Princeton University MAT 202 Spring 2008

Willie W. Wong

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- Today we introduce *continuous* linear dynamical systems. Like we did before for the discrete case, we'll set up a comparison with the one-variable (aka. compound interest) case.

- The compound interest scenario can be summarized in the following table, where r is the interest rate

	Recursive relation	Closed-form solution
Discrete	$x(t+1) = (1+r) \cdot x(t)$	$x(t) = (1+r)^t x(0)$
Continuous	$\frac{d}{dt}x = rx$	$x(t) = e^{rt}x(0)$

  We obtained the closed form solution in the continuous case by solving the ordinary differential equation

$$\begin{aligned} \frac{dx}{dt} &= rx \\ \int \frac{dx}{x} &= \int r\, dt \\ \ln(x) &= rt + c \\ x(t) &= e^{rt}e^c \end{aligned}$$

  By setting the initial condition $x(0) = e^{r\cdot 0}e^c$, we have that $e^c = x(0)$ and hence $x(t) = e^{rt}x(0)$.

- We can generalize the one-variable case. What we did for the discrete linear dynamical system becomes

	Recursive relation	"Closed-form solution"
Discrete	$\boldsymbol{x}(t+1) = A\boldsymbol{x}(t)$	$\boldsymbol{x}(t) = A^t\boldsymbol{x}(0)$

  and it remains to fill in the following

Continuous	??	??

  I placed "closed-form solution" in quotes since, as we all know by now, the expression $\boldsymbol{x}(t) = A^t\boldsymbol{x}$ is not really closed form as it requires the calculation of powers of A, a non-trivial task unless A can be diagonalized. But let's forget that fine distinction right now and just consider how to generalize the one-variable case to the multi-variable case for the continuous system.

- The naïve and (surprisingly) correct thing to do, is to just transfer the expressions wholesale from the single-variable case and complete the table as the following

	Recursive relation	"Closed-form solution"
Discrete	$\boldsymbol{x}(t+1) = A\boldsymbol{x}(t)$	$\boldsymbol{x}(t) = A^t\boldsymbol{x}(0)$
Continuous	$\frac{d}{dt}\boldsymbol{x} = B\boldsymbol{x}$	$\boldsymbol{x}(t) = e^{Bt}\boldsymbol{x}(0)$

So far we have just symbolically copied everything from the one-variable case. What we now need to do is to define what it means to take the derivative of a vector and to take the exponent of a matrix.

- Before this, a small digression, comparing against the one-variable case, we can consider the matrix B as equal to $A - I_M$ for all intents and purposes. To see this we first go back to the discrete one-variable case: to arrive from compound interest to continuously compound interest, what we do is we cut the time interval between t and $t+1$ into many small parts (say K of them), and at each of the small parts we let the "rate" be r/K. To be precise, we assume that during the continuum between t and $t+1$, the number in the bank account grows linearly, which means that for $0 \leq \tau \leq 1$, we assume, by the point slope form of a linear expression, that

$$x(t+\tau) - x(t) = \tau r x(t)$$

(check that the above equation indeed defines a line in the $x - t$ plane that passes through the points (t, y), $(t+1, (1+r)y)$.) By taking the derivative

$$\frac{d}{dt}x(t) = \lim_{\tau \to 0} \frac{x(t+\tau) - x(t)}{\tau} = \lim_{\tau \to 0} \frac{\tau r x(t)}{\tau} = r x(t)$$

we get the expression for the continuous case. Similarly, we can do this for the recursive relation in the multi-variable case: we start by re-writing

$$\boldsymbol{x}(t+1) - \boldsymbol{x}(t) = (A - I_M)\boldsymbol{x}(t)$$

Now by using the point-slope form of a line that passes through $(t, \boldsymbol{y})$ and $(t+1, \boldsymbol{z})$ as

$$\boldsymbol{x}(t+\tau) - \boldsymbol{y} = \tau(\boldsymbol{z} - \boldsymbol{y})$$

we have that the "line" described by the iterate of the recursive relation is

$$\boldsymbol{x}(t+\tau) - \boldsymbol{x}(t) = \tau(A - I_M)\boldsymbol{x}(t)$$

Taking the limit as $\tau \to 0$ we find the derivative

$$\frac{d}{dt}\boldsymbol{x} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_M}{dt} \end{bmatrix} = (A - I_M)\boldsymbol{x}(t) = B\boldsymbol{x}(t)$$

So here we've exhibited that B is, morally speaking, $A - I_M$ (a fact that will be in no doubt useful when you consider the eigenvalues of B), and what it means to take the time derivative of the vector $\boldsymbol{x}$ (you just take the time derivative of each component).

