MAT201 OVERVIEW OF CONTENTS AND SAMPLE PROBLEMS

The first main topic of MAT201 is vectors and the geometry of space. Using basic notions of distance and angle, as well as vector operations (dot and cross products), we can understand lines, planes, curves, quadric surfaces, and motions in $\mathbb{R}^3$.

Functions of several variables have important applications in approximation and optimization problems. We start by looking at their domain and range, and then their continuity and differentiability. In particular, the gradient plays an important role in the analysis of multivariate functions.

We then generalize the concept of integration to 2 and 3 dimensions. The main focus is on understanding the region of integration, setting up limits of integration for double and triple integrals, and integration techniques involving changing the order of integration or variables.

Finally, we study vector fields, i.e. vector-valued functions of several variables, and integrate vector fields along curves and surfaces. The course culminates in integral theorems (Green's, Stokes', Divergence Theorems) that generalize the Fundamental Theorem of Calculus.

All sample problems here come from past MAT201 quizzes and exams and are chosen to represent core concepts and techniques from the class corresponding to a B-level of knowledge.

Problems on Vectors and Basic Geometric Objects in $\mathbb{R}^3$

Example 1 (Vector Operations)

Let $A = (3, 3)$, $B = (-1, 4)$, $C = (1, -1)$ be three points in the plane.

(a) Find a point $P = (x, y)$ such that $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = \vec{0}$.

(b) Compute $\overrightarrow{PA} \times \overrightarrow{PB}$ and $\overrightarrow{PB} \times \overrightarrow{PC}$.

(c) Use properties of cross product to explain why $\overrightarrow{PA} \times \overrightarrow{PB} = \overrightarrow{PB} \times \overrightarrow{PC}$.

Example 2 (Lines and Planes)

Consider three points

$$P = (1, 1, 1), \qquad Q = (1, 2, 3), \qquad R = (0, 0, 1)$$

(a) Find an equation for the line L_1 that contains P and Q.

(b) Find an equation for the plane H that contains all three points P, Q and R.

(c) Let M be the plane $x + y - z = 2$. Find an equation for the line of intersection L_2 of the two planes H and M.

Example 3 (Curves and Quadric Surfaces)
Consider the curve C in space given by

$$\vec{r}(t) = \langle \sqrt{-t^2 + t}, \sqrt{t^2 + t} + 2, t - 1 \rangle, \qquad 0 \le t \le 1$$

(a) Find a parametric equation for the line tangent to C at $\vec{r}\left(\frac{1}{2}\right)$.

(b) Consider the quadric surface S given by

$$x^2 + y^2 - 4y - 2z + 2 = 0$$

Determine what type of quadric surface S is and sketch it.

(c) Verify that the curve C lies entirely in the surface S.

(d) The curve C also lies in a cone K with center at the point $P = (0, 2, -1)$. Find an equation for K and sketch it.

Example 4 (Motion in Space)
A bee flies along the curve

$$\vec{r}(t) = \langle t, e^{2t}, 2e^t \rangle.$$

Find the length of the path traveled by the bee from time $t = -1$ to $t = 1$.

Answers

1. **(Vector Operations)**

   (a) $P = (1, 2)$

   (b) Both are $\langle 0, 0, 6 \rangle$.

   (c)

$$\overrightarrow{PA} \times \overrightarrow{PB} = -(\overrightarrow{PB} + \overrightarrow{PC}) \times \overrightarrow{PB} = \overrightarrow{PB} \times (\overrightarrow{PB} + \overrightarrow{PC}) = \overrightarrow{PB} \times \overrightarrow{PC}$$

2. **(Lines and Planes)**

   (a) $\langle 1, 1+t, 1+2t \rangle$

   (b) $2x - 2y + z = 1$

   (c) $\langle 1+t, 3t, -1+4t \rangle$

3. **(Curves and Quadric Surfaces)**

   (a) $\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} + 2 + \frac{2}{\sqrt{3}}t, -\frac{1}{2} + t \right\rangle$

   (b) S is a paraboloid with vertext at $(0, 2, -1)$ opening up in the positive z-direction.

