Princeton University MAT 202 Spring 2008

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- We being our study of dynamical system by looking at the most classical example: two-species population interaction, a simplified version of Volterra's predator-prey model.
- On a remote island are pirates and ninjas (see figure). The ninjas survive by raiding the pirates, but they also increase their numbers by abducting young pirate-wannabes and brain-washing them. The pirates on the other hand, "recruits" from foreign shores and always return to the island because here they only have one group of enemies.
- Suppose that, during the year, 90% of the ninjas comes down from the mountain and raids the pirates. Each ninja has a 7 in 9 chance of success: securing ration for the year and, with 20% probability cut down a pirate; a failure is, of course, death. Suppose that also, during the year, ninjas come down from the mountain to look for fresh blood. The pirates, as we know, are not too picky about their recruits, so only 6% stand a chance surviving ninja-training. The pirates, on the other hand, recruit for quantity and not quality: they send 90% on shore leave and every 3 able-bodied men can effectively "convince" some landlubber that he really should join the pirates. Additionally, the ninjas and pirates both die from natural causes at a rate of 2%.
- This relationship we can write as a dynamical system. Let $N(t)$ be the number of ninjas at the tth year. Let $P(t)$ be the number of pirates at the tth year. Every year the ninjas loses $90\% \times (1 - 7/9)N$ in the raids, $0.02P$ naturally, kills $90\% \times 7/9 \times 0.2N$ pirates, and gains $0.06P$ to their numbers. The pirates gains $90\%P/3$ members per year from recruiting, and loses $0.02P$ members from natural death and $0.06P$ members from ninja kidnappings. This means

$$N(t+1) = N(t) - 0.2N(t) - 0.02N(t) + 0.06P(t)$$
$$P(t+1) = P(t) + 0.30P(t) - 0.03P(t) - 0.06P(t) - 0.14N(t)$$

or, in vector notation

$$\begin{bmatrix} N(t+1) \\ P(t+1) \end{bmatrix} = \begin{bmatrix} 0.78 & 0.06 \\ -0.14 & 1.22 \end{bmatrix} \begin{bmatrix} N(t) \\ P(t) \end{bmatrix}$$

