Please show all work. Books, notes, computers, calculators, cell phones, etc. are not permitted in this exam.

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Your name (print)

Write below and sign the Pledge: I pledge my honor that I have not violated the Honor Code during this examination.
1. (10 points total)

(a) Compute the gradient of the function

\[ f(x, y, z) = \ln(zx + \sin y) + ze^{xyz} \]

(b) Find an equation of the form \( ax + by + cz = d \) for the tangent plane to the level surface of \( f \) through \((1, 0, 1)\).

(c) Compute the directional derivative in direction \((-1, 2, 3)\) of \( f \) at \((1, 0, 1)\).

Solution:

(a) \[
\nabla f(x, y, z) = \begin{pmatrix}
\frac{z}{zx+\sin y} + yz^2e^{xyz} \\
\frac{\cos y}{zx+\sin y} + xz^2e^{xyz} \\
\frac{x}{zx+\sin y} + (1 + xyz)e^{xyz}
\end{pmatrix}
\]

(b) \[
\nabla f(1, 0, 1) = \begin{pmatrix}
1 \\
2 \\
2
\end{pmatrix}
\]

so the tangent plane is the set of \((x, y, z)\) such that

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} - \begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix} \cdot \begin{pmatrix}
1 \\
2 \\
2
\end{pmatrix} = 0
\]

or:

\[x + 2y + 2z = 3\]

(c) \[
D_{(-1,2,3)} f(1,0,1) = (1 2 2) \begin{pmatrix}
-1 \\
2 \\
3
\end{pmatrix} = 9
\]
2. (10 points total)

(a) Express the function

\[ f(x, y, z) = \left( \begin{array}{c}
(xy \cos z) \ln(1 + xy \cos z) + e^{x+y^2+z-1} \\
(xy \cos z) \ln(x + y^2 + z) + e^{xy \cos z}
\end{array} \right) \]

as a composition and use the chain rule to compute the total derivative \( Df \) at \((2, 0, -1)\).

(b) Let \( h(t) = g(t^2, t^3) \), where \( g \) is an unknown function. Write an expression for \( \frac{dh}{dt} h(t) \) in terms of the partial derivatives of \( g \).

Solution:

(a) Write \( f(x, y, z) = g(h(x, y, z)) \) where

\[ g(u, v) = \left( \begin{array}{c}
u \ln(1 + u) + e^{v-1} \\
u \ln v + e^u
\end{array} \right)\]

and

\[ h(x, y, z) = \left( \begin{array}{c}u(x, y, z) \\
v(x, y, z)
\end{array} \right) = \left( \begin{array}{c}xy \cos z \\
x + y^2 + z
\end{array} \right)\]

So

\[ Dg(u, v) = \left( \begin{array}{c}
\ln(1 + u) + \frac{u}{1+u} e^{v-1} \\
\ln v + e^u \frac{u}{v}
\end{array} \right)\]

and

\[ Dh(x, y, z) = \left( \begin{array}{ccc}
y \cos z & x \cos z & -xy \sin z \\
1 & 2y & 1
\end{array} \right)\]

So

\[ h(2, 0, -1) = \left( \begin{array}{c}0 \\
1
\end{array} \right)\]

and

\[ Dh(2, 0, -1) = \left( \begin{array}{ccc}0 & 2 \cos(-1) & 0 \\
1 & 0 & 1
\end{array} \right)\]

and

\[ Dg(0, 1) = \left( \begin{array}{c}0 \\
1
\end{array} \right)\]

so

\[ Df(2, 0, -1) = Dg(h(2, 0, -1))) Dh(2, 0, -1) = \left( \begin{array}{c}0 \\
1
\end{array} \right) \left( \begin{array}{c}0 & 2 \cos(-1) & 0 \\
1 & 0 & 1
\end{array} \right) = \left( \begin{array}{c}1 \\
0 & 2 \cos(-1) & 0
\end{array} \right)\]

(b) Since \( h \) is the composition of the function \( g \) and \( t \mapsto (t^2, t^3) \) we have