   (c) Just plug the equation for C into the equation for S, and check that it satisfies.

   (d) $\frac{1}{2}x^2 - \frac{1}{2}(y-2)^2 + (z+1)^2 = 0$

4. **(Motion in Space)**

$$2 + e^2 - e^{-2}$$

Problems on Functions of Several Variables

Example 1 (Limits)

For each of the following limits, compute their value or show they do not exist.

(a)

$$\lim_{(x,y)\to(2,1)} \frac{(x^2-4)(y-1)}{xy - x - 2y + 2}$$

(b) .

$$\lim_{(x,y)\to(0,0)} \cos\frac{2\pi x^2}{x^2+y^6}$$

(c)

$$\lim_{(x,y)\to(0,0)} \frac{xy^2 - 3x^2 - 3y^2}{x^2+y^2}$$

Example 2 (Gradient and Implicit Differentiation)

The function

$$F(x, y, z) = x^3 + 2z + e^{x+z} + 2z \ln y = 0$$

defines an implicit function $z = f(x, y)$.

(a) Compute ∇F.

(b) Find partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, 1, -1)$.

(c) Find the directional derivative of $z = f(x, y)$ at $(1, 1)$ in the direction $\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$.

Example 3 (Chain Rule)

$$f(x, y, z) = x^2 - x - y^2 + z^3$$

(a) Suppose a particle is moving on a trajectory given by

$$x = \sin t, \quad y = e^{t^2}, \qquad z = t^2$$

Compute $\frac{df}{dt}(0)$.

(b) Compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ at the point $(u, v) = \left(2, \frac{\pi}{2}\right)$ where

$$x = \ln(u \sin v), \qquad y = u \cos v, \qquad z = u$$

(c) Suppose z is defined implicitly as a function of x and y by $f(x, y, z) = 0$ for (x, y) near$(-1, 1)$. Note that $z(-1, 1) = -1$. Compute $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, and $\frac{\partial^2 z}{\partial x \partial y}$ at $(-1, 1)$.

Example 4 (Quadric Surfaces and Tangent Planes)

$$S_1 = \{2z^2 - x^2 - 2y^2 = 2\}$$

(a) Describe and sketch the surface S_1. You should label the intercepts on your sketch.

(b) Find an equation for the plane T that is tangent to S_1 at the point $(2, 1, 2)$.

(c) Let the surface S_2 be given by$4x^2 + y^2 + 4z^2 = 1$. Find two points on S_2 at which the tangent planes are parallel to T.

(d) Find the distance between the two parallel planes to S_2 from part (c).

Example 5 (Unconstrained Optimization)

$$f(x, y) = (x^2 + y^2 - 2y)e^{-y}$$

(a) Find and classify all critical points of $f(x, y)$.

(b) Find the absolute minimum and maximum of $f(x, y)$, or explain why they don't exist.
Hint: It could be helpful to look at $f(0, y)$ and see what happens when $y \to \pm\infty$.

Example 6 (Constrained Optimization)

$$S = \{x^2 + y^2 + 2z^2 = 1\}$$

Find the maximum and the minimum of the function $F(x, y, z) = e^{-2xyz}$ on S.

Answers

1. **(Multivariate Limits)**
   (a) 4
   (b) Limit does not exist.
   (c) -3

2. **(Gradient and Implicit Differentiation)**
   (a) $\left\langle 3x^2 + e^{x+z}, \frac{2z}{y}, 2 + e^{x+z} 2 \ln y \right\rangle$
   (b) $\frac{\partial z}{\partial x}(1, 1, -1) = -\frac{4}{3}$, $\frac{\partial z}{\partial y}(1, 1, -1) = \frac{2}{3}$
   (c) $-\frac{\sqrt{2}}{3}$

3. **(Chain Rule)**

(a) -1

(b) $\frac{\partial f}{\partial u} = \ln 2 + \frac{23}{2}$, $\frac{\partial f}{\partial v} = 0$

(c) $\frac{\partial z}{\partial x} = 1, \frac{\partial z}{\partial y} = \frac{2}{3}$, $\frac{\partial^2 z}{\partial x \partial y} = \frac{4}{3}$

4. **(Quadric Surfaces and Tangent Planes)**

   (a) S_1 is a hyperboloid of two sheets.