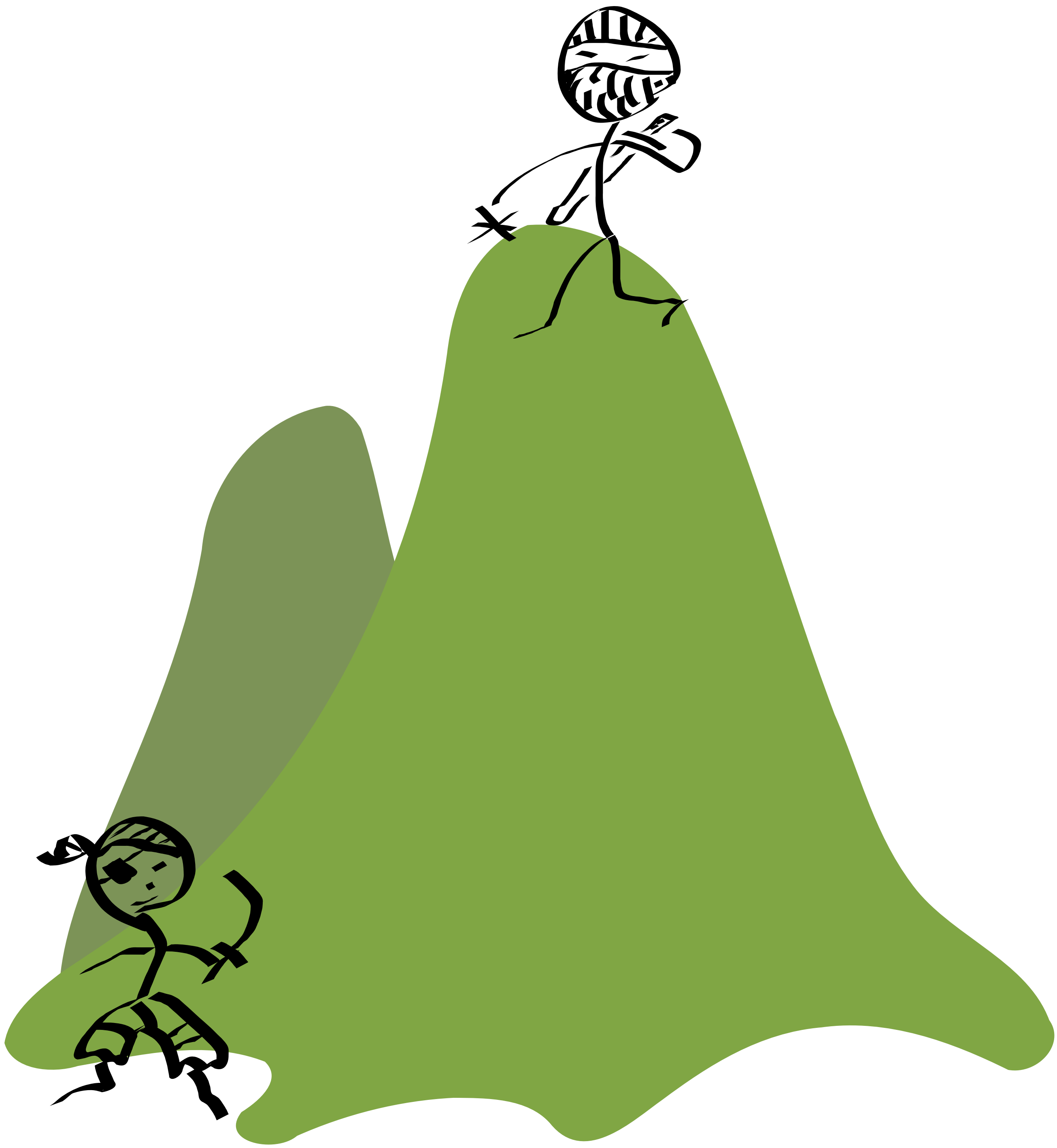

Figure 1: The perpetual rivalry between Ninjas and Pirates continue on this island far from (other) civilization. The Ninjas raid the Pirates for supplies, and occasionally kidnap a promising looking young cabin boy to train as the next generation of Ninjas. The pirates “recruit” on foreign shores and always come back to the island to hide from all their other enemies beside Ninjas (e.g. cowboys, zombies, robots, frat boys, and hippies).

- This is what we call a *discrete dynamical system* or *recursion relation*: an expression that expresses what "should be" in terms of what "is". The word "discrete" means that we are treating time in steps rather than a continuous flow: how the pirates and ninjas kill each other during the year is none of our concern, we are just interested in the population of ninjas and pirates on, say, January 2nd of every year. In general, a discrete dynamical system can always be written in the form

$$\boldsymbol{v}(t+1) = T(\boldsymbol{v}(t))$$

where, for each time t, the $\boldsymbol{v}(t)$ (which we call the *state vector*) is an element of $\mathbb{R}^M$ (which we call the *phase space* of the dynamical system), and $T : \mathbb{R}^M \to \mathbb{R}^M$ is a transformation (which we sometimes call the *flow*, why this is so will become clearer later) law that describes the dynamics.

- The simplest model of a dynamical system is something you have encountered in high school math classes: compound interest. The phase space is one dimensional: $\mathbb{R}^1$. The state vector, being a vector in 1 dimensional space, is but a number: the amount of money you have in the bank. The transformation law is the application of interest: x dollars this year becomes $(1+r)x$ next year, where r is the interest rate.

- Let's look at how our dynamical system behaves for some initial data points

t =0	1	2	3	4	5
$(N, P) = (200, 800)$	(204,948)	(216,1128)	(236,1346)	(265,1609)	(303,1926)
(200,400)	(180,460)	(168,536)	(163,630)	(165,746)	(174,887)
(200,200)	(168,216)	(144,240)	(127,273)	(115,315)	(109,368)
(200,100)	(162,94)	(132,92)	(108.94)	(90,99)	(76,108)
(200,75)	(161,64)	(129,55)	(104,49)	(84,45)	(68,43)
(200,50)	(159,33)	(126,18)	(99,4)	(77,-8)!	

Notice that at the fourth year for data starting with 200 ninjas and 50 pirates, we ended up with -8 pirates. That is why I placed an exclamation mark there: the evolution of the dynamical system has become unphysical. We started out with too many ninjas, who, in order to feed themselves looted the pirates too heavily until the pirates died out completely between years 3 and 4. After that, our model becomes nonsensical since the ninjas can no longer support themselves by looting the now-extinct pirates.

- Below I also plotted the trajectories on an XY plane, with the X-axis being the number of ninjas and the Y axis being the number of pirates. The *phase trajectories* shows how the dynamical system evolves. Notice that it is tempting to connect the dots and make curves of the trajectories; since our model is discrete, we cannot really tell what happens between two successive time-points, but connecting the dots will give us a sense of continuity, that the data "flows" from one point to another. This is why we sometimes call the transformation a "flow".

