Average 54180 ≈ 68%

Name ______________________ Instructor ______________________ Time your class meets _____

MATH 103 - Midterm Exam
October 28, 2009 (90 minutes)

This examination booklet contains 6 problems on 7 sheets of paper including the front cover. This
is a closed book exam. Do all of your work in this booklet. Show all your computations and
justify/explain your answers. Calculators are NOT allowed.

You have 90 minutes to complete the exam. The proctor will tell you when to begin
and when the proctor tells you that time is up, stop working, close your test booklet
and you may complete the honor pledge on this front cover sheet of your exam.

Students should sit one seat apart. Cell phones, headphones, laptops and other
electronic devices are not allowed.

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WRITE OUT AND SIGN PLEDGE:
I pledge my honor that I have not violated the Honor Code during this examination.

GRADERS WILL BE POSTED ON THE BLACKBOARD WEB SITE.
1. (16 points) Compute the following limits (show your work). Indicate limits that are infinite by $\infty$ or $-\infty$.

(a) \( \lim_{{x \to -2}} \frac{|x^2-4|}{{(x-2)(x+3)}} \)

(b) \( \lim_{{x \to -\infty}} \sqrt{x^2+x+1} - 3x \)

(c) \( \lim_{{x \to 1}} \frac{\sin(x-1)}{\sin(x^2-1)} \)

(d) \( \lim_{{x \to \infty}} \frac{(2x-4)^2(x+1)}{\sqrt{x^6+x^2}} \)

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**Alt. Solution to b)**

For \( x > 0 \)

\( \sqrt{x^2+x+1} - 3x = x \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} - 3x \)

\[ \rightarrow 1 \text{ as } x \to \infty \]

So \( x \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} - 3x \sim x - 3x = -2x \)

\[ \rightarrow -\infty \text{ as } x \to \infty \]

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For \( x < 0 \)

\( x \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} - 3x \sim x - x = 0 \)

\[ \rightarrow 0 \text{ as } x \to -\infty \]

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**a)** \( \lim_{{x \to -2^-}} \frac{|x^2-4|}{{(x-2)(x+3)}} = \lim_{{x \to 2^-}} \frac{|x+2||x-2|}{{(x-2)(x+3)}} = \lim_{{x \to 2^-}} \frac{|x+2|}{\frac{2+2}{2+3}} \cdot \lim_{{x \to 2^-}} \frac{|x-2|}{x-2} \)

\[ = \frac{4}{5} \lim_{{x \to 2^-}} \frac{x-2}{x-2} = \frac{4}{5} \]

**b)** \( \lim_{{x \to -\infty}} \sqrt{x^2+x+1} - 3x \)

\( \left( \frac{\sqrt{x^2+x+1} + 3x}{\sqrt{x^2+x+1} + 3x} \right) = \lim_{{x \to -\infty}} \frac{x^2+x+1 - 9x^2}{\sqrt{x^2+x+1} + 3x} \)

\[ = \lim_{{x \to -\infty}} \frac{x^2+1 - 8x^2}{\sqrt{x^2+1} + 3x} = \lim_{{x \to -\infty}} \frac{1 + \frac{1}{x} - 8x}{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + 3} \]

\[ \sim \lim_{{x \to -\infty}} \frac{1 - 8x}{3} = -\infty \]

**c)** \( \lim_{{x \to 1}} \frac{\sin(x-1)}{\sin(x^2-1)} \)

\( = \lim_{{x \to 1}} \frac{\sin(x-1)}{x-1} \cdot \frac{x-1}{x^2-1} \cdot \frac{x^2-1}{\sin(x^2-1)} \)

\[ \rightarrow 1 \] \( (\sin \theta \to 1 \text{ as } \theta \to 0) \]

**d)** \( \lim_{{x \to \infty}} \frac{(2x-4)^2(x+1)}{\sqrt{x^6+x^2}} \)

\( = \lim_{{x \to \infty}} \frac{(4x^2-16x+16)}{x^2} \cdot \frac{x+1}{\sqrt{x^8+1}} \)

Since \( |x| = \sqrt{x^6} \) when \( x \) is negative.

\[ \lim_{{x \to -\infty}} \frac{4}{-1} = -4 \]
2. (16 points) In this question you need not simplify your final answer after you have finished differentiating.
Compute $\frac{dy}{dx}$ when

