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There seems to be a lot of confusion about images and kernels; in particular, I refer to questions 31 and 34 in section 3.1 of the textbook (part of your 3rd homework set). Here I hope to clarify some things and give a solution for the two problems. For completeness, I include the two problems here:

Question 31: Give an example of a matrix A such that $im(A)$ is the plane with normal vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$.

Question 34: Give and example of a linear transformation whose kernel is the line spanned by $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$.

The most important thing I want to stress is the following fact: *the image of a linear transformation is a subspace of the co-domain, while the kernel of the linear transformation is a subspace of the domain*. Let's look at what this means in the context of the two problems. Question 31 specifies that the *image* is a subset in $\mathbb{R}^3$. This means that the *co-domain* should be $\mathbb{R}^3$. So the linear transformation should be $T : X \to \mathbb{R}^3$, where X hasn't been decided. For question 34, however, it is specified that the *kernel* is spanned by a line in $\mathbb{R}^3$. This suggests that the *domain* is $\mathbb{R}^3$. So the linear transformation should be $S : \mathbb{R}^3 \to Y$, where Y hasn't been decided.

Let the matrix to T (the transformation of question 31) be A, and the matrix to S (the transformation of question 34) be B. Our reasoning in the last paragraph means that the number of *rows* of A is 3, and the number of *columns* of B is 3. So supposing that X is $\mathbb{R}^M$ and Y is $\mathbb{R}^N$, the matrix A is $3 \times M$ and B is $N \times 3$.

Now, look at question 31 in more detail. We know that the image is a plane, so with some hindsight from section 3.3 (or some geometric intuition), we know that the image is spanned by at least two vectors. Now, what is the span of a linear transformation? It is the span of the column vectors of the corresponding matrix. So we write

$$A = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_M \end{bmatrix} \tag{1}$$

and then

$$im(A) = span\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M\} \tag{2}$$

The geometric intuition tells us that $M \geq 2$. Then the easiest thing to do is to try to find a set of 2 linearly independent vectors that span the requisite plane.

We want the column vectors of A to be in the plane. We are given that the plane is perpendicular (or normal) to this vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. So we are down to finding two linearly independent vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ with the condition that their respective dot products against the given vector is 0. Now, given a vector

$$\boldsymbol{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{3}$$

the condition that $\boldsymbol{u}$ is perpendicular to $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ is

$$\boldsymbol{u} \cdot \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = x + 3y + 2z = 0 \tag{4}$$

We can think of this as a *2-parameter family of solutions*! In particular, given a particular value of y and z, we can solve for a unique x (in other words y and z are the free parameters). So we plug in the two cases $y = 1, z = 0$ and $y = 0, z = 1$ (the same algorithm that we used to find the basis of the kernel for the example we did in class last Friday and this past Monday; indeed, the process of finding $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ that spans the image of A, with the requirement that the image of A is perpendicular to a given vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, is the same as finding the kernel of the linear transformation from $\mathbb{R}^3 \to \mathbb{R}^1$ given by

$$\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \boldsymbol{x}$$

but I digress), and find that we have two solutions

$$(x, y, z) = (-3, 1, 0) \qquad (x, y, z) = (-2, 0, 1) \tag{5}$$

You can check that these two vectors are indeed perpendicular to $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, and that they are linearly independent. Therefore they span the plane normal to $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. Plugging this back in for $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we have that

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6}$$

is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ that has the desired property.

Notice that you can trivially construct a linear transformation from $\mathbb{R}^M$ to $\mathbb{R}^3$ that satisfy the desired property by writing the column vectors $\boldsymbol{v}_3, \ldots, \boldsymbol{v}_M$ as linear combinations of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. But also notice that it is impossible to find a solution to question 31 *if you pick* $M = 1$. To see that, we can use the Rank-Nullity theorem we just learned on Monday. The image of A is a plane, which means that the image of A is 2 dimensional, which means that the rank of A is 2. The nullity, on the other hand, is a non-negative number. Which means that

$$\text{rank } + \text{ nullity } \geq 2 + 0 \neq 1 = \dim(\mathbb{R}^1)$$

This confirms the intuitive guess that there can be no linear map from a line into space such that the image of the line is a plane.

To summarise: in order to find the column vectors for a 3×2 matrix A that satisfy the desired property (the image is the plane perpendicular to some prescribed vector $\tilde{\boldsymbol{v}}$) we need to find 2 column vectors in $\mathbb{R}^3$ that are linearly independent and perpendicular to $\tilde{\boldsymbol{v}}$. To find those two vectors, we look for a *basis* of the kernel of the linear transformation corresponding to "dot product against $\tilde{\boldsymbol{v}}$ (remember that dot product against some vector is a linear transformation whose co-domain is $\mathbb{R}^1$). And to find the basis for the kernel, we solve

$$\tilde{\boldsymbol{v}} \cdot \boldsymbol{x} = 0$$

and write its solution as a parametrized family. By taking turns to set one of the free parameters as 1 and the rest 0, for each parameter we obtain a vector in the kernel. By construction (as shown in class on Monday), these vectors form a basis for the kernel of $\tilde{\boldsymbol{v}} \cdot \boldsymbol{x}$, and thus a basis for the desired plane in $\mathbb{R}^3$.