- To calculate each of the numbers in the table above, I have to recursively apply the transformation law for each step. This is, generally speaking, tedious: suppose I want to know what

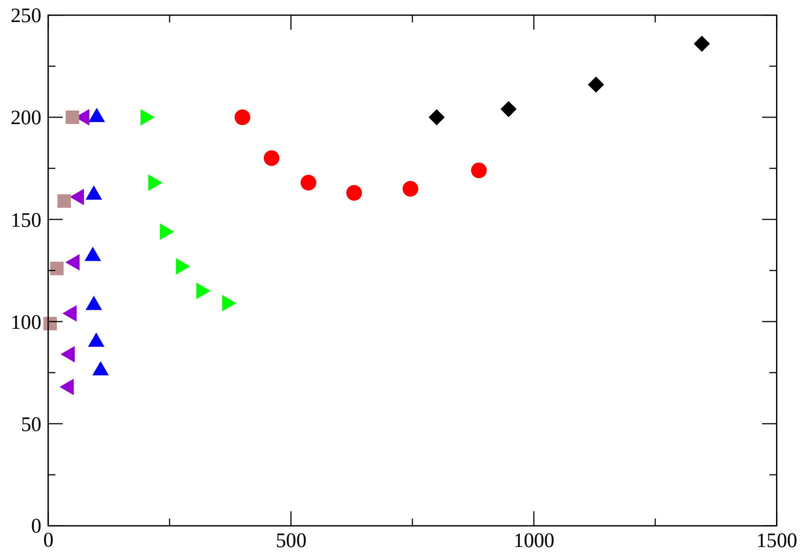

Figure 2: A phase-space portrait of the trajectories. The x-axis is the number of pirates, and the y-axis is the number of ninjas. Each color/shape represents the trajectory for a different starting initial data. The numerical data for these trajectories are given in the table above.

happens at time 20 for this system. This involves either applying the recursion formula 20 times, or calculating the matrix A^{20} where A is the matrix corresponding to the evolution of the dynamical system. Ideally we want a easier way to do this computation.

- Going back to the one-dimensional example of compound interest, we recall that for that calculation the solution is simple: at time t, the amount of money will be

$$x(t) = (1+r)^t x(0)$$

The important idea is that in this case: our transformation is just a multiplication by a number, and we know well how to multiply numbers successively.

- This leads us to the idea of *eigenvectors* and *eigenvalues*.

- We begin by looking at our ninja-vs-pirate example. What happens if we start with 100 ninjas and 700 pirates?

$$\begin{bmatrix} N(1) \\ P(1) \end{bmatrix} = \begin{bmatrix} 0.78 & 0.06 \\ -0.14 & 1.22 \end{bmatrix} \begin{bmatrix} 100 \\ 700 \end{bmatrix} = \begin{bmatrix} 120 \\ 840 \end{bmatrix} = 1.2 \begin{bmatrix} 100 \\ 700 \end{bmatrix}$$

So for this special starting state, we have a closed-form formula to compute the number of ninjas and pirates at an arbitrary time:

$$\begin{bmatrix} N(t) \\ P(t) \end{bmatrix} = A^t \begin{bmatrix} 100 \\ 700 \end{bmatrix} = A^{t-1} 1.2 \begin{bmatrix} 100 \\ 700 \end{bmatrix} = 1.2 A^{t-1} \begin{bmatrix} 100 \\ 700 \end{bmatrix} = \ldots = 1.2^t \begin{bmatrix} 100 \\ 700 \end{bmatrix}$$

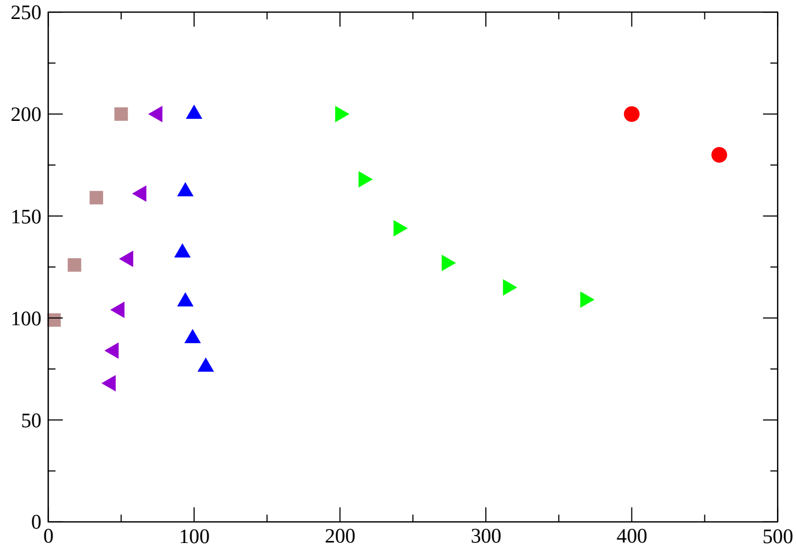

Figure 3: A zoomed-in portrait of the trajectories.

This is a case where the numbers of ninjas and pirates are such that they mutually thrive. We also have another special starting case with 300 ninjas and 100 pirates

$$\begin{bmatrix} N(1) \\ P(1) \end{bmatrix} = A \begin{bmatrix} 300 \\ 100 \end{bmatrix} = \begin{bmatrix} 240 \\ 80 \end{bmatrix} = 0.8 \begin{bmatrix} 300 \\ 100 \end{bmatrix}$$

for which we have

$$\begin{bmatrix} N(t) \\ P(t) \end{bmatrix} = 0.8^t \begin{bmatrix} 300 \\ 100 \end{bmatrix}$$

The case when the ninjas-to-pirates ratio is 3:1 is when the two destroys each other equally fast.

