REVIEW OF STOCHASTIC MECHANICS

References

Dynamical Theories of Brownian Motion

http://math.princeton.edu/~nelson/books/bmotion.pdf

Quantum Fluctuations

http://math.princeton.edu/~nelson/books/qf.pdf

together with references in the second book to the work of many people.

Classical mechanics

$\mathbb{R}^{dn}$ configuration space of n particles in d space dimensions

m_k mass of kth particle

$m_{ij} = m_k$ if $(k-1)d < i = j \leq kd$, 0 if $i \neq j$ Riemann metric

$T = \frac{1}{2} m_{ij} \dot{x}^i \dot{x}^j$ kinetic energy

(tensor notation with summation convention)

$L = T - V$ Lagrangian

$F_i = -\frac{\partial V}{\partial x^i} + \frac{d}{dt}\frac{\partial V}{\partial v^i}$ force

(x, v) state (position and velocity)

dynamical variable: function of (x, v)

$V = \varphi - A_i \dot{x}^i$ φ scalar potential, A covector potential

$S = -\int L$ Hamilton's principal function

From a variational principle, the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\nabla^i S - A^i\right)\left(\nabla_i S - A_i\right) + \varphi = 0$$

$F_i = m_{ij} \ddot{x}^j$ Newton equation

Stochastic mechanics — kinematics

A stochastic process is a Markov process in case past and future are conditionally independent given the present.

The theory is time symmetric.

In stochastic mechanics, the trajectory $x(t)$ is a Markov process:

$$dx(t) = dw(t) + b\big(x(t), t\big)dt$$

where b is the mean forward velocity.

w is a Wiener process (Brownian motion) with

$$\mathrm{E}_t\, dx^i(t) dx^j(t) = \hbar m^{ij} dt$$

where E_t is expectation with respect to the present at t.

This is just on the borderline of being falsifiable:

We can measure position at two times t_1 and t_2 with an error given by the uncertainty principle.

With a constant bigger than $\hbar$ in the diffusion tensor we could determine that particles do not move in the way predicted.

Since we have time symmetry, there is also

$b_*(x,t)$	mean backward velocity
$v = \dfrac{b + b_*}{2}$	current velocity
$u = \dfrac{b - b_*}{2}$	osmotic velocity
$\rho(x,t)$	probability density
$u^i = \dfrac{1}{2}\dfrac{\nabla^i \rho}{\rho}$	osmotic equation
$\dfrac{\partial \rho}{\partial t} = -\nabla_i(v^i \rho)$	current equation

Stochastic mechanics — dynamics

Classical dynamics comes from a variational principle applied to action integrals.

The trajectories of our Markov processes are nowhere differentiable.

How can we formulate the action?

$$\int \varphi\big(x(t), t\big)dt - \int A_j\big(x(t), t\big)dx^j(t) \quad \text{OK}$$

(ordinary Riemann integral and Fisk-Stratonovich time-symmetric stochastic integral)

The kinetic action is more subtle — work of Francesco Guerra and Laura Morato.

$$\frac{dx^i}{dt} \text{ is a difference quotient with } dt > 0 \text{ (not a derivative).}$$

Calculate

$$\mathrm{E}_t \, \frac{1}{2} \frac{dx^i}{dt} \frac{dx_i}{dt}$$

to o(1). Let

$$W^k = \int_t^{t+dt} [w^k(r) - w^k(t)] dr$$

We find

$$dx^i dx_i = b^i b_i dt^2 + 2b^i dw_i dt + 2\nabla_k b^i W^k dw_i + \mathrm{o}(dt^2)$$

First miracle: The term $2b^i dw_i dt$ is singular, of order $dt^{\frac{3}{2}}$, but

$$\mathrm{E}_t\, 2b^i dw_i dt = 0$$

Use the fact that w has orthogonal increments and calculate further. We find

$$\mathrm{E}_t\, \frac{1}{2}\frac{dx^i}{dt}\frac{dx_i}{dt} = \frac{1}{2}b^i b_i + \frac{1}{2}\nabla_i b^i + \frac{nd}{2dt} + \mathrm{o}(1)$$

Second miracle: The singular term $\dfrac{nd}{2dt}$ is a constant, independent of the trajectory, so it drops out in the variational principle.

Let $R = \frac{\hbar}{2} \log \rho$, so $\nabla^i R$ is the osmotic velocity u^i.

