# MAT204 OVERVIEW OF CONTENTS AND SAMPLE PROBLEMS

MAT204 introduces the notion of vector space and linear dependence concretely by looking at the solutions to a system of $m$ linear equations in $n$ variables using vectors and matrices which are somewhat familiar to many students. Gauss-Jordan elimination is the standard algorithm taught in both MAT202 and MAT204 that produces not only the solution to a linear system, but also the basis of the column space, row space, and null space of a matrix. Gaussian elimination is further employed to derive the inverse of a non-singular matrix. Matrices are then presented as representation of linear transformations between vector spaces with respect to bases of both the domain and range.

After a brief survey of of determinant, we proceed to introduce the notion of eigenvalue and eigenvector of a square matrix along with their applications to Markov processes and graph/network theory.

The course then focuses on both real and complex vector spaces with inner product structure. The Gram-Schmid Process is an algorithm to render any basis of a vector space into an orthonormal basis. Applications include Fourier analysis and linear regression. The Spectral Theorem is then applied to decompose any rectangular matrix into a product of orthogonal and diagonal matrices. Various applications of Singular Value Decomposition to statistics, PCA, and machine learning are explored in lecture and homework exercises.

Additional topics not covered in MAT202 also include normal matrices and generalized spectral theorem, Jordan basis to diagonalize a deficient matrix as much as possible, and applications to solving a system of first order differential equations with constant coefficients.

A common feature of MAT202 and MAT204 exams are True and False questions with which students often have most difficulty. MAT204 exam questions are often more theoretical than MAT202, and involve simple proofs using proof techniques such as proof by contradiction and proof by inductions.

All sample problems here come from past MAT204 quizzes and exams and are chosen to represent core concepts and techniques from the class corresponding to a B-level of knowledge.

Problems on Vector Space and Linear Transformations

**Example 1 (Gaussian Elimination)**

The following system is not linear in the variables $\alpha, \beta, \gamma$:

$$\begin{array}{rcrcrcl} \sin\alpha & + & \cos\beta & + & k\tan\gamma & = & 1 \\ 2\sin\alpha & + & 2\cos\beta & & & = & 0 \\ \sin\alpha & + & 2\cos\beta & + & 2k\tan\gamma & = & k \end{array}$$

Nevertheless, try to solve the system. In particular, determine for what values of $k$ does the system have

(a) no solution?

(b) a uniqe solutions?

(c) infinitely many solutions?

**Example 2 (Subspaces Associated to a Matrix)**

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 8 & 12 \\ 0 & 0 & 0 & 4 & 8 \end{bmatrix}$$

Find a basis for the row space $R(A)$, the column space $C(A)$, and the nullspace $N(\mathrm{A})$ of $A$.

**Example 3 (Subspace and Basis)**

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Let $V = \{B \in M(3,3) | AB = BA\}$. Show that $V$ is a vector space.

(b) Find a basis for $V$.

(c) Find the least positive integer $k$ such that $A^k = 0$. *Such matrices are called nilpotent matrices of order $k$.*

(d) Now suppose $A$ is a non-zero $n \times n$ nilpotent matrix of order $k$, where $k > 0$ is the least positive integer such that $A^k = 0$. Show that $C(A) \cap N(A) \neq \{\vec{0}\}$, where $C(A)$ is the column space of $A$ and $N(A)$ is the null space of $A$.

(e) Again suppose $A$ is an $n \times n$ nilpotent matrix of order $k$, show that $k \leq n$. *Hint: Since $A^{k-1} \neq 0$, we can choose $\vec{v} \in \mathbb{R}^n$ such that $A^{k-1}\vec{v} \neq \vec{0}$. What can you say about linear dependence of $\vec{v}, A\vec{v}, \cdots, A^{k-1}\vec{v}$?*

**Example 4 (Rank-Nullity Theorem)**

Suppose $A \in M(4,4)$ and $A^2 = 0$, determine (with proof) the largest possible value $k$ of $rank(A)$ and give an example of a matrix $A$ with $rank(A) = k$.

**Example 5 (LDU Decomposition)**

$$A = \begin{bmatrix} 1 & 1 & a+1 \\ 1 & a+1 & 1 \\ a+1 & 1 & 1 \end{bmatrix}, \qquad a \neq 0, -3$$

(a) Find the $LDU$-decomposition of $A$ where $L$ is unipotent lower triangular, $D$ is diagonal, and $U$ is unipotent upper triangular.