- How does just having these special cases help us? First, by the linearity of our dynamical system here, we see that it is the ratio of ninjas-to-pirates that matter, not the exact number. So let us define two vectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Notice that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent. Since we are working in $\mathbb{R}^2$, this means that they form a basis. Starting with an arbitrary state vector $\boldsymbol{v}(0)$, we can then decompose it in terms of this basis

$$\boldsymbol{v} = c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2$$

Using the linearity of the transformation, we have that

$$\boldsymbol{v}(t) = T(T(\cdots(\boldsymbol{v}(0)))\cdots) = c_1 A^t \boldsymbol{v}_1 + c_2 A^t \boldsymbol{v}_2 = c_1 \times 1.2^t \boldsymbol{v}_1 + c_2 \times 0.8^t \boldsymbol{v}_2$$

a closed-form expression for how $\boldsymbol{v}(t)$ behaves at arbitrary time t!

- (How this fact manifests itself in the phase space portrait: $\boldsymbol{v}_1$ gives an expanding ray and $\boldsymbol{v}_2$ gives a contracting ray; the rest of the phase space is "dragged along" by those two)

- Now, the theory: given a linear transformation $T : \mathbb{R}^M \to \mathbb{R}^M$ (or an $M \times M$ matrix A), we define

  – an *eigenvector* as a non-zero vector that remains parallel to itself after the transformation. In other words
  $$T(\boldsymbol{v}) = \lambda \boldsymbol{v}$$
  where λ is a scalar (it can be positive, negative, or even zero).

  – an *eigenvalue* is a scalar λ that is the scalar multiple for some eigenvector. In other words, there exists some non-zero vector $\boldsymbol{v}$ such that
  $$T(\boldsymbol{v}) = \lambda \boldsymbol{v}$$

- Example: let T be the identity transformation on $\mathbb{R}^M$, what are the eigenvalues and eigenvectors of of T? Answer: since for any vector $\boldsymbol{v}$
  $$T(\boldsymbol{v}) = I_M \boldsymbol{v} = \boldsymbol{v}$$
  we see that all vectors in $\mathbb{R}^M$ are eigenvectors with eigenvalue 1.

- Example: what about an orthogonal projection in $\mathbb{R}^2$ onto a line L? Answer: we solve this problem geometrically. One obvious answer is that any vector on the line L is an eigenvector with eigenvalue 1. This is because the orthogonal projection keeps vectors on line L fixed. We maybe temped to say that this is all the eigenvectors, since any vector not in L cannot be parallel to L. We would be forgetting something important: the 0 vector is parallel to everything! So if $\boldsymbol{w}$ is perpendicular to L, $T(\boldsymbol{w}) = 0 = 0\boldsymbol{w}$, showing that $\boldsymbol{w}$ is in fact an eigenvector with eigenvalue 0.

- Important point: the kernel of a linear transformation T is also the subspace of all eigenvectors of eigenvalue 0. This gives also another characterization of the invertibility of a matrix: a matrix A is invertible if and only if 0 is not an eigenvalue of A.

- Example: how about a rotation matrix? Answer: we still think geometrically, and realize that this depends on the angle. If we rotate by angles not a multiple of 180^o, then since the vector before and the vector after forms a non-flat angle, they are not parallel, so in this case, there are no eigenvectors and no eigenvalues. In the case where the rotation is by 180^o, every vector is send to its opposite: $T(\boldsymbol{v}) = -\boldsymbol{v}$, and so the only eigenvalue is -1 and every vector is an eigenvector.

- More about eigenvalues and eigenvectors will be described later. (In particular, how to find the eigenvalues and eigenvectors given a matrix.)

- We'll conclude today by writing down exactly how eigenvectors and eigenvalues help with dynamical systems.

- Given a discrete linear dynamical system (linearity means that the transformation from time t to time $t+1$ is a linear transformation on the phase space), we want to "solve it" or "integrate it" in the sense that we want to get a closed-form formula predicting what the system will look like at time t given some initial data. (Closed form formula means that the formula depends only on the time t and initial data, and *not*, on any intermediate steps [in particular, not a recursive formula].)

