1. (a) Show that a necessary condition for $2^m - 1$ to be a prime is that m is a prime.

   (b) For a positive integer n, let $\sigma(n) = \sum_{d|n} d$ be the sum of the positive divisors of n. An integer n is called *perfect* if $\sigma(n) = 2n$. Show that an even integer n is perfect if and only if $n = 2^{p-1}(2^p - 1)$ with p and $2^p - 1$ both primes.

2. (a) Explain why $x^2 + xy + 2y^2$ is the only reduced positive definite binary quadratic form of discriminant -7.

   (b) Show that an odd prime $p \neq 7$ can be expressed as $p = x^2 + xy + 2y^2$ with x and y integers if and only if $p \equiv 1, 2, 4 \bmod 7$.

3. Let $p > 2$ be a prime. Let $\left(\frac{\cdot}{p}\right)$ denote the Legendre symbol.

   (a) Show that
$$\sum_{k=1}^{p-1} \left(\frac{k(k+1)}{p}\right) = -1.$$

   (b) Assume that $p > 5$. Show that there are consecutive integers n and $n+1$ that are both quadratic residues modulo p.

4. Let $\pi(x)$ be the number of primes less than x. Suppose that the Prime Number Theorem holds:
$$\pi(x) \sim \frac{x}{\log x}.$$

   Show that for every constant $c > 1$ there exists $x(c) > 0$ such that if $x > x(c)$ then the interval $[x, cx]$ contains at least one prime number.