(b) Find $A^{-1}$

**Example 6 (Change of Basis)**.

In signal processing, a measuring device can not capture all the information in a signal which contains an infinite amount of data. Instead the device can only sample a finite amount, at some fixed times. For example, a measuring device might only sample a function $f(t)$, for $t = 0, 1, 2, 3$, this corresponds to sampling at $1Hz$ for 3 seconds.

$$\mathcal{S} : \mathcal{F}(\mathbb{R}) \to \mathbb{R}^4, \qquad \mathcal{S}(f) = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

where $\mathcal{F}(\mathbb{R})$ is the vector space of all real valued functions on $\mathbb{R}$. $\mathcal{S}$ transforms a signal $f(t)$ into a 4-dimensional vector consisting of $f$ sampled at 4 different times.

(a) Show that $\mathcal{S}$ is a linear transformation.

(b) Now restrict sampling to $P_3(\mathbb{R})$, the space of polynomials in $x$ with real coefficients of degree $\leq 3$ where

$$S : P_3(\mathbb{R}) \to \mathbb{R}^4, \qquad S(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \end{bmatrix},$$

find the matrix $A$ representing the linear transformation $S$ with respect to the basis $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$ and the standard basis of $\mathbb{R}^4$.

(c) Prove that $S$ is an invertible linear transformation.

(d) Find the matrix $B$ representing the linear transformation

$$T = S^{-1} : \mathbb{R}^4 \to P_3(\mathbb{R})$$

with respect to the basis $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$ and the standard basis of $\mathbb{R}^4$.

**Example 7 (True or False)**

For each of the following statements, select $\boxed{T}$ if the statement is *always* true or $\boxed{F}$ if the statement is *not always* true. **Justify your answers.**

(a) $\boxed{T}\,\boxed{F}$: Let $f_1(x), f_2(x), f_3(x) \in P_2(\mathbb{R})$, and $f_1(0) = f_2(0) = f_3(0) = 1$, then $f_1(x), f_2(x), f_3(x)$ must be linearly dependent.

(b) $\boxed{T}\,\boxed{F}$: If two $m \times n$ matrices $A$ and $B$ have the same row space and column space, then $A = B$.

(c) $\boxed{T}\,\boxed{F}$: If any $n-1$ vectors of $\vec{v}_1, \cdots, \vec{v}_n$ are linearly independent, then $\vec{v}_1, \cdots, \vec{v}_n$ are linearly independent.

(d) $\boxed{T}\,\boxed{F}$: Let $A \in M(m, n)$, if the columns of $A$ are linearly dependent, then the rows of $A$ are linearly dependent.

(e) $\boxed{T}\,\boxed{F}$: Suppose $U, V, W$ are three distinct 2015-dimensional vector subspaces of $\mathbb{R}^{2016}$, then $U \cap V \cap W$ must be at least 2013 dimensional.

(f) $\boxed{T}\boxed{F}$: If $A$ and $B$ are similar, namely $A = SBS^{-1}$, then $N(A) = N(B)$, where $N(A)$ denotes the null space of $A$.

(g) $\boxed{T}\boxed{F}$: Suppose $A, B$ are symmetric matrices, i.e. $A^t = A$ and $B^t = B$, then $AB$ is also symmetric.

## Answers

1. **(Gaussian Elimination)**

$$rref\left(\begin{bmatrix} 1 & 1 & k & | & 1 \\ 2 & 2 & 0 & | & 0 \\ 1 & 2 & 2k & | & k \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & | & 2-k \\ 0 & 1 & 0 & | & k-2 \\ 0 & 0 & k & | & 1 \end{bmatrix}$$

   (a) $k = 0$: the system has no solution.

   (b) $k \neq 0$:

   - $1 \leq k \leq 3$: the system as infinitely many solutions.
   - $k < 1$ or $k > 3$: the system has no solution.

   (c) The system never has a unique solution in $\alpha, \beta, \gamma$.

2. **(Subspaces Associated to a Matrix)**

$R(A)$ has basis $\left\{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}\right\}$; $C(A)$ has basis $\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}\right\}$;

and $N(A)$ has basis $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$.

3. **(Subspace and Basis)**

   (a) Check linearity conditions are satisfied.

(b)

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} = \{I, A, A^2\}$$

(c) $k = 3$

(d) Non-zero elements in the column space of $A^{k-1}$ is in $C(A)$ and $N(A)$.