- The exponent is more tricky. The only elementary operations we know how to do on matrices are multiplications and sums. Luckily, we have a way of writing the expression e^{rt} as only multiplications and sums: the Taylor series! Recall the Taylor series for e^x is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^N}{N!} + \cdots$$

we'll just simply define the exponent for the matrix to be

$$e^{Bt} = I_M + (Bt) + \frac{(Bt)^2}{2} + \cdots + \frac{(Bt)^N}{N!} + \cdots$$

- Notice that e^B is only defined for a square matrix B, and that the expression e^B stands for another square matrix.

- Now that the expression is defined, it is hardly useful: to compute e^{Bt} one needs to sum an infinite series, something that promises to take an infinite amount of time. Luckily, as you undoubtedly noticed, since matrix multiplications commutes with scalar multiplications, the term

$$(Bt)^N = B^N t^N$$

and we know how to compute powers of B if B were diagonal. So supposing that

$$B = \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & b_M \end{bmatrix}$$

we can write

$$e^{Bt} = \sum \frac{t^N}{N!} B^N = \sum \frac{t^N}{N!} \begin{bmatrix} b_1^N & 0 & \dots & 0 \\ 0 & b_2^N & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & b_M^N \end{bmatrix}$$

so

$$e^{Bt} = \begin{bmatrix} 1 + b_1 t + \frac{b_1^2 t^2}{2} + \dots & & \\ & 1 + b_2 t + \frac{b_2^2 t^2}{2} + \dots & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} e^{b_1 t} & & & \\ & e^{b_2 t} & & \\ & & \ddots & \\ & & & e^{b_M t} \end{bmatrix}$$

So we have the (not-so-)amazing fact that for a diagonal matrix, the exponent of the matrix is the matrix of the exponents.

- Recall that this miracle is only half the battle in the case of the discrete dynamical system: the other half comes from the nice fact that the powers of similar matrices are still similar, namely if $A = SFS^{-1}$, then $A^t = SF^tS^{-1}$. This way we transfer from the calculation of the unwieldy expression A^t to one that is much simpler to calculate, should A be diagonalizable (and hence F diagonal).

- We show now that this miracle still extends to the exponential: suppose B is diagonalizable, i.e. $B = SDS^{-1}$ for some invertible matrix S. We calculate

$$e^{Bt} = e^{SDS^{-1}t} = I_M + (SDS^{-1}t) + \frac{(SDS^{-1}t)^2}{2} + \cdots + \frac{(SDS^{-1}t)^N}{N!} + \cdots$$

which, using the relation between powers of similar matrices, we see becomes

$$= I_M + S(Dt)S^{-1} + S\frac{(Dt)^2}{2}S^{-1} + \cdots + S\frac{(Dt)^N}{N!}S^{-1} + \cdots$$

and thus

$$= S\left(I_M + Dt + \frac{(Dt)^2}{2} + \cdots + \frac{(Dt)^N}{N!} + \cdots\right)S^{-1} = Se^{Dt}S^{-1}$$

- Combining these all, we get the following fact for continuous linear dynamical systems: Let $\frac{d}{dt}\boldsymbol{x} = B\boldsymbol{x}$ be a continuous linear dynamical system, with $\boldsymbol{x} \in \mathbb{R}^M$, and B an $M \times M$ matrix. The equation can be written in integral form as

$$\boldsymbol{x}(t) = e^{Bt}\boldsymbol{x}(0)$$

where the $M \times M$ matrix e^{Bt} is defined using Taylor series. Moreover, supposing that B is diagonalizable, i.e., there exist an eigenbasis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M$ with eigenvalues $\lambda_1, \ldots, \lambda_M$ accordingly, so that letting

$$S = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_M \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_M \end{bmatrix}$$

we have $B = SDS^{-1}$, we can write

$$\boldsymbol{x}(t) = Se^{Dt}S^{-1}\boldsymbol{x}(0)$$

where

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_M t} \end{bmatrix}$$

- Phase space portraits for continuous dynamical system is more intuitive and more representative of the concept of the system as a *flow*.