Now we turn our attention to question 34. This question asks for a linear transformation whose kernel is exactly the line L spanned by $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Some of you asserted in your homework that such a transformation is given by $proj_L$, the projection onto L. This is incorrect, as the projection onto L will send any vector on L, in particular $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, to itself. This is the opposite of what we want: we want to send L to 0 and everything else to something that is non-zero. A simple geometric construction that does do what we want is therefore

$$S = \text{ identity transformation } - \underset{L}{proj} \tag{7}$$

Let us verify that this transformation indeed does what we want. First we want to check that S takes the line L to 0. This is true since the identity transformation takes any element $\boldsymbol{x}$ on the line L to itself, and so does the projection. So for $\boldsymbol{x} \in L$, we have

$$S(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{x} = 0 \tag{8}$$

Now suppose $\boldsymbol{y}$ is a vector that does not live on L. This means that it has a component that is "perpendicular" to L (see the lecture notes for Week 2 for details about the decomposition of a vector into a part that is perpendicular to L and a part that is parallel to L). In particular, this means that $proj_L(\boldsymbol{y}) \neq \boldsymbol{y}$. So

$$S(\boldsymbol{y}) = \boldsymbol{y} - \text{ something that is not equal to } \boldsymbol{y} \neq 0 \tag{9}$$

Now what does the matrix for this transformation look like? (If you had written something like what I had above, you would've received full credit. Since the question asked for a linear transformation, you don't have to explicitly calculate the matrix if you can define the transformation in terms of familiar operations [meaning, a mixture of the identity, scaling, projection, rotation, shear, and reflection transformations].) Remember that the identity transformation has, as its matrix, the identity matrix I_3 (since we are considering $S : \mathbb{R}^3 \to \mathbb{R}^3$). For the projection, we need to calculate: the formula for the projection operator is

$$\underset{L}{proj}(\boldsymbol{y}) = (\boldsymbol{u} \cdot \boldsymbol{y})\boldsymbol{u} \tag{10}$$

where $\boldsymbol{u}$ is an unit vector of L. We know that L is spanned by $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, so we calculate the unit vector by taking

$$\boldsymbol{u} = \frac{1}{\sqrt{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{bmatrix} \tag{11}$$

So

$$\underset{L}{proj}(\boldsymbol{y}) = \begin{bmatrix} -\frac{\sqrt{6}}{6}\boldsymbol{u} \cdot \boldsymbol{y} \\ \frac{\sqrt{6}}{6}\boldsymbol{u} \cdot \boldsymbol{y} \\ \frac{\sqrt{6}}{3}\boldsymbol{u} \cdot \boldsymbol{y} \end{bmatrix} \tag{12}$$

Now using the row-vector decomposition of a matrix and its associated way of multiplying against a vector through dot-products (from chapter 1), we see that the row vectors of the matrix for $proj_L$ has to be $-\frac{\sqrt{6}}{6}\boldsymbol{u}$, $\frac{\sqrt{6}}{6}\boldsymbol{u}$, $\frac{2\sqrt{6}}{6}\boldsymbol{u}$. Working out the arithmetic, we have the matrix is

$$\begin{bmatrix} \frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \tag{13}$$

Putting it all together, we have that the matrix B for S is

$$B = I_3 - \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \tag{14}$$

Now we can also apply a little bit of hindsight with the aid of the Rank-Nullity theorem. Since S is assumed to have kernel spanned by exactly 1 vector, we have that the nullity of S is 1. Now, we also know that $S : \mathbb{R}^3 \to \mathbb{R}^N$, so by the Rank-Nullity theorem

$$\text{rank } + 1 = 3 \tag{15}$$

where 3 is the dimension of the domain, and 1 is the nullity. This means that the rank of S has to be 2. But we also know that the rank of a linear transformation has to be simultaneously smaller than the number of rows and the number of columns in its associated matrix (in other words, the rank of a linear transformation is necessarily smaller than the dimension of its domain, and is necessarily smaller than the dimension of its co-domain). This means that the number of rows for the matrix of S (in other words, the number N) has to be greater than or equal to 2, further ruling out the possibility that the required transformation S can be given as a projection onto a single line.

Lastly, please understand that it is not required of you to write a five-page essay for two problems on the homework set. This document is only so long becaused I want to show you how every bit of reasoning from what we have learned so far in class fits into the solving of a seemingly innocuous problem. If you managed to read this far, I offer you my congratulations, since you just finished with some hefty mental acrobatics. I hope that this short note is helpful toward your understanding of the subject, and I regret that due to time constraints I probably cannot present such excruciating details during class time. As always, if you have any questions, do not hesitate to contact me.