth and Moon are S, E and M, respectively, and the lines SA, EB and MC are all perpendicular to SEM and intersect the circles representing the Sun, Moon and Earth at A, B and D, respectively. Then ABF is a good approximation to the tangent to the Sun and Earth and, hence, $|MC|$ is close to the radius of the shadow of the Earth at the distance of the Moon, which is $2r_M$ according to Aristarchus' fifth hypothesis.

Using the similarity of the triangles ASF, BEF and CMF we can deduce that

$$r_S d_M + 2r_M d_S = r_E(d_S + d_M),$$

and if we substitute into this equation Aristarchus' mean value for the relative sizes of the Sun and Moon (i.e. $d_S = 19d_M$, $r_S = 19r_M$) we obtain[24]

$$\frac{r_S}{r_E} = \frac{20}{3} \qquad \text{and} \qquad \frac{r_M}{r_E} = \frac{20}{57}.$$

The correct value for r_S/r_E is about 109. Aristarchus used his results to show that the volumes of the Sun and Earth are in the ratio of between 254 : 1 and

[23] Based on that given in Boyer (1989). This type of eclipse diagram played a leading role in distance determinations until the seventeenth century.

[24] Aristarchus actually carried out his geometrical calculations maintaining the consequences of the inequalities for the relative sizes of Sun and Moon and obtained the estimates

$$\frac{19}{3} < \frac{r_S}{r_E} < \frac{43}{6} \qquad \text{and} \qquad \frac{43}{108} < \frac{r_M}{r_E} < \frac{19}{60}.$$

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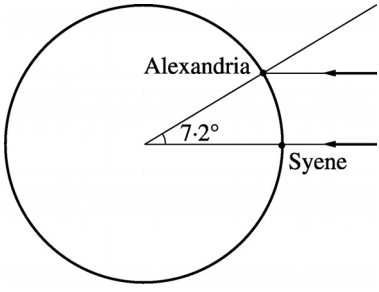

Fig. 2.10. Eratosthenes' method for determining the size of the Earth.

368 : 1. It may be that the fact that the Sun was thus such a massive object compared to the Earth was one of the reasons Aristarchus was led to believe that the Sun was at rest with the Earth orbiting round it.

While the relative sizes of the Sun, Moon and Earth were understood poorly by the ancient Greeks, the actual size of the Earth was determined with some accuracy. In *On the Heavens* Aristotle credits 'the mathematicians' with the figure of 400 000 *stade*s for the circumference of the Earth, but left no indication of how, or by whom, this figure was determined. A *stade* was a unit of distance the value of which varied from one place to another so it is hard to judge the accuracy of Aristotle's figure, but it would appear to be a significant overestimate.[25]

This value was improved significantly in the following century by Eratosthenes, the director of the Library at Alexandria (see Figure 2.10). He found that at noon on the summer solstice the Sun was directly overhead at Syene (present-day Aswan) while at the same time in Alexandria, which he assumed to be about 5000 *stade*s due north, the Sun was $7\frac{1}{5}^\circ$ (i.e. one-fiftieth of a circle) from the zenith.[26] Eratosthenes concluded that the circumference of the Earth is $5000 \times 50 = 250\,000$ *stade*s. This was later changed to 252 000, possibly in order to have a round figure of 700 *stade*s per degree. Depending on the interpretation of a *stade*, this could be anything from about 5 per cent too small to 25 per cent too big.[27]

[25] According to Diller (1949), 1 *stade* could be anything between about 150 and 200 m. Thus, 400 000 *stade*s is somewhere between 60 000 and 80 000 km. The Earth's circumference is actually about 40 000 km.

[26] Actually the difference in latitude between Aswan and Alexandria is slightly less than Eratosthenes' value, and there is a difference in longitude of approximately 3°.

[27] Eratosthenes' original work on the measurement of the Earth is lost and our knowledge of his technique comes from a number of later sources, notably Cleomedes, a popular writer on astronomical matters. There has been much debate about the precise details of Eratosthenes'

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Eratosthenes is credited[28] also with measuring the obliquity of the ecliptic, arriving at a value of 22/83 of a right angle,[29] or about $23^\circ\,51'$, which is within 1 per cent of the true value for that time.