   (b) $x + y - 2z = -1$

   (c) $\left(\pm\frac{1}{6}, \pm\frac{2}{3}, \mp\frac{1}{3}\right)$

   (d) $\sqrt{\frac{3}{2}}$

5. **(Unconstrained Optimization)**

   (a) $(0, 2+\sqrt{2})$ is a saddle, $(0, 2-\sqrt{2})$ is a local minimum.

   (b) No absolute maximum, $(0, 2-\sqrt{2})$ is absoluteminimum.

6. **(Constrained Optimization)**

   Maximum is $e^{\frac{2}{\sqrt{54}}}$ and minimum is $e^{-\frac{2}{\sqrt{54}}}$.

Problems on Multiple Integrals

Example 1 (Double Integral - Order of Integration)

Write the following sum of two double integrals as one double integral using the order $dxdy$. No need to evaluate.

$$\int_1^2 \int_0^{2x-2} x^2 y\,dy\,dx + \int_2^4 \int_0^{4-x} x^2 y\,dy\,dx$$

Example 2 (Double Integral - Polar Coordinates)

Let D be the region in the xy-plane where

$$x^2 + y^2 \le 2x, \qquad \text{and} \qquad x^2 + y^2 \le 1 \qquad \text{and} \qquad y \ge 0$$

For any function $f(x, y)$, set up the computation of $\int\int_D f dA$ in both rectangular and polar coordinates.

Example 3 (Triple Integral - Rectangular Coordinates)
Let R be the region in space bounded by the surfaces

$$x = 0, \qquad y = 0, \qquad x + z = 4, \qquad z = 1, \qquad y = z^2$$

(a) Set up but do not evaluate the triple integral $\int\int\int_R f dV$ with the order of integration $dxdydz$.

(b) Set up but do not evaluate the triple integral $\int\int\int_R f dV$ with the order of integration $dxdzdy$.

Example 4 (Triple Integral - Cylindrical Coordinates)
Consider the following surfaces

$$S_1: \qquad 2z = x^2 + y^2 - 1 \qquad \text{for} \qquad z \leq 0$$

$$S_2: \qquad x^2 + y^2 + z^2 = 1 \qquad \text{for} \qquad z \geq 0$$

Find the volume of the solid region D enclosed by S_1 and S_2.

Example 5 (Triple Integral - Spherical Coordinates)
Compute the volume of the region in the first octant (where $x, y, z \geq 0$) outside the cone $z = \sqrt{3}\sqrt{x^2 + y^2}$ and inside the sphere of radius 2 centered at the origin.

Example 6 (Change of Variable Formula)
Let R be the region where $2 \leq x + y \leq 4$ and $1 \leq x - y \leq 2$. Sketch the region R and then compute $\int\int_R (x + y)\sin(x^2 - y^2) dA$.

Answers

1. **(Double Integral - Order of Integration)**

$$\int_0^2 \int_{1+\frac{y}{2}}^{4-y} x^2 y dx dy$$

2. **(Double Integral - Polar Coordinates)** Rectangular coordinates:

$$\int_0^{\frac{1}{2}} \int_0^{\sqrt{2x-x^2}} f(x,y)dydx + \int_{\frac{1}{2}}^1 \int_0^{\sqrt{1-x^2}} f(x,y)dydx$$

or

$$\int_0^{\frac{\sqrt{3}}{2}} \int_{1-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dxdy$$