(e) First show that $\vec{v}, A\vec{v}, \cdots, A^{k-1}\vec{v}$ are linearly independent in $\mathbb{R}^n$, so $k \leq n$ must hold by Replacement Theorem.

4. **(Rank-Nullity Theorem)**

$$rank(A) \leq 2, \qquad A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. **(LDU Decomposition)**

(a)

$$\begin{bmatrix} 1 & 1 & a+1 \\ 1 & a+1 & 1 \\ a+1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ a+1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a(a+3) \end{bmatrix} \begin{bmatrix} 1 & 1 & a+1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$A^{-1} = \frac{1}{a^2(a+3)} \begin{bmatrix} -1 & -1 & a+2 \\ -1 & a+2 & -1 \\ a+2 & -1 & -1 \end{bmatrix}$$

6. **(Change of Basis)**

(a) Check linearity conditions.

(b)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix}$$

(c) Show $S$ is injective and surjective.

(d)

$$B = \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 & 0 \\ -11 & 18 & -9 & 2 \\ 6 & -15 & 12 & -3 \\ -1 & 3 & -3 & 1 \end{bmatrix} = A^{-1}$$

7. **(True or False)**

(a) F

(b) F

(c) F

(d) F

(e) T

(f) F

(g) T

Problems on Determinants, Eigenvalues and Eigenvectors

**Example 1 (Determinant)**

(a) Let $M$ be a $6\times 6$ invertible matrix such that $M^4+2M=0$. Find $\det(M)$.

(b) Let $A$ be an $n\times n$ matrix. Suppose $AA^t = I_n$ and $\det(A)<0$, show

$$\det(A+I_n)=0.$$

(c) Compute the determinant of the $n \times n$ matrix $A_n$ with diagonal entries 3 and 2's just above and 1's just below the diagonal.

$$\begin{bmatrix} 3 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 3 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 3 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 3 \end{bmatrix}$$

**Example 2 (Eigenvalues and Diagonalization)**

$$A = \begin{bmatrix} a+1 & 1 & 1 \\ 1 & a+1 & 1 \\ 1 & 1 & a+1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & a+1 \\ 1 & a+1 & 1 \\ a+1 & 1 & 1 \end{bmatrix}$$

(a) Find all eigenvalues and a basis for each corresponding eigenspace of

$$A = \begin{bmatrix} a+1 & 1 & 1 \\ 1 & a+1 & 1 \\ 1 & 1 & a+1 \end{bmatrix}, \qquad a \in \mathbb{R}$$

*Hint: You are advised not to use brute force. What are the obvious eigenvalues?*

(b) Deduce $\det(A)$ and $\det(B)$.
*Hint: You are advised not to use brute force.*

(c) Find all eigenvalues and a basis for each corresponding eigenspace of $B$.
*Hint: You are advised not to use brute force! What is an obvious eigenvalue? How can part (b) help?*

(d) When are $A$ and $B$ similar? Namely, for what values of $a \in \mathbb{R}$, if any, does there exist invertible $S$ such that $A = SBS^{-1}$?

(e) When are $A$ and $B$ simultaneously diagonalizable? Namely, for what values of $a \in \mathbb{R}$, if any, does there exist invertible $S$ such that $A = SDS^{-1}$ and $B = SES^{-1}$ where $D$ and $E$ are diagonal matrices.

**Example 3 (Complex Eigenvalues)**

Suppose $A_n$ is an $n \times n$ matrix with $a_{j,j+1} = 1$ for $1 \leq j \leq n-1$ and $a_{n1} = b^n$ and $a_{ij} = 0$ otherwise. Namely,

$$A_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ b^n & 0 & 0 & \cdots & 0 \end{bmatrix}$$

(a) Find the eigenvalues and corresponding eigenvectors of $A_n$ over $\mathbb{C}$.
*Hint: You may wish to start with the case $n = 3$ and then figure out the general solution.*

(b) Find $A_n^{-1}$.

**Example 4 (True or False)**

For each of the following statements, select $\boxed{T}$ if the statement is *always* true or $\boxed{F}$ if the statement is *not always* true. **Justify your answers.**

(a) $\boxed{T}\boxed{F}$: Suppose $\det(A) = \det(B) = 0$, then $\det\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = 0$.

(b) $\boxed{T}\boxed{F}$: Let $A$ be an $n \times n$ matrix, if $I_n - A^{2018}$ is invertible, then $I_n - A$ is also invertible.