- In particular, assume we have the system

$$\boldsymbol{x}(t+1) = A\boldsymbol{x}(t)$$

or, equivalently

$$\boldsymbol{x}(t) = A^t\boldsymbol{x}(0)$$

Suppose there exists a set of eigenvectors of A that forms a basis of the phase space $\mathbb{R}^M$, i.e. vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M$ that form a basis of $\mathbb{R}^M$ with

$$A\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_1\ , \quad A\boldsymbol{v}_2 = \lambda_2\boldsymbol{v}_2\ , \quad \ldots\ , \quad A\boldsymbol{v}_M = \lambda_M\boldsymbol{v}_M$$

then if

$$\boldsymbol{x}(0) = c_1\boldsymbol{v}_1 + \cdots + c_M\boldsymbol{v}_M$$

we can write the closed-form formula

$$\boldsymbol{x}(t) = c_1\lambda_1^t\boldsymbol{v}_1 + c_2\lambda_2^t\boldsymbol{v}_2 + \cdots + c_M\lambda_M^t\boldsymbol{v}_M$$

- This decomposition can be better visualized using coördinates. Let $\mathcal{B}$ denote the basis given by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M$. Then in the coördinates the linear transformation for the phase space can be written as the diagonal matrix

$$B = \big[[T(\boldsymbol{v}_1)]_\mathcal{B} \quad \ldots \quad [T(\boldsymbol{v}_M)]_\mathcal{B}\big] = \big[[\lambda_1\boldsymbol{v}_1]_\mathcal{B} \quad \ldots \quad [\lambda_M\boldsymbol{v}_M]_\mathcal{B}\big] = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & \lambda_M \end{bmatrix}$$

Let S be the matrix

$$S = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_M \end{bmatrix}$$

then our change of coördinate formula from before is

$$A = SBS^{-1}$$

Furthermore

$$S^{-1}\boldsymbol{x}(0) = [\boldsymbol{x}(0)]_\mathcal{B} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_M \end{bmatrix}$$

so

$$\boldsymbol{x}(t) = A^t\boldsymbol{x}(0) = SB^tS^{-1}\boldsymbol{x}(0)$$

Using the fact that B is a diagonal matrix, B^t is just each individual component raised to the tth power, so

$$\boldsymbol{x}(t) = S\begin{bmatrix} \lambda_1^t & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^t & 0 & \dots & 0 \\ 0 & 0 & \lambda_3^t & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & \lambda_M^t \end{bmatrix}[\boldsymbol{x}(0)]_{\mathcal{B}}$$

which is precisely what our formula above gives.

- (Possible phase portraits of two-dimensional dynamical systems based on eigenvalues: attractors, repellers, saddle-points in the case of two eigenvectors; the cases of fewer eigenvectors.)

APR 9, 2008

- Today we talk about eigenvalues
- Recall the definition of an eigenvalue: a number λ is an eigenvalue of a matrix A if there is some nonzero vector $\boldsymbol{v}$ such that

$$A\boldsymbol{v} = \lambda\boldsymbol{v}$$

- We rearrange the terms, and use the fact that $\boldsymbol{v} = I_M\boldsymbol{v}$ if $\boldsymbol{v} \in \mathbb{R}^M$:

$$A\boldsymbol{v} - \lambda I_M\boldsymbol{v} = 0$$

This expression is saying that $\boldsymbol{v}$ is in the kernel of the matrix $B = A - \lambda I_M$ or

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1M} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2M} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3M} \\ \vdots & \vdots & & \ddots & \vdots \\ b_{M1} & b_{M2} & b_{M3} & \dots & b_{MM} \end{bmatrix} = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \dots & a_{1M} \\ a_{21} & a_{22}-\lambda & a_{23} & \dots & a_{2M} \\ a_{31} & a_{32} & a_{33}-\lambda & \dots & a_{3M} \\ \vdots & \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & a_{M3} & \dots & a_{MM}-\lambda \end{bmatrix}$$

- Since $\boldsymbol{v}$ is nonzero, the kernel of B is not trivial. So B is not an invertible matrix. So $\det(B) = 0$.
- We therefore have: for an eigenvalue λ, the *characteristic equation*

$$\det(A - \lambda I_M) = 0$$

is satisfied.

- Example: consider the 2×2 square matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

its characteristic equation is

$$\det(A - \lambda I_2) = \det(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc = 0$$

- We define the *trace* of a matrix to be the sum of its diagonal terms and write it $tr(A)$, then for a 2×2 matrix, the characteristic equation is

$$\lambda^2 - tr(A)\lambda + \det(A) = 0$$

a quadratic equation in λ. We can apply the quadratic formula to the equation to get its roots, which will be the eigenvalues.

- Example:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \det(A - \lambda I_2) = \lambda^2 - 4\lambda - 5 = 0$$

The characteristic equation can be factored in this case:

$$\lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$$

so the roots are $\lambda = \{5, -1\}$. Those two numbers are the eigenvalues (notice we haven't a clue yet what their corresponding eigenvectors are).

- Example: the rotation matrix (we claimed on Monday that it has no eigenvalues unless the angle is a multiple of 180^o)

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Its characteristic equation is

$$\lambda^2 - tr(A)\lambda + \det(A) = \lambda^2 - (2\cos\theta)\lambda + 1 = 0$$

We use the quadratic formula

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

and notice that the discriminant $4\cos^2\theta - 4$ is only non-negative when $\cos\theta = \pm 1$; for all the other cases the quadratic equation has no (real) solutions. And $\cos\theta = \pm 1$ precisely when θ is a multiple of 180^o.