- Before starting, we first introduce the concept of a vector field: whereas a vector is given by a direction and a length, a vector field is the attachment of vectors to points in space. Physically, every linear transformation represents a vector field. Since points in $\mathbb{R}^M$ can be identified with vectors that begin at the origin, associated to each point in $\mathbb{R}^M$, through a linear transformation $T : \mathbb{R}^M \to \mathbb{R}^M$, is a second vector.

- In the analogy with fluids, the point in $\mathbb{R}^M$ conveys the position of a given particle, while the second vector associated to it (the *vector field evaluated at that point*) can be thought of as its velocity.

- Then, expanding this analogy, the dynamical system

  $$\frac{d}{dt}\boldsymbol{x} = A\boldsymbol{x}$$

  prescribes, via the vector field $A\boldsymbol{x}$, an underlying current in phase-space (one can think of the phase space as filled with fluids or aether, which flows steadily, with the velocity at each point in accord to the vector field). Prescribing an initial data is then dropping a small particle into the fluid to be carried by the current. The path taken by the particle, which is carried by the current, gives the phase-space trajectory of the initial data.

- Now, to qualitatively describe the phase space portrait: suppose we have the continuous linear dynamical system described by

  $$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

  on the phase space $\mathbb{R}^2$. The behaviour of the system is, as seen on Monday, given by

  $$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) = \begin{bmatrix} e^t x_1 \\ e^{-t} x_2 \end{bmatrix}$$

  This tells us a few things: the eigenvectors of A, as it is already diagonalized, are the standard vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$; since e^t is increasing with time, $\boldsymbol{e}_1$ is an expanding direction, and since e^{-t} is decreasing with time, $\boldsymbol{e}_2$ is a contracting direction (from this we can draw the basics of the phase space portrait, based on the eigen-directions); as $t \to \infty$, only the x_1 part will dominate, since the other part approaches 0, while the reverse is true as $t \to -\infty$, and this tells us how to draw the portrait.

- Notice that unlike the discrete case, the change between expanding and contracting directions is at eigenvalue 0, not eigenvalue 1, recall that the eigenvalue of the transformation matrix is:

	Discrete dynamical system	Continuous dynamical system
Expanding direction	$\lvert\lambda\rvert > 1$	$\lambda > 0$
Contracting direction	$\lvert\lambda\rvert < 1$	$\lambda < 0$
Stationary direction	$\lambda = 1$	$\lambda = 0$

The case of negative eigenvalues for discrete systems is a tad delicate: for such systems the trajectory jumps (alternating) between two portions of the phase space. This type of systems do not have a continuous analogue.

- To give another example, suppose

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

The dynamics is given by, again since A is diagonal

$$\boldsymbol{x}(t) = \begin{bmatrix} e^{3t}x_1 \\ e^{2t}x_2 \end{bmatrix}$$

Here both of the eigenvalues are positive, indicating that both of the standard directions corresponds to expanding eigen directions (draw). The question is then, how do we draw the trajectory of a point off of the standard directions. To do so we look at the long time behaviour as $t \to \pm\infty$. In the positive direction, we notice that e^{3t} grows much quicker than e^{2t}, so in the limit the eigendirection for eigenvalue 3 will dominate that with eigenvalue 2. This tells us that the direction of the trajectory will grow more and more tangent to the (in our case) $\boldsymbol{e}_1$ direction. On the other hand, in the negative direction, first we notice that as $t \to -\infty$, both components of the vector approaches 0. So the trajectory should end at the origin. But the rate at which they approach is not the same: we see that e^{3t} decays also much quicker than e^{2t}, which means that at very negative times, the $\boldsymbol{e}_2$ direction will dominate, and so as the trajectory approaches the origin, it will be tangent to $\boldsymbol{e}_2$.

- This description is also more easily seen by treating the trajectory as a *parametrized curve in* $\mathbb{R}^2$. To be more precise, we want to plot the curve (with y being the $\boldsymbol{e}_1$ direction and z being the $\boldsymbol{e}_2$ direction)

$$\begin{cases} y &= e^{3t}x_1 \\ z &= e^{2t}x_2 \end{cases}$$

from $-\infty < t < \infty$. Now, making the substitution $u = e^t$, we have that the graph is the same as that of

$$\begin{cases} y &= u^3 x_1 \\ z &= u^2 x_2 \end{cases}$$

from $0 < u < \infty$. Solving the implicit equation we get

$$u = (\frac{y}{x_1})^{1/3} \Rightarrow z = y^{2/3}\frac{x_2}{x_1^{2/3}}$$

remembering that x_1 and x_2 are but constants, we can graph this easily enough.

