Stochastic Mechanics of Particles and Fields

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The configuration space of a physical system is a differentiable manifold M. The state of a system is given by a point x in M and a tangent vector v at x, the velocity of the configuration. Call the tangent bundle TM of M the state space of the system (though the phrase "velocity phase space" is often used). A dynamical variable is a possibly time-dependent function on state space.

Newtonian mechanics. The kinetic energy T of a system is given, using tensor notation and the summation convention, by

$$(1) \qquad T = \frac{1}{2} m_{ij} v^i v^j$$

where m_{ij} is a Riemannian metric, called the mass tensor. Using the Riemannian connection we can define acceleration. If $x(t)$ is the configuration of the system at time t, then in local coordinates the acceleration a is given by $a^i = \dot{v}^i + \Gamma^i_{jk} v^j v^k$ (where the dot is the time derivative). A force F is a possibly time-dependent covector field. Now we can express Newton's law:

$$(2) \qquad F_i = m_{ij} a^j$$

If F is given, we have the equations of motion for the system in state space TM, expressed in local coordinates by

$$(3) \qquad \begin{aligned} \dot{x}^i &= v^i \\ \dot{v}^i &= m^{ij} F_j - \Gamma^i_{jk} v^i v^k \end{aligned}$$

defining a flow, at least a local flow, on TM.

Lagrangian mechanics. A Lagrangian L is a dynamical variable with the dimensions of energy. Given a path X in configuration space M with velocity vector $\dot{X}$, define the action I by

$$(4) \qquad I = \int_{t_0}^{t_1} L(X, \dot{X}, t)\, dt$$

For an isolated system, L in time-independent. Hamilton's principle of least action is that I should be stationary under variations of the path with t_0 and t_1 fixed. This leads to the Euler-Lagrange equation

$$(5) \qquad \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial v^i} = 0$$

Now let us put Newtonian and Lagrangian mechanics together. Let M be a Riemannian manifold with mass tensor m_{ij} and kinetic energy (1), and define the potential energy V by $L = T - V$.

Given a point x on a Riemannian manifold M, it is possible to choose normal coordinates (NC) at x so that the first derivatives of m_{ij} and the Christoffel symbols are 0 at x. Thus

$$(6) \qquad \frac{\partial L}{\partial x^i} = -\frac{\partial V}{\partial x_i} \quad \text{(NC)}$$

By (5) and (6),

$$(7) \qquad -\frac{\partial V}{\partial x^i} = \frac{d}{dt}\left(m_{ij}v^j - \frac{\partial V}{\partial v^i}\right) \quad \text{(NC)}$$

By (2),

$$(8) \qquad \frac{d}{dt}(m_{ij}v^j) = F_i \quad \text{(NC)}$$

so that

$$(9) \qquad F_i = -\frac{\partial V}{\partial x^i} + \frac{d}{dt}\frac{\partial V}{\partial v^i} \quad \text{(NC)}$$

But F_i is a dynamical variable, a function of position and velocity, so $\frac{\partial V}{\partial v^i}$ must be independent of the velocity. That is, the Lagrangian must be of the form

$$(10) \qquad L = \frac{1}{2}m_{ij}v^i v^j - \varphi + A_i v^i$$

This is a tensor equation, so it holds globally in general coordinates. Call φ the scalar potential and A_i the covector potential, and call a Lagrangian of the form (10) a basic Lagrangian.

Let $X(s, x, t)$ be the configuration at time s of the system starting at x at time t. Hamilton's principal function is

$$(11) \qquad S(x,t) = -\int_t^{t_1} L\big((X(s,x,t), \dot{X}(s,x,t), s\big)\, ds$$

A second form of the principle of least action is that S be stationary when the flow is perturbed by a time-dependent vector field. This leads to the Hamilton-Jacobi equation

$$(12) \qquad \frac{\partial S}{\partial t} + \frac{1}{2}(\nabla^i S - A^i)(\nabla_i S - A_i) + \varphi = 0$$

with the same equations of motion (3). When the covector potential A_i is 0, so that $\varphi = V$, this becomes

$$(12') \qquad \frac{\partial S}{\partial t} = \frac{1}{2} \nabla^i S \nabla_i S + V$$

All of this is well known in classical deterministic mechanics, and an extended exposition is in Chapter II of [1].