(c) $\boxed{T}\boxed{F}$: Suppose $A \in M(n,n)$, then $A$ and $A^t$ have the same eigenvectors.

(d) $\boxed{T}\boxed{F}$: Suppose $A$ and $B$ are two $n \times n$ matrices, the $AB$ and $BA$ must be similar.

**Answers**

1. **(Determinant)**
   (a) 4
   (b) Write $I_n$ as $AA^t$ and use multiplicative property of determinant.
   (c) $\det(A_n) = 2^{n+1} - 1$.

2. **Eigenvalues and Diagonalization)**

   (a) $a+3$-eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $a$-eigenbasis is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

   (b)
   $$\det(A) = a^2(a+3), \qquad \det(B) = -a^2(a+3)$$

   (c) $a+3$-eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $a$-eigenbasis is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

   (d) $a = 0$

   (e) For all $a$.

3. **(Complex Eigenvalues)**

   (a)
   $$A_n \begin{bmatrix} 1 \\ b\omega^k \\ b^2\omega^{2k} \\ \vdots \\ b^{n-1}\omega^{(n-1)k} \end{bmatrix} = b\omega^k \begin{bmatrix} 1 \\ b\omega^k \\ b^2\omega^{2k} \\ \vdots \\ b^{n-1}\omega^{(n-1)k} \end{bmatrix}, \qquad \omega = e^{i2\pi/n}, \qquad 0 \le k \le n-1$$

   (b)
   $$A_N^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & b^{-n} \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

4. **(True or False)**

   (a) F

   (b) T

   (c) F

(d) F

## Problems on Inner Product Space

**Example 1 (QR Decomposition)**

$$M = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(a) Find the QR decomposition $M = QR$, where $Q$ is an orthogonal matrix and $R$ an upper triangular matrix with positive diagonal entries.

(b) Find $\det(M^tM)$ and $\det(MM^t)$. *You are advised not to use brute force.*

**Example 2 (Singular Value Decomposition)**

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \\ 2\sqrt{2} & 0 & 0 \end{bmatrix}$$

(a) Find the Singular Value Decomposition (SVD) of $A$ where

$$A = VDU^t$$

where $V$ and $U$ are orthogonal matrices and $D$ is diagonal.

(b) Find the least squares solution to $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$.

(c) Find the pseudo inverse $A^+$ of $A$, and compute $A^+A$ and $AA^+$.

(d) Find the least squares solution of minimal norm to $A^t\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

**Example 3 (Hermitian Matrices)**

Suppose $A$ is an $n \times n$ Hermitian matrix. Prove that

(a) $A - iI_n$ is invertible, where $i^2 = -1$ and $I_n$ is $n \times n$ identity matrix.

(b) $B = (A - iI_n)^{-1}(A + iI_n)$ is unitary.

(c) $B - I_n$ is invertible.

**Example 4 (Positive Definiteness)**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & x & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(a) For what values of $x$ is $A$ positive definite?

(b) For what values of $x$ is $A^2$ positive definite?
*Hint: You do not need to compute $A^2$.*

**Example 5 (True or False)**

For each of the following statements, select $\boxed{T}$ if the statement is *always* true or $\boxed{F}$ if the statement is *not always* true. **Justify your answers.**

(a) $\boxed{T}\boxed{F}$: $I - 2P$ is an orthogonal matrix if $P$ is an orthogonal projection matrix.

(b) $\boxed{T}\boxed{F}$: For any $n \times n$ matrix $A$ with $A^2 = A$, $\|A\vec{v}\| \leq \|\vec{v}\|$ for all $\vec{v} \in \mathbb{R}^n$.

(c) $\boxed{T}\boxed{F}$: If $A$ is a skew-symmetric $n \times n$ real matrix, then $Q_A(\vec{v}) = \vec{v}^t A\vec{v} = 0$ for all $\vec{v} \in \mathbb{R}^n$.

(d) $\boxed{T}\boxed{F}$: If $A$ is a skew-Hermitian $n \times n$ complex matrix, then $Q_A(\vec{v}) = \vec{v}^* A\vec{v} = 0$ for all $\vec{v} \in \mathbb{C}^n$.

(e) $\boxed{T}\boxed{F}$: If $A$ is Hermitian, and $A^{2015} = I$, then $A = I$.

(f) $\boxed{T}\boxed{F}$: If $A$ has only positive eigenvalues, then $A$ is positive definite.