\[ \frac{dh}{dt} h(t) = \left( \frac{\partial}{\partial x} g(t^2, t^3) \right) \left( \frac{\partial}{\partial y} g(t^2, t^3) \right) \left( \begin{array}{c} 2t \\
3t^2
\end{array} \right) = 2t \cdot \frac{\partial}{\partial x} g(t^2, t^3) + 3t^2 \cdot \frac{\partial}{\partial y} g(t^2, t^3) \]
3. (20 points total)
Consider the system of equations
\[
\begin{align*}
xy + zy^2 + xz^2 &= 1 \\
x - 2xyz - z^3 &= 2
\end{align*}
\]
and the solution \( p = (1, -1, 1) \).
(a) Show that the solutions \((x, y, z)\) of the system can be expresses near \( p \) as \((x, y(x), z(x))\).
(b) Compute \( y', z' \) at the points corresponding to the solution \( p \).
(c) Estimate a solution \( q \) to the equations whose first coordinate is 0.9.
Solution:
(a) Consider the function
\[
F(x, y, z) = \left( \begin{array}{c} xy + zy^2 + xz^2 \\ x - 2xyz - z^3 \end{array} \right)
\]
Then
\[
DF(x, y, z) = \left( \begin{array}{ccc} y + z^2 & x + 2yz & y^2 + 2xz \\ 1 - 2yz & -2xz & -3z^2 - 2xy \end{array} \right)
\]
so
\[
DF(1, -1, 1) = \left( \begin{array}{ccc} 0 & -1 & 3 \\ 3 & -2 & -1 \end{array} \right)
\]
In order to express \( y, z \) in terms of \( x \) the implicit function theorem tells us that the matrix whose columns are the columns of \( DF(1, -1, 1) \) corresponding to \( y, z \) must have full rank. In this case this is the matrix
\[
A = \left( \begin{array}{cc} -1 & 3 \\ -2 & -1 \end{array} \right)
\]
which is indeed non-singular because \( \det A = -3 + 6 = 3 \neq 0 \).
(b) By the implicit function theorem,
\[
A \left( \begin{array}{c} y'(1) \\ z'(1) \end{array} \right) = - \left( \begin{array}{c} 0 \\ 3 \end{array} \right)
\]
so
\[
\left( \begin{array}{c} y'(1) \\ z'(1) \end{array} \right) = -A^{-1} \left( \begin{array}{c} 0 \\ 3 \end{array} \right) = - \left( \begin{array}{cc} -1/7 & -3/7 \\ 2/7 & -1/7 \end{array} \right) \left( \begin{array}{c} 0 \\ 3 \end{array} \right) = \left( \begin{array}{c} 9/7 \\ -3/70 \end{array} \right)
\]
(c)
\[
q \approx \left( \begin{array}{c} 0.9 \\ -1 + (-0.1)y'(-1) \\ 1 + (-0.1)z'(1) \end{array} \right) = \left( \begin{array}{c} 0.9 \\ -1 - 9/70 \\ 1 - 3/70 \end{array} \right) = \left( \begin{array}{c} 0.9 \\ -79/70 \\ 67/70 \end{array} \right)
\]
4. (20 points total)

(a) Find the critical points of the function

\[ f(x, y, z) = x^3 - 4x^2 - 2y^2 - z^2 - 2xy - 2yz \]

(b) Classify the critical points as local minimum, local maximum, saddle point or "can’t tell".

(c) Which, if any, of the points you found in (b) is a global extremum?

Solution:

(a) The derivative is

\[ \nabla f(x, y, z) = \begin{pmatrix} 3x^2 - 8x - 2y \\ -4y - 2x - 2z \\ -2z - 2y \end{pmatrix} \]

so \( \nabla f = 0 \) if and only if either

\[
\begin{align*}
x &= -y \\
z &= -y \\
3x^2 - 6x &= 0
\end{align*}
\]

and \( x = 2, y = -2, z = 2 \), or else \( x = y = z = 0 \).

(b) The Hessian is

\[
\begin{pmatrix}
6x - 8 & -2 & 0 \\
-2 & -4 & -2 \\
0 & -2 & -2
\end{pmatrix}
\]

When \( x = 0 \) this matrix is

\[
\begin{pmatrix}
-8 & -2 & 0 \\
-2 & -4 & -2 \\
0 & -2 & -2
\end{pmatrix}
\]

and is negative semi-definite, so \( (0, 0, 0) \) is a local max. For \( x = 2 \), it is

\[
\begin{pmatrix}
4 & -2 & 0 \\
-2 & -4 & -2 \\
0 & -2 & -2
\end{pmatrix}
\]

which is indefinite, so \( (2, -2, 2) \) is a saddle.

(b) The function has no global max and min.
5. (20 points total)

Maximize \( x^2 + y \) subject to \( xy \geq 1 \) and \( 2x + 2y \leq 5 \) (you may assume that local maxima are global).

Solution: The constraints are \( g_1(x, y) = -xy \leq -1 \) and \( g_2(x, y) = 2x + 2y \leq 5 \).

First check degenerate points: the matrix of gradients is

\[
\begin{pmatrix}
-y & -x \\
2 & 2
\end{pmatrix}
\]

The second row is non-zero so when only the 2nd eq. is binding there is no degeneracy. When only the 1st is binding, that is \( xy = 1 \), the first row of the matrix cannot be zero, so there is no degeneracy. Finally, when both are binding we combine \( xy = 1 \) and \( 2x + 2y = 5 \) to find that \( x(\frac{1}{2}(5 - x)) = 1 \), or: \( x^2 - 5x + 2 = 0 \). There are two solutions but one can check that for each solution \( x, y = \frac{1}{2}(x - 5) \neq x \) and so the rank of the matrix of gradients is 2. Hence there are no degeneracies.