Basic stochasticization. The word "stochasticization" is cacophonous, but it is a more accurate description of the procedure in stochastic mechanics than "stochastic quantization", because the physics remains classical despite the appearance of $\hbar$. (Also, the latter phrase has two distinct meanings.) The basic equations of stochastic mechanics will be derived here under two simplifying assumptions (see [1] for the general case). First, take M to be $\mathbb{R}^n$ and the mass tensor to be a constant diagonal matrix giving the masses of the various particles making up the configuration. Then M is flat and the Christoffel symbols Γ^i_{jk} are 0. But tensor notation is still useful; for example, v^i is the velocity and v_i is the momentum, and the Laplacian $\Delta = \nabla^i \nabla_i$ has the appropriate mass coefficients in it. Second, take the covector potential A_i to be 0.

Let w be the Wiener process on M, the stochastic process of mean 0 characterized by

$$(13) \qquad dw^i dw_i = \hbar dt + \mathrm{o}(dt)$$

We postulate that the motion of the configuration is a Markov process governed by the stochastic differential equation

$$(14) \qquad dX^i = b^i \big(X(t), t \big) dt + dw^i$$

where b^i is the forward velocity. Thus the fluctuations are of order $dt^{\frac{1}{2}}$, and with a value larger than $\hbar$ in (13) this postulate could be falsified by experiment, without violating the Heisenberg uncertainty principle.

Now let us compute the expected kinetic action of this process. Let $dt > 0$ and let $df = f(t + dt) - f(t)$ (the increment rather than the differential, which does not exist if f is not differentiable). From (14),

$$(15) \qquad dX^i = \int_t^{t+dt} b^i \big(X(r), r \big) \, dr + dw^i$$

Apply this equation to itself, i.e. to $X(r)$, giving

$$(16) \qquad \begin{aligned} dX^i &= \int_t^{t+dt} b^i \left(X(t) + \int_t^r b\big(X(s), s\big) \, ds + w(r) - w(t), r \right) dr + dw^i \\ &= b^i dt + \nabla_k b^i W^k + dw^i + \mathrm{O}(dt^2) \end{aligned}$$

where

$$(17) \qquad W^k = \int_t^{t+dt} [w^k(r) - w^k(t)] \, ds$$

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From this it follows that

$$(18) \qquad \frac{1}{2}dX^i dX_i = \frac{1}{2}b^i b_i dt^2 + b^i dw_i dt + \nabla_i b^i dt^2 + \frac{\hbar dt}{2} + \mathrm{o}(dt^2)$$

Let $\mathbb{E}_t$ be the conditional expectation given the configuration at time t. *First miracle*: the term $b_i dw_i dt$ in (18) is singular, of order $dt^{\frac{3}{2}}$, but by the Markov property $\mathbb{E}_t b^i dw_i dt = b^i \mathbb{E}_t dw_i dt = 0$. Hence the expected energy is

$$(19) \qquad \mathbb{E}_t \frac{1}{2}\frac{dX^i}{dt}\frac{dX_i}{dt} = \frac{1}{2}b^i b_i + \frac{1}{2}\nabla_i b^i + \frac{\hbar}{2dt} - V\left(X(t)\right) + \mathrm{o}(1)$$

Second miracle: the singular term $\frac{\hbar}{2dt}$ in (19) is a constant not depending on the path, so it drops out when taking the variation—form the Riemann sum for the action, take the variation with the singular term dropping out, and then pass from the Riemann sum to the integral. The stochastic principal function is

$$(20) \qquad S(x,t) = -\mathbb{E}_{x,t}\int_t^{t_1}\left(\frac{1}{2}b^i b_i + \frac{\hbar}{2}\nabla_i b^i - V\right)\left(X(s),s\right)ds$$

where $\mathbb{E}_{x,t}$ is the expectation conditioned by $X(t) = x$.

