Stochastic Mechanics of Particles and Fields

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These slides are posted at

http://math.princeton.edu/~nelson/papers/xsmpf.pdf

A preliminary draft of the paper is at

http://math.princeton.edu/~nelson/papers/smpf.pdf

configuration space: M manifold

state space: TM tangent bundle

dynamical variable: function on $TM \times \mathbb{R}$

Newtonian mechanics

mass tensor: m_{ij}

kinetic energy: $T = \frac{1}{2} m_{ij} v^i v^j$

Newton's law: $F_i = m_{ij} a^j$

equations of motion:

$$\begin{aligned} \dot{x}^i &= v^i \\ \dot{v}^i &= m^{ij} F_j - \Gamma^i_{jk} v^j v^k \end{aligned}$$

Lagrangian mechanics

Lagrangian: dynamical variable $L = L(x, v, t)$

path: $X\colon \mathbb{R} \to M$ with velocity $\dot{X}$

action: $I = \displaystyle\int_{t_0}^{t_1} L(X, \dot{X}, t)\, dt$

Hamilton's principle of least action: I stationary under variation of path

Euler-Lagrange equation: $\dfrac{\partial L}{\partial x^i} - \dfrac{d}{dt}\dfrac{\partial L}{\partial v^i} = 0$

Basic mechanics

Basic = Newtonian $\cap$ Lagrangian

potential energy: V defined by $L = T - V$

Assume that the m_{ij} are constant, so the metric is flat and the Christoffel symbols Γ^i_{jk} are 0. (This can always be achieved at a single point in normal coordinates, so the following holds in general.)

$$\frac{\partial L}{\partial x^i} = -\frac{\partial V}{\partial x_i}$$

$$-\frac{\partial V}{\partial x^i} = \frac{d}{dt}\left(m_{ij}v^j - \frac{\partial V}{\partial v^i}\right)$$

$$\frac{d}{dt}(m_{ij}v^i) = F_i$$

$$F_i = -\frac{\partial V}{\partial x^i} + \frac{d}{dt}\frac{\partial V}{\partial v^i}$$

But F_i is a dynamical variable, a function of position and velocity, so $\dfrac{\partial V}{\partial v^i}$ must be independent of the velocity. That is, the Lagrangian must be a

basic Lagrangian:

$$L = \frac{1}{2}m_{ij}v^iv^j - \varphi + A_iv^i$$

scalar potential: φ

covector potential: A_i

Hamilton's principal function:

$$S(x,t) = -\int_t^{t_1} L(\big(X(s,x,t), \dot{X}(s,x,t), s\big)\, ds$$

A second form of the principle of least action is that S be stationary when the flow is perturbed by a time-dependent vector field.

Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla^i S - A^i)(\nabla_i S - A_i) + \varphi = 0$$

When the covector potential A_i is 0, so that $\varphi = V$, this becomes

$$\frac{\partial S}{\partial t} = \frac{1}{2}\nabla^i S \nabla_i S + V$$

Basic stochasticization

Two simplifying assumptions: First, take M to be $\mathbb{R}^n$ and the mass tensor to be a constant diagonal matrix giving the masses of the various particles making up the configuration. Second, take the covector potential A_i to be 0.

Let w be the Wiener process on M, the stochastic process of mean 0 characterized by

$$dw^i dw_i = \hbar dt + \mathrm{o}(dt)$$

We postulate that the motion of the configuration is a Markov process governed by the stochastic differential equation

$$dX^i = b^i\big(X(t), t\big)dt + dw^i$$

where b^i is the forward velocity.

Thus the fluctuations are of order $dt^{\frac{1}{2}}$, and with a value larger than $\hbar$ this postulate could be falsified by experiment, without violating the Heisenberg uncertainty principle.

Now let us compute the expected kinetic action of this process. Let $dt > 0$ and let $df = f(t+dt) - f(t)$ (the increment rather than the differential, which does not exist if f is not differentiable). Then

$$dX^i = \int_t^{t+dt} b^i\big(X(r), r\big)\, dr + dw^i$$

Apply this equation to itself, i.e. to $X(r)$, giving

$$\begin{aligned} dX^i &= \int_t^{t+dt} b^i \left(X(t) + \int_t^r b\big(X(s), s\big)\, ds + w(r) - w(t), r \right) \\ &= b^i dt + \nabla_k b^i W^k + dw^i + \mathrm{O}(dt^2) \end{aligned}$$

where

$$W^k = \int_t^{t+dt} [w^k(r) - w^k(t)]\, ds$$

From this it follows that

$$\frac{1}{2}dX^i dX_i = \frac{1}{2}b^i b_i dt^2 + b^i dw_i dt + \nabla_i b^i dt^2 + \frac{\hbar dt}{2} + \mathrm{o}(dt^2)$$

Let $\mathbb{E}_t$ be the conditional expectation given the configuration at time t.

First miracle:

The term $b_i dw_i dt$ is singular, of order $dt^{\frac{3}{2}}$, but by the Markov property $\mathbb{E}_t b^i dw_i dt = b^i \mathbb{E}_t dw_i dt = 0$.