Next we solve the lagrange equations

\[
\begin{align*}
2x + \lambda y - 2\mu &= 0 \\
1 + \lambda x - 2\mu &= 0 \\
\lambda(xy - 1) &= 0 \\
\mu(2x + 2y - 5) &= 0 \\
\lambda &\geq 0 \\
\mu &\geq 0 \\
xy &\geq 1 \\
2x + 2y &\leq 5
\end{align*}
\]

If \( \lambda = 0 \) and \( \mu = 0 \) then (2) is violated.

If \( \lambda = 0 \) and \( \mu \neq 0 \) then from (1),(2) we have \( x = 1/2 \) and from (4), \( y = 2 \). Then \( \mu = 1/2 \) by (1) so this is a candidate.

If \( \mu = 0 \) and \( \lambda \neq 0 \) then from (3) \( x, y \neq 0 \). By (2) \( \lambda = -1/x \) so by (1) \( 2x^2 - y = 0 \), By (3), \( 2x^3 - 1 = 0 \) or: \( x = 1/\sqrt[3]{2} \). Then \( \lambda < 0 \), which contradicts (5)

Finally, if \( \lambda, \mu \neq 0 \) we have the system

\[
\begin{align*}
xy &= 1 \\
2x + 2y &= 5 
\end{align*}
\]

which has two solutions, \((\frac{1}{2}, 2)\) and \((2, \frac{1}{2})\). The second of these gives a larger value of \( x^2 + y \) so this is the max.
6. (20 points total)

Find the maximum of $xy$ subject to $x^2 + y^2 = 1$, $x - y \leq 0$ and $x \geq -1/2$.

Solution: We check for degenerate points. The gradient matrix of the constraints is

\[
\begin{pmatrix}
2x & 2y \\
1 & -1 \\
0 & -1
\end{pmatrix}
\]

which has full rank already. Since the last two rows are non-zero and LI of each other, we need to check the combinations of rows 1 alone, 1 and 2, 1 and 3, and all three rows.

The first row alone is LD only if $x = y = 0$ which violates the first constraint.

Row 1 and 2 are LD if and only if either $x = -y$. This leads to a degenerate point if the first two constraints are satisfied equalities. But if $x - y = 0$ then $x = y$. Since also $x = -y$ we find $x = y = 0$ which violates $x^2 + y^2 = 1$.

Row 1 and 3: these are LD if and only if $x = 0$. If this is true and the 1st and 3rd constraints are binding, but then the 3rd says $x = -1/2$ which is a contradiction.

All 3: if all 3 constraints are binding we find $x = -1/2$, $y = x = -1/2$, and so $x^2 + y^2 = 1$ is violated.

In summary, there are no degenerate points.

Next we solve

\[
\begin{align*}
y - 2\lambda x - \mu_1 + \mu_2 &= 0 \\
x - 2\lambda y + \mu_1 &= 0 \\
\mu_1(x - y) &= 0 \\
\mu_2(x + 1/2) &= 0 \\
\mu_1 &\geq 0 \\
\mu_2 &\geq 0 \\
x - y &\leq 0 \\
x &\geq -\frac{1}{2} \\
x^2 + y^2 &= 1
\end{align*}
\]

If $\mu_1 = 0$ and $\mu_2 = 0$, we have the system

\[
\begin{align*}
y - 2\lambda x &= 0 \\
x - 2\lambda y &= 0 \\
x^2 + y^2 &= 1
\end{align*}
\]

Clearly $x, y, \lambda \neq 0$. Dividing the first two eq. gives $x/y = y/x$ or $x^2 = y^2$ so $x^2 = y^2 =$
1/2 and $x, y = \pm \frac{1}{\sqrt{2}}$ indep. signs. Of these solutions, only $(1/\sqrt{2}, 1/\sqrt{2})$ satisfies the inequality constraints.

If $\mu_1 \neq 0, \mu_2 = 0$ then $x = y$, and we get the same solution again.

If $\mu_2 \neq 0, \mu_1 = 0$ then $x = -1/2$ so $y = \pm \sqrt{3}/2$. The $-$ case does not satisfy $x - y \leq 0$. The $+$ case does, and we get the point $(-1/2, \sqrt{3}/2)$.

If $\mu_1, \mu_2 \neq 0$ we have $x = 1/2$ and $y = 1/2$ which does not satisfy $x^2 + y^2 = 1$.

Evaluating $xy$ at the points we found, we find that the maximum is attained at $(1/\sqrt{2}, 1/\sqrt{2})$. 