- Example: A is an upper triangular matrix. Now we notice that λI_M is also an upper triangular matrix. Therefore, $A - \lambda I_M$ is also an upper triangular matrix. Its determinant, as done last week, must be the product of the diagonal entries, so

$$\det(A - \lambda I_M) = (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) \cdots (a_{MM} - \lambda) = 0$$

is the characteristic equation for A. We notice that this characteristic equation is already factored! So its roots can be simply read off from the equation, and we have that *the eigenvalues of an upper triangular matrix are precisely the diagonal entries*.

- In general, given an $M \times M$ matrix A, its characteristic equation will be

$$f_A(\lambda) = 0$$

where f_A is a polynomial of degree M and looks like

$$f_A(\lambda) = (-\lambda)^M + tr(A)(-\lambda)^{M-1} + \ldots + \det(A)$$

f_A is called the *characteristic polynomial of* A

- (The proof of the claim will not be given here. See book for details.)

- Knowing that the characteristic equation is a polynomial equation allows us to use tools from basic algebra to study the eigenvalues:

  – The fundamental theorem of algebra says that a degree M polynomial has at most M distinct roots. This implies that a $M \times M$ matrix A has at most M distinct eigenvalues.

  – Also, by the intermediate value theorem, an odd-degree polynomial must have at least 1 root (its graph must crosses the x-axis at least once), so if A is a $M \times M$ matrix with M odd, A has at least one eigenvalue. (If M is even, we need not have eigenvalues: see the case of the rotational matrix.)

- The first statement can be strengthened if we introduce the concept of *algebraic multiplicity*: given a polynomial, we can factor it if we know a root

$$f(\lambda) = (\lambda - c)g(\lambda)$$

if $f(c) = 0$. (The factorisation can be accomplished using synthetic division.) Starting from the polynomial f, we factor it as much as possible

$$f(\lambda) = (\lambda - c_1)(\lambda - c_2)(\lambda - c_3) \cdots (\lambda - c_k)h(\lambda)$$

where $h(\lambda)$ is a polynomial that cannot be factored (in other words, $h(\lambda \neq 0$ for any λ). The multiplication factors in front can be grouped by multiplying terms with $c_i = c_j$ together: so the factoring gives

$$f(\lambda) = (\lambda - b_1)^{d_1}(\lambda - b_2)^{d_2} \cdots (\lambda - b_j)^{d_j} h(\lambda)$$

the numbers d_i, which represents the maximal number of times we can divide $f(\lambda)$ by $(\lambda - b_i)$, is called the *algebraic multiplicity* of the root b_i.

- The algebraic multiplicity of an eigenvalue is the algebraic multiplicity of that number as a root to the characteristic polynomial f_A.
- The fundamental theorem of algebra says that if we count the number of roots of a polynomial with multiplicity (this means that if $f_A(\lambda)$ contains a factor $(\lambda - b_i)^{d_i}$, the root b_i is counted d_i times), the number is at most the degree of the polynomial. So this means that if we count the number of eigenvalues of a $M \times M$ matrix A with multiplicity, the total number is at most M.
- A hint of things to come: the fundamental theorem of algebra also says that the total number of *real* and *complex* roots of a polynomial, when counted together with multiplicity, is exactly the degree of a polynomial. This means that the cases where we have fewer eigenvalues (with multiplicity) than the dimension of the matrix is in fact cases where some of the eigenvalues are complex numbers. We will treat this next week. For now just think of eigenvalues are real numbers.
- The polynomial expression for the characteristic equation also tells us the following: suppose A is a matrix with eigenvalues $\lambda_1, \ldots, \lambda_M$ listed with multiplicity. Then the polynomial f_A can be completely factored

  $$f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_M - \lambda)$$

  A bit of elementary calculation shows that, by expanding the factors,

  $$f_A(\lambda) = (-\lambda)^M + (\lambda_1 + \lambda_2 + \ldots + \lambda_M)(-\lambda)^{M-1} + \ldots + \lambda_1\lambda_2\lambda_3 \cdots \lambda_M$$

  comparing this against the previous expression for the characteristic polynomial, we have that (when A has the maximum number of eigenvalues allowed [with multiplicity])

  - the product of the eigenvalues (with multiplicity) is equal to $\det(A)$
  - the sum of the eigenvalues (with multiplicity) is equal to $tr(A)$

APR 11, 2008

- Now that we've learned about how to find eigenvalues, we now try to find eigenvectors.
- Remember how we derived the characteristic equation for a matrix: we look at the expression

  $$\det(A - \lambda I_M) = 0$$

  because the existence of an eigenvalue/eigenvector pair implies that the matrix

  $$A - \lambda I_M$$

  is not invertible, i.e., it has a non-trivial kernel. With this in mind:

- Let $\lambda_1, \dots \lambda_j$ be distinct eigenvalues of A (not listed with multiplicity). ($j \le M$ as the number of distinct eigenvalues of A must be not greater than the dimension of A.) We define the following spaces:

$$\begin{aligned} E_{\lambda_1} &= \ker(A - \lambda_1 I_M) \\ E_{\lambda_2} &= \ker(A - \lambda_2 I_M) \\ \vdots &= \vdots \\ E_{\lambda_j} &= \ker(A - \lambda_j I_M) \end{aligned}$$

  and we call them the *eigenspaces* associated to the eigenvalues λ_1 and so on.