- Now suppose the matrix is
$$A = \begin{bmatrix} 10/3 & -2/3 \\ 2/3 & 5/3 \end{bmatrix}$$
We first look of its eigenvalues: its characteristic polynomial is
$$f_A(\lambda) = \lambda^2 - 5\lambda + 6$$
(trace is 5, determinant is 6). This polynomial factors
$$f_A(\lambda) = (\lambda - 3)(\lambda - 2)$$
(alas! the same eigenvalues as before [yes, I planned it this way]). And next we find out about the eigenvectors: the eigenvector for eigenvalue 3 is the basis of $\ker(A - 3\lambda)$, which we solve
$$\begin{bmatrix} 1/3 & -2/3 & | & 0 \\ 2/3 & -4/3 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$
and so an eigenvector is $(2, 1)$. Similarly, we find that the eigenvector for the eigenvalue 2 is $(1, 2)$, so we have the diagonalization
$$S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} , \quad A = S \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} S^-1$$
Now, to draw the phase space portrait, we first draw the two eigendirections (draw), both are expanding (add arrows), furthermore from our reasoning from before, at $t \to \infty$, the trajectory curves will want to tend toward the $(2, 1)$ direction, and as $t \to -\infty$, as the trajectory approaches the origin, it wants to be tangent to $(1, 2)$, so we have (draw).

- So far we have only considered the case when the eigenvalues are real. Now we shall re-introduce the concept of complex eigenvalues. But before doing that, let me quickly recall the concept of complex-valued functions.

- We begin by letting $z(t)$ be a function that associates to each value $t \in \mathbb{R}$ a complex number z. Remembering that we can re-write the complex number z as $x + iy$, where x is its real part and y its imaginary part, it is natural to expect that $z(t) = x(t) + iy(t)$. And in fact, this can be done.

- Normal arithmetic operations on complex numbers becomes normal arithmetic operations on complex-valued functions: if $z(t)$ and $w(t)$ are two complex valued functions such that $z(t) = x(t) + iy(t)$ and $w(t) = u(t) + iv(t)$, then
$$\begin{aligned} z + w &= x + u + i(v + y) \\ zw &= xu - vy + i(xv + yu) \\ &\vdots \end{aligned}$$

as usual. Derivatives of complex valued functions operate in much the same way:

$$\frac{d}{dt}z = \frac{d}{dt}x + i\frac{d}{dt}y$$

and the chain rule still applies if we can write t as a function of s:

$$\frac{d}{ds}z = \frac{dz}{dt}\frac{dt}{ds}$$

- The important fact we aim to derive is that, if λ is a complex number, then if $z(t) = e^{\lambda t}$, we have

$$\frac{d}{dt}z = \lambda z$$

- We quickly verify that this fact is compatible with Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Let $\theta = tq$, then on the one hand

$$\frac{d}{dt}e^{itq} = iqe^{itq}$$

on the other

$$\frac{d}{dt}(\cos(tq) + i\sin(tq)) = -q\sin(tq) + iq\cos(tq) = iq(\cos(tq) + i\sin(tq)) = iqe^{itq}$$

- Armed with this fact, we assert that the case for dynamical systems with complex eigenvalues extends, a la what we did for the real eigenvalues case, to the continuous setting: Suppose we are looking at the continuous dynamical system

$$\frac{d}{dt}\boldsymbol{x} = B\boldsymbol{x}$$

where B is complex-diagonalizable, i.e., we have the diagonal matrix D whose entries $d_{ii} = \lambda_i$ are complex eigenvalues and complex eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M$ such that writing $S = \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_M\end{bmatrix}$ we have

$$B = SDS^{-1}$$

then the closed form expression for the trajectory is given by

$$\boldsymbol{x}(t) = Se^{Dt}S^{-1}\boldsymbol{x}(0)$$

where the matrix e^{Dt} is given by

$$\begin{bmatrix} e^{\lambda_1 t} & & & & \\ & e^{\lambda_2 t} & & & \\ & & e^{\lambda_3 t} & & \\ & & & \ddots & \\ & & & & e^{\lambda_M t} \end{bmatrix}$$

- Recall that when we first looked at complex eigenvalues: the complexity of eigenvalues is characterized by an oscillation behaviour. Now, since the behaviour of the system will be described by, more or less
$$z(t) = e^{pt}e^{iqt}$$
and further more we see that e^{pt} is a positive real function that either grows or decays steadily, the oscillation is only manifested in the complex part of the eigenvalue, namely the e^{iqt} part. In fact, we have that *the real part of the complex eigenvalue controls the growth or decay in the lengths of vectors, whereas the imaginary part controls the oscillation [in other words, rotation in phase space] in the directions of vectors.*