Eccentric circles and epicycles

Some 150 years after Eudoxus, Apollonius of Perga (who spent most of his life in Alexandria and is now most famous for his work on conic sections) devised a new solution to Plato's challenge of saving the phenomena using uniform circular motions. No astronomical works of Apollonius survive; we know of his writing only through the later work of Ptolemy. It was well known that Eudoxus' scheme of homocentric spheres failed to account for certain readily observable phenomena (e.g. the varying brightness of the planets) and Apollonius suggested a scheme based on eccentric circles and epicycles in order to get round this difficulty.[30]

He developed two equivalent models for the motion of the Sun (illustrated in Figure 2.11) that were designed to account for the fact that the time between the vernal equinox and summer solstice (measured by Callippus as 94 days) is longer than that between the summer solstice and the autumnal equinox (92 days). In the first scheme, the annual motion of the Sun S was around a circle (the dashed circle in the diagram) the centre of which was not the Earth E, but a point D called the 'eccentre', displaced away from the Earth, the distance $|DE|$ being known as the 'eccentricity'. Alternatively, we can consider the Sun positioned on a small circle (the epicycle) the centre C of which rotates uniformly about the Earth around another circle (the deferent). The epicycle rotates in the opposite sense to that with which C rotates around the deferent, but with the same angular speed, so that $EDSC$ remains a parallelogram. The path traced out by the Sun S will again be the dashed circle. The point C, which rotates around the Earth at a uniform rate at the same average speed as the Sun, represents the mean position of the Sun.

method; see Dutka (1993) and the references cited therein; also, Fischer (1975), Rawlins (1982), Goldstein (1984). Posidonius used a similar method a century later, involving Alexandria and Rhodes, to obtain a similar estimate for the Earth's size. On the relative merits of Posidonius' and Eratosthenes' contributions (see, for example, Taisbak (1974)).

[28] By Ptolemy, *Almagest*, Book I, 12 (see Toomer (1984)).

[29] An explanation for this curious ratio is given in Fowler (1987), pp. 51–2.

[30] Apollonius was probably not the first person to develop an epicycle theory; indeed, van der Waerden (1974, 1982) provides evidence that the Pythagoreans were in possession of a primitive epicycle theory. However, as far as we know, Apollonius was the first to show how epicycles could be used to model all the main features of planetary motion, including retrograde motion.

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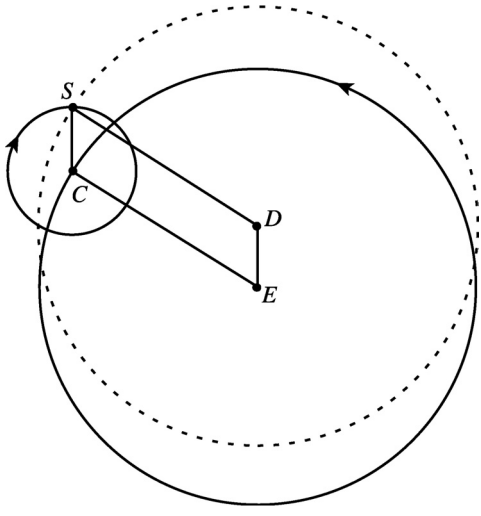

Fig. 2.11. Eccentric circles and epicycles for the Sun.

In order to use these models it is necessary to determine the length and direction of the line ED so that the position of the Sun relative to the Earth matches the observations. This requires the ability to solve the triangle DES which in turn requires trigonometry. It was precisely this sort of need that led to the later development of that subject by the astronomers Hipparchus and Ptolemy.

Apollonius also made the crucial realization that the epicyclic model could be used to account for the retrograde motions of the planets if the epicycle rotates in the same sense as that with which its centre rotates around the deferent; this is illustrated schematically in Figure 2.12. If the radii and the speeds of rotation of the two circles are chosen suitably, the combined effect produces motion that is predominantly anticlockwise, but with short periods during which the planet appears from the Earth to move in the opposite direction. It was well known in Apollonius' time that the superior planets appear brighter during their periods of retrograde motion and the epicycle model explained this quite naturally, because these were precisely the places at which the planets were closest to the Earth. The epicycle–deferent construction was a significant improvement over Eudoxus' homocentric spheres, but in its basic form it was still incapable of reproducing the complex motions of the planets with any real accuracy.