- In the case we have a zero eigenvalue, E_0 is exactly the kernel of the matrix A.

- Example: let A be the matrix for the orthogonal projection onto a subspace V of $\mathbb{R}^M$. It has two eigenvalues: 1 and 0. What is E_1? It is the kernel of $A - I_M$, i.e., the set of all vectors for which $A\boldsymbol{v} = \boldsymbol{v}$. This is quite clearly (from the geometric interpretation of the orthogonal projection) the subspace V itself: remember from when we studied the method of least squares that for any vector $\boldsymbol{w}$, the length of the orthogonal projection of $\boldsymbol{w}$ onto a subspace V is at most the length of $\boldsymbol{w}$ itself

$$|A\boldsymbol{w}| \le |\boldsymbol{w}|$$

  with equality only when $\boldsymbol{w} \in V$. So we know that any vector *outside* of V cannot satisfy $A\boldsymbol{v} = \boldsymbol{v}$. On the other hand, since A acts on vectors in V like the identity, any vector in V satisfy the equation. So V (no more, no less) is exactly the eigenspace E_1. Now, E_0, being the kernel of A, is just the orthogonal complement space $V^\perp$.

- Example: find the eigenspaces for the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

  First we note that this matrix is upper triangular, so its eigenvalues are just the diagonal entries: $1, 0, 1$. So this matrix has two eigenvalues: 0 with algebraic multiplicity 1, and 1 with algebraic multiplicity 2. We first look at E_0, it is the kernel of A. So we do this like we did earlier in this course

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

  So using $y = s$ a free parameter, we show that the solution must be

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

so the space E_0 is the span of the vector $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. The space E_1 is found by looking for the kernel of

$$A - 1I_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

By finding the RREF, the solution to $(A - I_3)\boldsymbol{v} = 0$ can be given by taking $x = s$ the free parameter

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So E_1 is the span of the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

- Here we come to an important point: the eigenvectors for a matrix do not need to span the entire space (they are necessarily linearly independent between different eigenvalues). The space spanned by the eigenvectors in the eigenspaces E_0, E_1 is only a plane in $\mathbb{R}^3$.

- Definition: the *geometric* multiplicity of an eigenvalue is the dimension of its corresponding eigenspace.

- In our example above, both E_0 and E_1 have dimension 1, so the geometric multiplicity of both the 0 and 1 eigenvalues are 1. Notice that *the geometric multiplicity is not necessarily the same as the algebraic multiplicity*. In fact, we can only say that *the geometric multiplicity is no more than the algebraic multiplicity*.

- Keeping in mind our application of the eigenvectors and eigenvalues to the problem of discrete linear dynamical systems, we see that we are interested in those cases where the eigenvectors of A form a basis: we give this a name.

- Give A an $M \times M$ matrix. A basis of $\mathbb{R}^M$ is called an *eigenbasis* of A if all of the basis vectors are eigenvectors of A.

- So we have the following:

  - Given a matrix A with distinct eigenvalues $\lambda_1, \ldots, \lambda_J$. For each eigenvalue we find its corresponding eigenspace E_{λ_i}. For each of the eigenspace we find a basis. Then if we take all the vectors from the bases of all of the eigenspaces, the list of vectors is linearly independent. (This fact follows because, picking out a random vector $\boldsymbol{v}$ from this list, it will be linearly independent from the other vectors that lies in the same eigenspace as itself [since they form a basis of the eigenspace], and it cannot be linearly dependent on the vectors outside of its eigenspace because they have different eigenvalues.

- So, given a matrix A, we have an eigenbasis for A only when the total number of vectors in the above list is M, the dimension of the space. This is equivalent to saying that *the sum of the geometric multiplicities of the distinct eigenvectors is exactly* M.
- As an addendum, if an $M \times M$ matrix A has exactly M distinct eigenvalues, then A has an eigenbasis. This is because the geometric multiplicity of an eigenvalue is always at least 1. (If the dimension of the eigenspace is 0, the eigenspace does not exist at all, and the number cannot be an eigenvalue.) In fact, each of the eigenspaces has dimension exactly 1.

• Let us go back to the Ninja-Pirates problem. Suppose that now Cowboys have invaded the island. For some reason or another, the Cowboys have the knack of keeping their numbers on the island constant, and as a result, we found that the interaction of the three-population system to be governed by

$$\begin{bmatrix} N(t+1) \\ P(t+1) \\ C(t+1) \end{bmatrix} = \begin{bmatrix} 0.78 & 0.06 & 0.16 \\ -0.14 & 1.22 & -0.08 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N(t) \\ P(t) \\ C(t) \end{bmatrix}$$

We ask: what kind of long-term behaviour can the system exhibit?