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- Recall a fact about 2×2 matrices with complex eigenvalues: they are similar to rotation-scaling matrices. Remember the following: if B is a real 2×2 matrix with eigenvalue $p + iq$ with eigenvector $\boldsymbol{v} + i\boldsymbol{w}$ (recall that this means $p - iq$ is the other eigenvalue with eigenvector $\boldsymbol{v} - i\boldsymbol{w}$, then
$$B = SRS^{-1}$$
where
$$S = \begin{bmatrix}\boldsymbol{w} & \boldsymbol{v}\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$$
- We'll do a similar calculation here: first we start with the following: B is complex diagonalizable means that
$$B = PDP^{-1}$$
where
$$P = \begin{bmatrix}\boldsymbol{v} + i\boldsymbol{w} & \boldsymbol{v} - i\boldsymbol{w}\end{bmatrix}, \quad D = \begin{bmatrix} p + iq & 0 \\ 0 & p - iq \end{bmatrix}$$
This is the definition of complex diagonalizable. So we can write the solution to
$$\frac{d}{dt}\boldsymbol{x} = B\boldsymbol{x}$$
as
$$\boldsymbol{x}(t) = Pe^{Dt}P^{-1}\boldsymbol{x}(0)$$
Now, let's look more closely at e^{Dt}:
$$\begin{aligned} e^{Dt} &= \begin{bmatrix} e^{pt}e^{iqt} & 0 \\ 0 & e^{pt}e^{-iqt} \end{bmatrix} \\ &= e^{pt}\begin{bmatrix} \cos(qt) + i\sin(qt) & 0 \\ 0 & \cos(qt) - i\sin(qt) \end{bmatrix} \\ &= e^{pt}T\begin{bmatrix} \cos(qt) & -\sin(qt) \\ \sin(qt) & \cos(qt) \end{bmatrix}T^{-1} \end{aligned}$$

where

$$T = \frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

(Do check that this is true!) Then we have that

$$\boldsymbol{x}(t) = e^{pt}(PT)\begin{bmatrix} \cos(qt) & -\sin(qt) \\ \sin(qt) & \cos(qt) \end{bmatrix}(PT)^{-1}$$

- Now we quickly calculate what PT is:

$$\begin{aligned} PT &= \frac{1}{2i}\begin{bmatrix} v_1 + iw_1 & v_1 - iw_1 \\ v_2 + iw_2 & v_2 - iw_2 \end{bmatrix}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \\ &= \frac{1}{2i}\begin{bmatrix} 2iw_1 & 2iv_1 \\ 2iw_2 & 2iv_2 \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{w} & \boldsymbol{v} \end{bmatrix} = S \end{aligned}$$

- Notice that this means that (IMPORTANT!!!) we can write the solution as

$$\boldsymbol{x}(t) = e^{Bt}\boldsymbol{x}(0) = Se^{Rt}S^{-1}$$

where

$$e^{Rt} = e^{pt}\begin{bmatrix} \cos(qt) & -\sin(qt) \\ \sin(qt) & \cos(qt) \end{bmatrix}$$

In particular this implies that we can write $\boldsymbol{x}(t)$ now as a product of real matrices and a real vector (re-affirming our claim on Wednesday that $\boldsymbol{x}(t)$ is, amazingly, still a real vector), and gives us a good way of interpreting the trajectory: here we made explicit the condition that *in the basis given by $\boldsymbol{w}, \boldsymbol{v}$, where $\boldsymbol{v} + i\boldsymbol{w}$ is a complex eigenvector with eigenvalue $p + iq$, the trajectory is given by a rotation and a scaling.*

- Let's look at an example now: let

$$B = \begin{bmatrix} 9 & 17 \\ -2 & 15 \end{bmatrix}$$

and try to draw the phase-space portrait. To start, we find the eigenvalues of B:

$$f_B(\lambda) = \lambda^2 - 24\lambda + (135 + 34)$$

Applying the quadratic formula

$$\lambda = \frac{24 \pm \sqrt{576 - 4 * 169}}{2} = 12 \pm \sqrt{144 - 169} = 12 \pm 5i$$