The definition of a Markov process is that given the present, the past and the future are conditionally independent. This is a time-symmetric notion. In addition to the forward velocity b^i there is the backward velocity b^i_*. The current velocity v^i and the osmotic velocity u^i are defined by

$$(21) \qquad v^i = \frac{b^i + b^i_*}{2}$$

$$(22) \qquad u^i = \frac{b^i - b^i_*}{2}$$

The osmotic velocity depends only on the time-dependent probability density ρ. Let

$$(23) \qquad R = \frac{\hbar}{2}\log\rho$$

Then

$$(24) \qquad u^i = \frac{1}{\hbar}\nabla^i R$$

Computation shows that

$$(25) \qquad \frac{\partial S}{\partial t} + \frac{1}{2}\nabla^i S\nabla_i S + V - \frac{1}{2}\nabla^i R\nabla_i R - \frac{\hbar}{2}\Delta R = 0$$

$$(26) \qquad \frac{\partial R}{\partial t} + \nabla_i R\nabla^i S + \frac{\hbar}{2}\Delta S = 0$$

Here (25) is the stochastic Hamilton-Jacobi equation. There is no deterministic analogue of (26) since $R = 0$ when $\hbar = 0$. These two coupled nonlinear partial differential equations determine the process X. *Third miracle*: with

(27) $$\psi = e^{(R+iS)}$$

these equations are equivalent to the Schrödinger equation

(28) $$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}\left(-\frac{1}{2}\Delta + V\right)\psi$$

This derivation is that of Guerra and Morato [2], but using the classical Lagrangian. The result extends to the general case, when there is a covector potential A_i and M is not necessarily flat; see [1].

Stochastic mechanics of particles. The wave function ψ describes the Markov process completely; $|\psi|^2$ is the time-dependent probability density ρ, $u^i = \nabla^i \Re \log \psi$, and $v^i = \nabla^i \Im \log \psi$. Stochastic mechanics has been developed by many people, especially in Italy and the US. There are discussions of energy, nodes, interference, bound states, statistics (Bose or Fermi), and spin in [1], together with references to the original work.

The original hope that stochastic mechanics would provide a realistic alternative to quantum mechanics has not been realized by the theory in its present form. This is because the Markov process lives on configuration space M, and a point in M may consist of widely separated particles in physical space. This leads to an unphysical nonlocality—instantaneous signaling between widely separated particles—if the trajectories of the process are regarded as physically real; see the discussion in [3].

Stochastic mechanics of fields. There are two motivations for applying stochastic mechanics to fields. One is that fields live on physical spacetime and nonlocality problems may be avoided. The other is that it may provide useful technical tools in constructive quantum field theory.

The strategy is to apply basic stochasticization to a basic field Lagrangian. So far as I know, this approach has not been tried before.

Consider a real scalar field φ on d-dimensional spacetime. Choose a spacelike hyperplane $\mathbb{R}^s$, where s, the number of space dimensions, is $d - 1$. The configuration space is a set of scalar functions φ on $\mathbb{R}^s$. Denote a velocity vector by π and define the kinetic energy by

(29) $$\int_{\mathbb{R}^s} [(\nabla \varphi)^2 + \pi^2]\, dx_1 \ldots dx_s$$

Then the classical motion with zero potential energy satisfies the wave equation. Now we have the setup to apply stochastic mechanics, with a basic Lagrangian. There are problems both in the classical and quantum theories due to the infinite number of degrees of freedom in field theory. All I have to report at present is this plan for research. The hope is that making the Markov processes, rather than the quantum field, the focus of

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investigation will prove easier and more fruitful than the usual Hamiltonian approach of constructive quantum field theory (whether in Minkowski spacetime or via the Euclidean method).

References

[1] Edward Nelson, *Quantum Fluctuations*, Princeton University Press, Princeton, New Jersey, 1985. http://www.math.princeton.edu/~nelson/books/qf.pdf

[2] Francesco Guerra and Laura M. Morato, "Quantization of dynamical systems and stochastic control theory", *Physical Review D*, Vol. 27, No. 3, pp. 1774-1786, 1983. http://prd.aps.org/abstract/PRD/v27/i8/p1774_1

[3] Edward Nelson, "Review of Stochastic Mechanics", *Journal of Physics: Conference Series, EmerQuM 11: Emergent Quantum Mechanics 2011 (Hans von Foerster Congress) 10-13 November 2011, Vienna, Austria*, ed. Gerhard Grössing, Vol. 361, 2012. http://iopscience.iop.org/1742-6596/361/1/012011/