- Firstly, we know that we can qualitatively describe the long-term behaviour of a linear discrete dynamical system if we have a decomposition along an eigenbasis. So we first look at the eigenvalues of the matrix of evolution. The characteristic equation is

$$\det(\begin{bmatrix} 0.78-\lambda & 0.06 & 0.16 \\ -0.14 & 1.22-\lambda & -0.08 \\ 0 & 0 & 1-\lambda \end{bmatrix}) = (1-\lambda)[(0.78-\lambda)(1.22-\lambda) - 0.06 * 0.14)]$$

  Now

$$(0.78-\lambda)(1.22-\lambda) + 0.06 * 0.14 = \lambda^2 - 2\lambda - 0.9516 - 0.084 = \lambda^2 - 2\lambda + 0.96$$

  which can be factored as

$$(\lambda - 1.2)(\lambda - 0.8)$$

  So the characteristic equation is

$$(1-\lambda)(1.2-\lambda)(0.8-\lambda) = 0$$

  and the eigenvalues are $1, 1.2, 0.8$.
- Now, since we have 3 distinct eigenvalue for a dynamical system whose phase space is $\mathbb{R}^3$, we can conclude that each of the eigenspaces has dimension 1, and the matrix has an eigenbasis.

- This means that, letting $\boldsymbol{v}_{0.8}, \boldsymbol{v}_{1.2}, \boldsymbol{v}_1$ be the eigenbasis vectors with eigenvalues $0.8, 1.2, 1$ respectively, in the coördinates defined by this basis, the transformation matrix looks like

$$\begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  and we can write the evolution of the system, for a starting vector $\boldsymbol{x}(0)$ as

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{v}_{0.8} & \boldsymbol{v}_{1.2} & \boldsymbol{v}_1 \end{bmatrix} \begin{bmatrix} 0.8^t & 0 & 0 \\ 0 & 1.2^t & 0 \\ 0 & 0 & 1^t \end{bmatrix} [\boldsymbol{x}(0)]_{\mathcal{B}}$$

- This tells us that the system has the following behaviour:
  - If the second coefficient of $[\boldsymbol{x}(0)]_{\mathcal{B}}$ is positive, then as $t \to +\infty$, 1.2^t becomes really big, so

$$0.8^t c_1 \boldsymbol{v}_{0.8} + 1.2^t c_2 \boldsymbol{v}_{1.2} + 1 c_3 \boldsymbol{v}_1$$

    will be mostly in the $\boldsymbol{v}_{1.2}$ direction, since its coefficient becomes a lot bigger than the coefficients of anything else. So we will have an exponential population growth on the island approaching the proportions given by $\boldsymbol{v}_{1.2}$.
  - If the second coefficient is negative, then as $t \to +\infty$, 1.2^t becomes really large, so again the population tends to the $-\boldsymbol{v}_{1.2}$ direction. This means that somewhere along the line at least one of the three populations will vanish and become negative (why?), so as $t \to \infty$ we entered the unphysical regime for this dynamical system, meaning that at some finite time t, our equations no longer adequately describe the behaviour of the system.
  - If the second coefficient is 0, then the system reduces to

$$0.8^t c_1 \boldsymbol{v}_{0.8} + c_3 \boldsymbol{v}_1$$

    and since 0.8^t tends to 0 as t tends to $+\infty$, the contribution of the first term becomes smaller and smaller. So the system will eventually want to stabilize in the direction $\boldsymbol{v}_1$.
- As a side note, you can check that, computing the appropriate kernels, that we can take (up to a scalar multiple)

$$\boldsymbol{v}_{0.8} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_{1.2} = \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Lastly, some facts about Eigenvalues and similarity: suppose A and B are similar matrices, then
  - A, B have the same characteristic polynomial: $f_A(\lambda) = f_B(\lambda)$

- the rank and the nullity of A and B are the same
- A and B have the same eigenvalues (and hence the same trace and same determinants); but they need not have the same eigenvectors.

- This means that you have a quick preliminary check to see whether two matrices A and B can be similar: by looking at their trace and determinant.

- Example: Are the matrices

$$\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

  similar?

  - Quickly we check their trace: $1 + 6 = 3 + 4 = 7$. Okay, so their trace is the same. So they may be similar.
  - check the determinant: $6 - 8 = -2$ for the first, and $12 - 2 = 10$ for the second. The determinants are not equal, so the two matrices cannot be similar.

- Notice that having the trace and determinant the same does not imply that the two matrices are similar:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  have the same trace and determinant, but they are not similar (since the identity matrix is only similar to itself).

Homework for this week

7,1: 2, 12, 30, 32, 34, 36

7.2: 10, 18, 20, 23, 24, 26, 32