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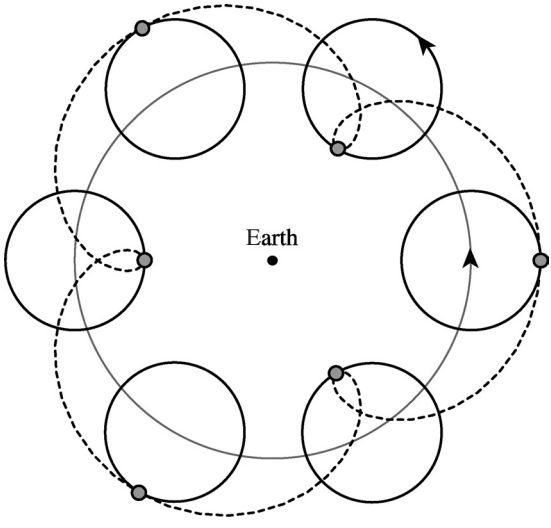

Fig. 2.12. Epicycles and retrograde motion.

Apollonius did more than just indicate that epicycles can be used to produce retrograde motion. He also proved an important theorem that provides a condition that the rotation rates and radii of the epicycle and deferent must satisfy for stationary points (i.e. points where the planet appears to change direction) to occur, and this is illustrated in Figure 2.13. The epicycle rotates anticlockwise about its centre C, with angular speed ω_2, while the point C rotates in the same sense with angular speed ω_1 around the deferent, the centre of which is the Earth, E. The resulting motion for the planet is the dashed curve, and there are two points at which the motion appears stationary as seen from the Earth, one of which is the point P.

The velocity of the planet at P can be decomposed into two parts, one due to the rotation of C around the deferent (the vector with magnitude u) and one due to the rotation of the epicycle (the vector with magnitude v). Clearly $u = \omega_1|EP|$ and $v = \omega_2|CP|$. For the point P to be stationary as seen from the Earth, the sum of these two velocities must lie along EP and, hence, we must have $u = v\cos\angle APD$. Now, since $\angle APC$ and $\angle BPD$ are both right angles it follows that $\angle APD = \angle CPB$ and, hence, that $\cos\angle APD = |BP|/2|CP|$. Putting all this together gives the condition for stationary points as

$$\omega_1|EP| = \tfrac{1}{2}\omega_2|BP|.$$

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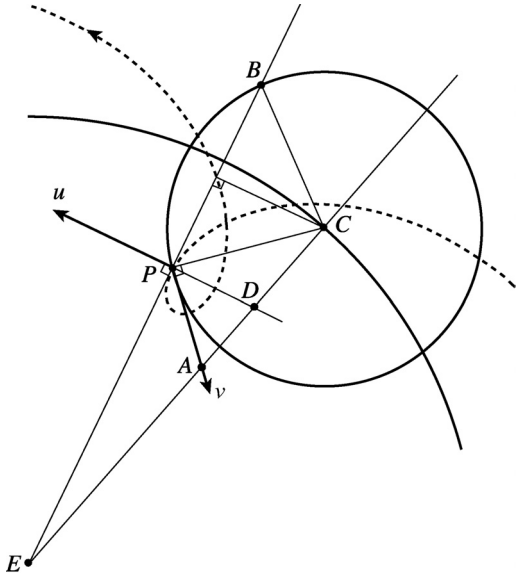

Fig. 2.13. Apollonius' theorem on stationary points.

Apollonius' proof was rather different (and rather more complicated) because he did not have modern trigonometrical methods at his disposal.[31] Now, $1/2|BP| < |CP|$ and $|EP| + |CP| > |EC|$, so a consequence of this theorem is that retrograde motion can occur only if the angular speeds and the radii of the epicycle and deferent satisfy the inequality[32]

$$\frac{\omega_1}{\omega_2} < \frac{|CP|}{|EC| - |CP|}.$$

In the hands of the ancient Greeks, the epicycle–deferent mechanism was employed skillfully to account for many different irregularities in the motions of the heavenly bodies. By altering the relative sizes of the two circles and their directions of rotation, different effects could be produced, and by placing epicycles upon epicycles, extremely complex motions could be generated. This basic mechanism was the cornerstone of all quantitative planetary theories until the time of Kepler. Even Copernicus, who succeeded in removing the need to

[31] More details can be found in Neugebauer (1955, 1959).

[32] It is not known whether Apollonius used his theorem to determine the numerical values of the parameters necessary to reproduce the observed phenomena, but Ptolemy later used it in Book XII of the *Almagest* to construct a table of stationary points even though it is not strictly valid for Ptolemy's more complicated scheme.

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model retrograde motion using epicycles, still made extensive use of them to account for other irregularities.[33]

[33] The power and versatility of the epicycle is described in Hanson (1960). Hanson shows that an orbit of any given shape can be reproduced to any given accuracy by a model built up from a finite number of epicycles placed one upon the other. Mathematically this is equivalent to approximating an arbitrary continuous function by a finite truncation of a Fourier series.

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