Problem 1. Consider the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & (k^2+6) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ k \end{bmatrix}$$

For what values of k does this system has no solution, exactly one solution, or infinitely many solutions? For each of the three cases find all possibilities for k.

Problem 2.

a) Find the matrix A describing the following linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$: First rotate clockwise by $\frac{\pi}{6}$, then scale by a factor of 2, then reflect about the x_2-axis.

b) Is the matrix A in part a) invertible? If so describe its inverse; if not explain why not.

Problem 3. Find the matrix of the orthogonal projection onto the plane $x+y-z=0$ in $\mathbb{R}^3$.

Problem 4. Let $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ -1 & 2 & 3 & 5 \\ 0 & 1 & k & k \end{bmatrix}$, where k is a real number. After applying a few elementary row operations to A we reach the matrix $\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & k-1 & k-1 \end{bmatrix}$.

a) Find the rank of A. (Your answer should be in terms of k.)

b) Which values of k is the image of A not equal to $\mathbb{R}^3$? For every such value of k, find a vector $\vec{b}$ in $\mathbb{R}^3$ not in $\text{im}(A)$.

c) Find a non-zero vector $\vec{c}$ in $\mathbb{R}^4$ which lies in $\ker(A)$ for every real number k.

Problem 5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

a) Find the matrix A of T with respect to the standard basis of $\mathbb{R}^2$.

b) Find the matrix B of T with respect to the basis

$$\mathcal{B} = \left\{ \vec{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

c) Let $\vec{w}$ have $\mathcal{B}$-coordinates $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $T(\vec{w})$.

Problem 6. Let $A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 7 \\ 2 \\ 4 \end{bmatrix}$.

a) Find the least-squares solution to the equation $A\vec{x} = \vec{b}$.

b) Find the vector in $\text{im}(A)$ closest to $\vec{b}$.

Problem 7. Find $\det A$, where A is the $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Problem 8. Let A be a 3×3 matrix with eigenvalues $-1, 0$, and 1.

a) Is A diagonalizable? Explain.

b) Show that $A^3 = A$.

c) Is A^2 similar to the matrix $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$?

Problem 9. True or False?
If B is a 6×6 matrix with characteristic polynomial $\lambda^3(\lambda-3)^2(\lambda+5)$, then $\text{rank}(B)$ is at least 3.

Problem 10.
a) Show that the quadratic form $q(\vec{v}) = 5(x^2+y^2+z^2) - 4(xy+yz+zx)$ is positive definite.

b) Find the semimajor and the semiminor axes of the ellipse $5x^2 + 8y^2 + 4xy = 1$.

c) What is the area of $5x^2 + 8y^2 + 4xy \leq 1$?

Problem 11. Consider the matrix $A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

a) Find the singular values of A along with their multiplicities.

b) Find a singular value decomposition $A = U\Sigma V^T$ of A.

Problem 12. Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Find the solution to the 3×3 continuous dynamical system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \text{ with initial condition } \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solutions

Problem 1. If $k = 1$, the system has no solution. If $k = -1$, the system has infinitely many solutions. If $k \neq 1, -1$, the system has a unique solution.

Problem 2. a) $A = \begin{bmatrix} -2\cos(-\pi/6) & 2\sin(-\pi/6) \\ 2\sin(-\pi/6) & 2\cos(-\pi/6) \end{bmatrix}$.

b) Yes, A is invertible and $A^{-1} = \begin{bmatrix} -\frac{1}{2}\cos(-\pi/6) & \frac{1}{2}\sin(-\pi/6) \\ \frac{1}{2}\sin(-\pi/6) & \frac{1}{2}\cos(-\pi/6) \end{bmatrix}$

Problem 3. $P = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

Problem 4. a) $\text{rank}(A) = 2$ if $k = 1$, and $\text{rank}(A) = 3$ if $k \neq 1$.

b) $k = 1$. The vector $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in $\text{im}(A)$.

c) The vector $\vec{c} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ is in $\ker(A)$ for every k.

Problem 5. a) $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

b) $B = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$.

c) $T(\vec{w}) = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$.

Problem 6. a) $\vec{x}^* = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

b) $A\vec{x}^* = \begin{bmatrix} -1 \\ 1 \\ 4 \\ 2 \end{bmatrix}$.

Problem 7. $\det(A) = 2$.

Problem 8. a) Yes. Since A is a 3×3 matrix and has three distinct eigenvalues, they each have algebraic and geometric multiplicity 1. Therefore there is an eigenbasis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for A where $A\vec{v}_1 = -\vec{v}_1, A\vec{v}_2 = \vec{0}, A\vec{v}_3 = \vec{v}_3$.

b) If S is the 3×3 matrix wth columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$, then A diagonalizes as

$$A = S \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} S^{-1}.$$

Hence

$$A^3 = S \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^3 S^{-1} = S \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} S^{-1} = A.$$

c) No, A^2 is not similar to B.

Problem 9. True.

Problem 10. a) $q(\vec{v}) = (x - 2y)^2 + (y - 2z)^2 + (z - 2x)^2 \geq 0$ with equality if and only if $x = y = z = 0$, so q is positive definite.

b) The semiminor is in the direction of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and the semimajor is in the direction of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

c) $\pi/6$.

Problem 11. a) $\sigma_1 = \sqrt{5}, \sigma_2 = \sqrt{2}, \sigma_3 = \sqrt{2}$.

b) $A = U\Sigma V^T$, where

$$U = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, V = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2/\sqrt{5} & 0 & 0 & 1/\sqrt{5} \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & 0 & -2/\sqrt{5} \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \sqrt{5} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix}.$$

Problem 12. $\vec{x}(t) = \begin{bmatrix} e^{-t}\cos(2t) \\ e^{-t}(\sin 2t + \cos 2t) \\ 1 \end{bmatrix}$.