Look at the $+$ one, we solve for the complex eigenvector

$$\ker\begin{bmatrix} -3 - 5i & 17 \\ -2 & 3 - 5i \end{bmatrix} \Rightarrow \begin{bmatrix} -(3 + 5i)(3 - 5i) & 17(3 - 5i) \\ -2 & 3 - 5i \end{bmatrix} = \begin{bmatrix} -34 & 17(3 - 5i) \\ -2 & 3 - 5i \end{bmatrix}$$

So the eigenvector is

$$\begin{bmatrix} 3-5i \\ 2 \end{bmatrix}$$

In terms of the notation above, we have

$$p = 12, q = 5, \boldsymbol{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$$

So

$$S = \begin{bmatrix} -5 & 3 \\ 0 & 2 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} -1/5 & 3/10 \\ 0 & 1/2 \end{bmatrix}$$

and the evolution is given by

$$\boldsymbol{x}(t) = e^{12t} S \begin{bmatrix} \cos(5t) & -\sin(5t) \\ \sin(5t) & \cos(5t) \end{bmatrix} S^{-1} \boldsymbol{x}(0)$$

Because the exponential factor ($p = 12$) is relatively large, the picture is a out-ward spiral that grows rapidly.

- The phase portrait for these spirals can be categorized thus
  - If $p > 0$, the portrait is an outward spiral
  - If $p < 0$, the portrait is an inward spiral
  - If $p = 0$, the portrait is an ellipse

  where

  - The shape of the spiral/ellipse is determined by the components of the eigenvectors $\boldsymbol{v}$ and $\boldsymbol{w}$: the two determine the linear distortion applied to the otherwise circular spiral.
  - The relative density of the spiral (how much change of radius between successive loops) is given by the absolute value $|p/q|$, the smaller the number the denser the spiral, the bigger the number the wider the spiral.
  - Our categorization by first splitting into the cases where we have real/complex eigenvalues and then into contracting and expanding portraits can be summarized thus, because the eigenvalues of 2×2 matrices are directly connected to the two quantities "trace" and "determinant".
    * If $\det(A) \leq (tr(A)/2)^2$, then we have real eigenvalues; otherwise, the eigenvalues are complex.
    * If $\det(A) < 0$, then we must have real eigenvalues, and the two real eigenvalues must have opposite signs, so the portrait must be a saddle point.
    * If $\det(A) = 0$, then at least one of the eigenvalues are zero, so at least one of the directions will stay "fixed".
    * If $\det(A) > 0$ then:

  · If $tr(A) > 0$, the picture is expanding (either outward spiral or outward star)
  · If $tr(A) < 0$, the picture is contracting
  · If $tr(A) = 0$, then we are necessarily in the complex eigenvalue case, and the portrait is an ellipse.

(summarize with a picture)

- Lastly we want to consider equilibrium and stability: we say a solution is an equilibrium if $\frac{d}{dt}\boldsymbol{x} = 0$. Notice that since the equation is

$$\frac{d}{dt}\boldsymbol{x} = A\boldsymbol{x}$$

this implies that to have an equilibrium (also called a stationary solution), this requires $A\boldsymbol{x} = 0$, and so the initial data must be in the kernel of A. This tells us that A is not invertible, and so $\det(A) = 0$, and $\boldsymbol{x}$ is an eigenvector with 0 eigenvalue, unless $\boldsymbol{x} = 0$. So we have that

  – The 0 vector is always an equilibrium
  – Any other vector is an equilibrium only when it is in $\ker(A)$.

- We call an equilibrium solution *stable* if, given an initial data near it, it will get closer to it; *metastable* if, given an initial data near it, it will not get farther for it; and *unstable* if there is some initial data near it that will fly off to infinity.

- Notice that the case where $\det(A) = 0$ (A has a kernel) cannot be stable: it $\boldsymbol{x}$ is an equilibrium, and let $\boldsymbol{y}$ be a very short vector in the same direction, since $\boldsymbol{x} + \boldsymbol{y}$ is still in $\ker(A)$, it is another equilibrium point. So for all time, the two vectors will stay the same distance from each other.

- So we restrict to looking at the case where A is invertible, and therefore the only equilibrium is the 0 point. It is quite easy to see that the origin is stable only if the overall picture is contracting, considering the spectral decomposition, this means that *the origin is a stable equilibrium if and only if the real part of each of the eigenvalues of A is negative*.

Homework for this week

9.1: 22, 23, 26, 28, 30

9.2: 6, 27, 29