Apply the variational principle to the expected action. We find the stochastic Hamilton-Jacobi equation:

$$(1)\ \frac{\partial S}{\partial t} + \frac{1}{2}(\nabla^i S - A^i)(\nabla_i S - A_i) + \varphi - \frac{1}{2}\nabla^i R \nabla_i R - \frac{\hbar}{2}\nabla^i \nabla_i R = 0$$

Write the current equation in terms of R and S:

$$(2)\ \frac{\partial R}{dt} + \nabla_i R(\nabla^i S - A^i) + \frac{\hbar}{2}\nabla^i \nabla_i S - \frac{\hbar}{2}\nabla_i A^i = 0$$

(1) and (2) are a system of coupled nonlinear partial differential equations.

Let $\psi = e^{\frac{1}{\hbar}(R+iS)}$

Third miracle: (1) and (2) are equivalent to the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{1}{2}\left(\frac{\hbar}{i}\nabla^j - A^j\right)\left(\frac{\hbar}{i}\nabla_j - A_j\right) + \varphi\right]\psi$$

m_{ij} on $\mathbb{R}^{dn}$ is a flat Riemannian metric. We can apply the same procedure to any Riemannian manifold. We find the Schrödinger equation with an additional term

$$\frac{\hbar^2}{12}\bar{R} \quad \text{where } \bar{R} \text{ is the scalar curvature.}$$

This is the Bryce DeWitt term.

The mean forward derivative $\mathrm{D}F(t)$ of a stochastic process F is defined by

$$\mathrm{D}F(t) = \lim_{dt \to 0+} \mathrm{E}_t \frac{F(t+dt) - F(t)}{dt}$$

and the mean backward derivative is

$$\mathrm{D}_* F(t) = \lim_{dt \to 0+} \mathrm{E}_t \frac{F(t) - F(t-dt)}{dt}$$

The mean acceleration is

$$a = \frac{1}{2}(\mathrm{D}\mathrm{D}_* x + \mathrm{D}_* \mathrm{D} x)$$

We have the stochastic Newton equation

$$F_i = m_{ij} a^j$$

Successes of stochastic mechanics

• classical derivation of the Schrödinger equation — Francesco Guerra and Laura Morato

• the probability density ρ of the Markov process agrees with $|\psi|^2$ at all times

• stochastic explanation of the relation between momentum and the Fourier transform of the wave function — David Shucker, Eric Carlen

• existence of the Markov process under the physically natural assumption of finite action — Eric Carlen; this is the most technically demanding work in the entire subject

• a stochastic explanation of why identical particles satisfy either Bose-Einstein or Fermi-Dirac statistics if $d \geq 3$, with parastatistics possible if $d = 2$

• a stochastic explanation of spin and why it is integral or half-integral — Thaddeus Dankel, Daniela Dohrn and Francesco Guerra

• if the force is time-independent, conservation of the expected stochastic energy $\mathrm{E}_t \left(\frac{1}{2} u^i u_i + \frac{1}{2} v^i v_i + \varphi \right)$

• a stochastic picture of the two slit experiment, explaining how particles have trajectories going through just one slit but produce a probability density as for interfering waves

Failures of stochastic mechanics

• extreme non-locality: with two dynamically uncoupled particles, a force applied to one can immediately affect the other, in a way independent of their spatial separation

see §23 of Quantum Fluctuations

• wrong predictions for measurements at different times: with two dynamically uncoupled particles, measurements of their positions at two different times disagree with quantum mechanics

see the Afterword (Chapter Ten) of "Diffusion, Quantum Theory, and Radically Elementary Mathematics", ed. William G. Faris, Mathematical Notes # 47, Princeton University Press (2006)

Puzzle: How can a theory be so right and yet so wrong? Is stochastic mechanics an approximation to a correct theory? If so, what?

Open questions

The spin-statistics theorem is a result of relativistic quantum field theory. Can stochastic mechanics give an explanation in terms of nonrelativistic particle mechanics?

The solution of the Schrödinger equation is computationally difficult. Is there a way in stochastic mechanics to study the motion of an ensemble of configurations only along the most probable trajectories, thus giving a stochastic method for solving the Schrödinger equation? This seems unlikely, but a positive solution would be of major computational importance.

These notes are posted at

http://www.math.princeton.edu/∼nelson/papers/sm.pdf