Hence the expected energy is

$$\mathbb{E}_t \frac{1}{2}\frac{dX^i}{dt}\frac{dX_i}{dt} = \frac{1}{2}b^i b_i + \frac{1}{2}\nabla_i b^i + \frac{\hbar}{2dt} - V\big(X(t)\big) + \mathrm{o}(1)$$

Second miracle:

The singular term $\frac{\hbar}{2dt}$ is a constant not depending on the path, so it drops out when taking the variation—form the Riemann sum for the action, take the variation with the singular term dropping out, and then pass from the Riemann sum to the integral.

The stochastic principal function:

$$S(x,t) = -\mathbb{E}_{x,t} \int_t^{t_1} \left(\frac{1}{2} b^i b_i + \frac{\hbar}{2} \nabla_i b^i - V \right) \big(X(s), s \big) ds$$

where $\mathbb{E}_{x,t}$ is the expectation conditioned by $X(t) = x$.

In addition to the forward velocity b^i there are the

backward velocity: b_*^i

current velocity: $v^i = \dfrac{b^i + b_*^i}{2}$

osmotic velocity: $v^i = \dfrac{b^i - b_*^i}{2}$

The osmotic velocity depends only on the time-dependent probability density ρ. Let

$$R = \frac{\hbar}{2} \log \rho$$

Then

$$u^i = \frac{1}{\hbar} \nabla^i R$$

Computation shows that

$$\frac{\partial S}{\partial t} + \frac{1}{2}\nabla^i S \nabla_i S + V - \frac{1}{2}\nabla^i R \nabla_i R - \frac{\hbar}{2}\Delta R = 0$$
$$\frac{\partial R}{\partial t} + \nabla_i R \nabla^i S + \frac{\hbar}{2}\Delta S = 0$$

The first equation is the stochastic Hamilton-Jacobi equation. There is no deterministic analogue of the second equation since $R = 0$ when $\hbar = 0$. These two coupled nonlinear partial differential equations determine the process X.

Third miracle:

With

$$\psi = e^{(R+iS)}$$

these equations are equivalent to the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}\left(-\frac{1}{2}\Delta + V\right)\psi$$

This derivation is that of Guerra and Morato [2], but using the classical Lagrangian. The result extends to the general case, when there is a covector potential A_i and M is not necessarily flat; see [1].

Stochastic mechanics of particles.

The wave function ψ describes the Markov process completely:

$$\rho = |\psi|^2$$

$$u^i = \nabla^i \Re \log \psi$$

$$v^i = \nabla^i \Im \log \psi$$

Stochastic mechanics has been developed by many people, especially in Italy and the US. There are discussions of energy, nodes, interference, bound states, statistics (Bose or Fermi), and spin in [1], together with references to the original work.

The original hope that stochastic mechanics would provide a realistic alternative to quantum mechanics has not been realized by the theory in its present form.

This is because the Markov process lives on configuration space M, and a point in M may consist of widely separated particles in physical space.

This leads to an unphysical nonlocality if the trajectories of the process are regarded as physically real; see the discussion in [3].

Stochastic mechanics of fields

There are two motivations for applying stochastic mechanics to fields. One is that fields live on physical spacetime and nonlocality problems may be avoided. The other is that it may provide useful technical tools in constructive quantum field theory.

The strategy is to apply basic stochasticization to a basic field Lagrangian. So far as I know, this approach has not been tried before.

Consider a real scalar field φ on d-dimensional spacetime. Choose a spacelike hyperplane $\mathbb{R}^s$, where s, the number of space dimensions, is $d - 1$. The configuration space is a set of scalar functions φ on $\mathbb{R}^s$. Denote a velocity vector by π and define the kinetic energy by

$$\int_{\mathbb{R}^s} [(\nabla\varphi)^2 + \pi^2]\, dx_1 \ldots dx_s$$

Then the classical motion with zero potential energy satisfies the wave equation. Now we have the setup to apply stochastic mechanics, with a basic Lagrangian.

There are problems both in the classical and quantum theories due to the infinite number of degrees of freedom in field theory. All I have to report at present is this plan for research.

The hope is that making the Markov processes, rather than the quantum field, the focus of investigation will prove easier and more fruitful than the usual Hamiltonian approach of constructive quantum field theory.

The Markov process is governed by the action, but Hamiltonian methods require the exponential of the action, which is far harder to control.

References

[1] Edward Nelson, *Quantum Fluctuations*, Princeton University Press, Princeton, New Jersey, 1985.
http://www.math.princeton.edu/~nelson/books/qf.pdf

[2] Francesco Guerra and Laura M. Morato, "Quantization of dynamical systems and stochastic control theory", *Physical Review D*, Vol. 27, No. 3, pp. 1774-1786, 1983.
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[3] Edward Nelson, "Review of Stochastic Mechanics", *Journal of Physics: Conference Series, EmerQuM 11: Emergent Quantum Mechanics 2011 (Hans von Foerster Congress) 10-13 November 2011, Vienna, Austria*, ed. Gerhard Grössing, Vol. 361, 2012.
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