(a) $y = (x + e^{2x})(\sin(x/3))$

(b) $y = \frac{\sec(x^2 + 4)}{x^3}$

(c) $y = \sqrt{x + \sqrt{x^3 + 5x}}$

(d) $x = t^2 + 4t$ and $y = t^3 - 12t$

\[ a) \quad \frac{d}{dx} \left( x + e^{2x} \right) \left( \sin \left( \frac{x}{3} \right) \right) = \frac{1}{3} \left( x + e^{2x} \right) \cos \left( \frac{x}{3} \right) + \sin \left( \frac{x}{3} \right) \left[ 1 + 2e^{2x} \right] \]

\[ b) \quad \frac{d}{dx} \frac{\sec(x^2 + 4)}{x^3} = \frac{x^3 \left[ \sec(x^2 + 4)(\tan(x^2 + 4))(2x) \right] - 3x^2 \sec(x^2 + 4)}{x^6} \]

\[ c) \quad \text{Since } \frac{d}{dx} \sqrt{u} = \frac{1}{2\sqrt{u}} \frac{du}{dx} \text{ we have} \]

\[ \frac{d}{dx} \sqrt{x + \sqrt{x^3 + 5x}} = \frac{1}{2 \sqrt{x + \sqrt{x^3 + 5x}}} \frac{d}{dx} \left( x + \sqrt{x^3 + 5x} \right) \]

\[ = \frac{1}{2 \sqrt{x + \sqrt{x^3 + 5x}}} \left( 1 + \frac{1}{2 \sqrt{x^3 + 5x}} \cdot (3x^2 + 5) \right) \]

\[ d) \quad \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{3t^2 - 12}{2t + 4} \text{ which can be simplified if you like} \]

\[ \frac{3 (t+2)(t-2)}{2(t+2)} = \frac{3}{2} (t-2). \]
3. (12 points) Consider the function \( f(x) = x^3 - 6x^2 + 4 \).

(a) Find all points \((x, y)\) on the graph of \( f \) where the tangent line is horizontal.
(b) Show that there are at least 3 values of \( x \) where \( f(x) = 1 \).

\[ a) \quad f'(x) = 3x^2 - 12x = 3x(x - 4) = 0 \]

when \( x = 0 \) and when \( x = 4 \)

\[ y = 4^3 - 6(4^2) + 4 = -28 \]

So there are two points where the tangent line is horizontal at \((0, 4)\) and at \((4, -28)\).

\[ b) \quad \text{We use the Intermediate Value Theorem since } f \text{ is continuous everywhere.} \]

\[ f(0) = 4 \]

\[ f(-1) = -1 - 6 + 4 = -3 \]

\[ f(1) = 1 - 6 + 4 = -1 \]

\[ f(10) = 1000 - 600 + 4 = 404 \]

\(-1 < 1 < 404\) \( \Rightarrow \) there is an \( x \) in \((1, 10)\) where \( f(x) = 1 \).

(of course other answers are also possible)
4. (12 points) Consider the curve $C$ defined by the equation
\[ x^2 + y^2 - e^{x^2 - y^2} = 1. \]

(a) Compute $\frac{dy}{dx}$.

(b) Find all points where the line $y = x$ intersects the curve $C$.

(c) For each point in (b) find the equation to the line normal to $C$ at that point.

\begin{align*}
\text{(a)} & \quad 2x + \int y \frac{dy}{dx} - e^{x^2 - y^2} (\int x - \int y \frac{dy}{dx}) = 0 \\
& \quad x \left(1 - e^{x^2 - y^2}\right) = \frac{dy}{dx} \left(-y - y e^{x^2 - y^2}\right) \\
& \quad \frac{dy}{dx} = -\frac{x}{y} \frac{1 - e^{x^2 - y^2}}{1 + e^{x^2 - y^2}}
\end{align*}

\begin{align*}
\text{(b)} & \quad x^2 + y^2 - e^{x^2 - y^2} = 1 \quad \Rightarrow \quad 2x^2 - e^0 = 1 \\
& \quad \text{and} \quad y = x \\
& \quad 2x^2 = 2 \\
& \quad x^2 = 1 \\
& \quad x = 1 \text{ or } x = -1.
\end{align*}