Polar coordinates:

$$\int_0^{\frac{\pi}{3}} \int_0^1 f(r\cos\theta, r\sin\theta)rdrd\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta)rdrd\theta$$

or

$$\int_0^1 \int_0^{\arccos\frac{r}{2}} f(r\cos\theta, r\sin\theta)rd\theta dr$$

3. **(Triple Integral - Rectangular Coordinates)**

   (a) $\int_0^1 \int_0^{z^2} \int_0^{4-z} f dxdydz$

   (b) $\int_0^1 \int_{\sqrt{y}}^1 \int_0^{4-z} f dxdydz$

4. **(Triple Integral - Cylindrical Coordinates)**

$$\frac{11}{12}\pi$$

5. **(Triple Integral - Spherical Coordinates)**

$$\frac{2}{\sqrt{3}}\pi$$

6. **(Change of Variable Formula)**

$$-\frac{1}{4}\sin(8) + \frac{3}{4}\sin(4) - \frac{1}{2}\sin(2)$$

Problems on Integration of Vector Fields

Example 1 (Conservative Vector Fields)
Consider the vector field

$$\vec{F}(x,y) = \langle axy + y^2, x^2 + bxy\rangle$$

(a) For which a and b is $\vec{F}$ conservative?

(b) Using the values of a and b from the previous part, find a function $f(x,y)$ such that $\nabla f = \vec{F}$.

(c) Using the previous parts, give an equation of a line L in the plane with the property that for any segment C of L,

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

Example 2 (Line Integral)
Let C be the curve of intersection of the surfaces $x^2 + y^2 = 1$ and $z = x^8 + y^8$. It is oriented counterclockwise when viewed from the point $(0,0,10)$. Compute $\oint_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F}(x,y,z) = \langle 2xz + y, 2yz + 4x, x^2 + y^2\rangle$$

Example 3 (Flux Integral)
Let

$$\vec{F} = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle,$$

C_1 be the circle of radius 1 centered at the origin and C_2 comprise the sides of the square of side length 2 centered at the origin, both oriented counter-clockwise.

(a) Compute $div\vec{F}$.

(b) Compute $\int_{C_1} \vec{F} \cdot \vec{n} ds$.

(c) Compute $\int_{C_2} \vec{F} \cdot \vec{n} ds$.
Hint: Use Divergence Theorem on a suitable region.

Example 4 (Green's Theorem)

Let C be a simple closed curve given by

$$\vec{r}(t) = \langle 4\cos t + \cos(2t), 4\sin t + \cos t\rangle, \qquad 0 \le t \le 2\pi$$

Use Green's Theorem to find the area enclosed by C.

Example 5 (Stokes' Theorem)

Let

$$\vec{F} = \langle z, 2x, 3y\rangle.$$

and $S = S_1 + S_2 + S_3$ be the surface consisting of three sides:

$$S_1: \qquad y = z^2 - 1, \qquad y \le 0, \qquad -1 \le x \le 1,$$

$$S_2: \qquad x = 1, \qquad z^2 - 1 \le y \le 0$$

$$S_3: \qquad x = -1, \qquad z^2 - 1 \le y \le 0$$

C is the boundary of S.

(a) Evaluate

$$\int\int_S \vec{F} \cdot \vec{n} d\sigma$$

where $\vec{n}$ is the unit normal vector on S pointing away from the origin.

(b) Use Stokes' Theorem to evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where C is oriented clockwise when viewed from the the point $(0, 1, 0)$.

Example 6 (Divergence Theorem)

Let

$$\vec{F}(x, y, z) = \langle x, y, z + y^2\rangle$$

and S be the upper half of the unit sphere $x^2 + y^2 + z^2 = 1$. Evaluate the flux integral

$$\int\int_S \vec{F} \cdot \vec{n} d\sigma$$

where S is oriented upward.

Answers

1. **(Conservative Vector Fields)**
   (a) $a = b = 2$.
   (b) $f(x, y) = xy^2 + x^2y + c$
   (c) L is $x + y = 0$, or $x = 0$ or $y = 0$.

2. **(Line Integral)**
$$3\pi$$

3. **(Flux Integral)**
   (a) 0
   (b) 2π
   (c) 2π

4. **(Green's Theorem)**
$$16\pi$$

5. **(Stokes' Theorem)**
   (a) 0
   (b) -4

6. **(Divergence Theorem)**
$$\frac{9}{4}\pi$$