**Answers**

1. **(QR Decomposition)**

(a)

$$M = \begin{bmatrix} \frac{1}{2} & -\frac{3}{\sqrt{12}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{2\sqrt{6}}{3} \end{bmatrix}$$

(b)

$$\det(M^t M) = 32, \qquad \det(MM^t) = 0$$

2. **(Singular Value Decomposition)**

(a)

$$A = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 & -\sqrt{15} & -\sqrt{6} & \sqrt{5} \\ 2 & \sqrt{15} & -\sqrt{6} & \sqrt{5} \\ 2 & 0 & -\sqrt{6} & -2\sqrt{5} \\ 3\sqrt{2} & 0 & 2\sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 0 & 1 \\ -1 & 0 & 3 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

(c)

$$A^+ = \frac{1}{24} \begin{bmatrix} 0 & 0 & 0 & 6\sqrt{2} \\ -8 & -8 & -8 & 6\sqrt{2} \\ -12 & 12 & 0 & 0 \end{bmatrix}, \qquad A^+ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AA^+ = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{5}{6} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d)

$$\frac{1}{24} \begin{bmatrix} -20 \\ 4 \\ -8 \\ 12\sqrt{2} \end{bmatrix}$$

3. **(Hermitian Matrices)**
   (a) Show the eigenvalues of $A - iI$ are never 0.
   (b) Show $BB^* = I_n$.
   (c) Eigenvalues of $B$ are never 0.
4. **(Positive Definiteness)**
   (a) $x > \frac{3}{2}$
   (b) $x \neq \frac{3}{2}$.
5. **(True or False)**
   (a) T
   (b) F
   (c) T
   (d) F
   (e) T
   (f) F

Problems on Jordan Basis and Differential Equations

**Example 1 (Jordan Basis)**
Recall that $P_3(\mathbb{R})$ is the vector space of all polynomials in $x$ with real coefficients of degree $\leq 3$. Fix some $a \in \mathbb{R}$, let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformation where

$$T(1) = 2, \qquad T(x) = 2x+1, \qquad T(x^2) = 2x^2+a, \qquad T(x^3) = 2x^3+x^2-ax+a^2.$$

(a) Find the $4 \times 4$ matrix $A$ representing the linear transformation $T$ with respect to the basis $\{1, x, x^2, x^3\}$.

(b) Find the Jordan normal form of $A$ and a Jordan chain basis for $A$.

(c) Find $T^3(x^3)$.

**Example 2 (System of Differential Equations)**

Consider the following system of differential equations

$$\begin{aligned} x'(t) &= y(t) \\ y'(t) &= -2x(t) + 3y(t) + z(t) \\ z'(t) &= x(t) - y(t) \end{aligned}$$

Given the initial conditions

$$x(0) = 1, \qquad y(0) = 0, \qquad z(0) = 1,$$

solve for $x(t), y(t)$ and $z(t)$.

**Example 3 (True or False)**

For each of the following statements, select $\boxed{T}$ if the statement is *always* true or $\boxed{F}$ if the statement is *not always* true. **Justify your answers.**

(a) $\boxed{T}$ $\boxed{F}$: If $A \in M(4,4)$ satisfies $0 < rank(A^3) < rank(A^2) < rank(A)$, then the Jordan normal form of $A$ is $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ regardless of what $A^4$ is.

(b) $\boxed{T}$ $\boxed{F}$: Any $n \times n$ matrix $A$ is similar to its transpose $A^t$.

(c) $\boxed{T}$ $\boxed{F}$: If $A$ and $B$ have Jordan normal forms $J_A$ and $J_B$ respectively, then $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ has Jordan normal form $\begin{bmatrix} J_A & 0 \\ 0 & J_B \end{bmatrix}$.

**Answers**

1. **(Jordan Basis)**

   (a)

$$A = \begin{bmatrix} 2 & 1 & a & a^2 \\ 0 & 2 & 0 & -a \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(b)

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

A chain basis consists of two chains: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ -a \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -a \\ 1 \end{bmatrix} \right\}.$

(c)

$$T^3(x^3) = 2^3x^3 + 3 \cdot 2^2x^2 - 3 \cdot 2^2ax + 3 \cdot 2^2a^2$$

2. **(System of Differential Equations)**

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} e^t - te^t \\ -te^t \\ e^t \end{bmatrix}$$

3. **(True or False)**

(a) F

(b) T

(c) F