There are two intersection points \((1,1)\) and \((-1,-1)\).

\begin{align*}
\text{(c)} & \quad \text{The slope of } C \text{ at } (1,1) \text{ is } 0 \text{ because } 1 - e^{1-1} = 0. \\
& \quad \text{(and the denominator } y \left(1 + e^{x^2 - y^2}\right) \text{ will be } 1(1+e^0) = 2) \\
& \quad \text{Therefore the normal line is vertical; its equation is } \boxed{x = 1}. \\
& \quad \text{Similarly the slope of } C \text{ at } (-1,-1) \text{ is also zero and } \\
& \quad \text{the normal line is the vertical line } \boxed{x = -1}. 
\end{align*}
5. (12 points) Alice and Bob are racing to the corner as shown in the diagram below. Right now, Alice is 400 meters from the corner and Bob is 300 meters from the corner. Right now, Alice runs with speed 10 meters per second and Bob runs with speed 6 meters per second. Find the instantaneous rate of change \( \frac{d\theta}{dt} \) for \( \theta \) as shown in the diagram.

Let \( A = \) Alice's distance from the corner.  
\( B = \) Bob's distance from the corner.

\[
\tan \theta = \frac{B}{A}
\]

\[
\Rightarrow (\sec^2 \theta) \frac{d\theta}{dt} = \frac{\frac{dB}{dt} - \frac{dA}{dt}}{A^2}
\]

\[
\Rightarrow \frac{d\theta}{dt} = \cos^2 \theta \left( \frac{A \frac{dB}{dt} - B \frac{dA}{dt}}{A^2} \right)
\]

Where \( \cos \theta = \frac{A}{\sqrt{A^2 + B^2}} \) so \( \cos^2 \theta = \frac{A^2}{A^2 + B^2} \).

\[
\Rightarrow \frac{d\theta}{dt} = \frac{A \frac{dB}{dt} - B \frac{dA}{dt}}{A^2 + B^2}
\]

Right now, \( A = 400, \frac{dA}{dt} = -10, B = 300, \frac{dB}{dt} = -6 \)

so now
\[
\frac{d\theta}{dt} = \frac{(400)(-6) - (300)(-10)}{400^2 + 300^2} = \frac{600}{(500)^2} = \frac{6}{2500}
\]

(radians/sec)
6. (12 points)

(a) Consider the triangle $T_1$ formed by the $x$-axis, the $y$-axis and the tangent line to the curve $y = 1/x$ at the point $(1,1)$. Draw $T_1$ together with the curve $y = 1/x$ for $x > 0$ and find the area of the triangle $T_1$.

\[ \frac{d}{dx} \frac{1}{x^2} = -\frac{1}{x^2} \Rightarrow \text{slope at } (1,1) \text{ is } -1 \]

\[ \Rightarrow \text{the tangent line crosses the } x \text{-axis at } x = 2 \text{ and the } y \text{-axis at } x = 2 \]

\[ \Rightarrow \text{the area of } T_1 = \frac{1}{2} (2)(2) = 2 \]

(b) For any point $(a, 1/a)$ with $a > 0$ find a formula for the area of the triangle $T_a$ formed in this way, by the tangent line to $y = 1/x$ at $x = a$, the $x$-axis and the $y$-axis.

The slope of $y = \frac{1}{x}$ at $(a, 1/a)$ is $-\frac{1}{a^2}$

Eqn of tangent line $y - \frac{1}{a} = -\frac{1}{a^2} (x-a)$

Crosses the $x$-axis when $y = 0$, so $-\frac{1}{a} = -\frac{x}{a^2} + \frac{1}{a}$

\[ \Rightarrow \frac{x}{a^2} = \frac{2}{a} \Rightarrow x = 2a \]

Crosses the $y$-axis when $x = 0$, so $y - \frac{1}{a} = -\frac{1}{a^2} (-a)$, so $y = \frac{1}{a} + \frac{1}{a} = \frac{2}{a}$

\[ \Rightarrow \text{the area is } \frac{1}{2} (2a)(\frac{2}{a}) = 2 \text{ base } \frac{2}{a} \]

\[ \Rightarrow \text{The area of } T_a \text{ is always equal to } 2. \]