# Princeton University MAT 202 Spring 2008

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February 4 - 8

FORMAL MATTER/PAPERWORK

- This is MAT 202, Linear Algebra with Applications
- Instructor: Willie W. Wong
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  - Office: Fine 408
  - Office hours: Thursday 10a-11a; 2p-3:30p; or by appointment
- Grader: Neil Katuna (`nkatuna@princeton.edu`)
- Course website(s)
  - The main course website is at `http://blackboard.princeton.edu`
  - Some additional material (read: lecture notes) will be posted at `http://www.math.princeton.edu/~wwong/teach.html`
- Hand-outs: 1) Course information 2) Homework assignments ... Read the hand-outs
  - Review sessions: Tuesdays 7:30pm - 9:30pm in Fine 314 with Adrian Banner, with additional ones before Midterms and Finals TBA
  - Grades for quizzes and homeworks will be curved at the end taking in account which section you are in: because quizzes and homeworks will be graded separately by section, where midterm and final will be graded together.
  - Quizzes: IMPORTANT! The questions for the quizzes are the same for all section. There will be 2 quizzes (tell them this is noted on the homework hand-out): First on Wednesday, Feb 27, covering chapters 1 and 2. The second on Wednesday, April 16, covering chapters 5 and 6, as well as sections 7.1 and 7.2. The quizzes will be 50 minutes long and in class. There will be NO make-up except for extremely compelling reason (e.g. a doctors note proving that you broke your leg that morning). If you already know you will be away on those dates, let me know ASAP. Otherwise you'll just get a zero on your quiz.

- Homeworks: cross out that line about "plastic mailbox outside instructor's office". I don't have one. Homeworks are due on Mondays, I will collect them at the end of class. If, for some reason, you have to miss class, please make sure that you leave your homework in my mailbox (the one labeled "Wong") outside Room 314 (this one) by 2pm the latest. Late homeworks will not be accepted. The two lowest grades (out of the eleven total) (lowest in terms of percentage score; each homework set will be weighed the same) will be dropped when calculating your final grade, so that should take care of any accidents and special situations. As long as my grader keeps up, I will return the homeworks on Friday of the same week.

- Homework solutions will be posted on Blackboard on Tuesday mornings.

## FEB 4, 2008

- What is *Linear Algebra*?
  - Algebra: the study of variables and their relations. More precisely, given one (or more) relation between (one or more) undefined variables, we want to know *what values can the variables take on that simultaneously satisfies all the relations*.
    - Example: a set of 1 equation of 1 variable

      $$x^2 + 2x + 1 = 0$$

      The value $x = -1$ will satisfy this relation.
    - Example: a set of 2 equations of 1 variable

      $$\begin{aligned} x^2 - 2x - 3 &= 0 \\ x^2 - 1 &= 0 \end{aligned}$$

      The first one is satisfied by the two values $x = 3$ and $x = -1$. The second one is satisfied by $x = 1$ and $x = -1$. So the only value that will satisfy *both* of the equations is $x = -1$.
    - Example: the relation doesn't have to be equations. They can be

      $$\begin{aligned} x + 3y - z &> 0 \\ 2x - y &< -4 \end{aligned}$$

  - Linear: we say that an algebraic equation is linear if its solutions contain lines. Remember for elementary geometry that two points determine a line. So an algebraic equation (between two or more variables) is linear if, given two different, known solutions, all values on the line determined by the two solutions are solutions.

* Example:

$$y = x^2$$

is not linear. The points $x = 0, y = 0$ and $x = 1, y = 1$ are solutions. (Plot on board.) But the point $x = 0.5$ and $y = 0.5$, whose coördinates are half-way between the two given solutions (and hence is on the line between the two solutions), is not a solution.

* Example:

$$3y + 2x = 0$$

is linear.

* If you graph the solution of a linear algebraic equation, you should get something straight and flat. A linear equation of two variables has as its graph a line in the plane. A linear equation of three variables has the graph of a plane in space. A linear equation of four variables has the graph of a hyperplane in 4-dimensional space.

– So linear algebra is the study of systems of (one or more) linear equations on variables.

- Let's look at a system of linear equations:

$$\begin{array}{cccc} 3x & +2y & -z & = 5 \\ 2x & -y & +3z & = 3 \\ -x & +y & +z & = 6 \end{array}$$

(Solve by elimination).

- Let's look at this solving by elimination method: what's the idea? We want to "transform"/"extract" from the given set of equations a set of equations that says

$$\begin{array}{cccc} x & & & =? \\ & y & & =? \\ & & z & =? \end{array}$$

- Geometric interpretation: the number of variables you have represent the number of dimensions. If I just say that I have the three variables $(x, y, z)$, without specifying any relations, they can be any point in space. By specifying a single linear equation, I am saying, oh, by the way, I want $(x, y, z)$ that sits on "such and such" plane. Therefore, for the system of linear equations above, we are saying: $(x, y, z)$ not only needs to sit on such-and-such plane, it also needs to sit on this-and-that plane, and which-and-what plane. In other words, $(x, y, z)$ needs to sit on the intersection of the three planes.

- Geometric interpretation: eliminating the $z$ variable by adding the first and third equation means finding a plane that passes through the intersection line between the planes defined by the first and third equation and does not depend on the $z$ variable.

- Equations can have a unique solution, no solutions, or infinitely many (why can't it have just 2?). What does this mean graphically?
- More examples.

## FEB 6, 2008

- Keep in mind the following system:

$$\begin{aligned} 2x + y + z &= 5 \\ 4x - 6y(+0z) &= -2 \\ -2x + 7y + 2z &= 9 \end{aligned}$$

- To solve a system of linear equations, we are allowed to use three operations:
  - Permutation: changing the order of the lines
  - Multiplication: multiplying a relation by a nonzero number
  - Summing-and-Replacement: replacing an equation by the sum of two equations.

  These operations are *reversible* and do not change the solution set of the system of equations.

- If two systems of linear equations can be made identical through uses of the three operations, we say they are "equivalent".
- The goal to solving a system of linear equations is to replace a complicated system with a simpler, but equivalent one. The simplest system is of course the diagonal system specifying $x, y, z$ so the answer can be read off directly.
- Notice that in all three of the operations, the symbols $x, y, z$ are just placeholders that are not changed in the operation. To make life simpler, we write the equation as a rectangle without the symbols to save time.

$$\begin{bmatrix} 2 & 1 & 1 & \vdots & 5 \\ 4 & -6 & 0 & \vdots & -2 \\ -2 & 7 & 2 & \vdots & 9 \end{bmatrix}$$

- The rectangular array has 3 rows, one for each of the equations. The portion of the array in front of the dots (the first three columns in this case), corresponds to the coefficients $x, y$ and $z$ in the original equation. Each symbol to its column. The three columns together is called the "coefficient matrix". The last column corresponds to the "right hand side" of the original system. The entire array is called the "augmented matrix" of the system.

- The three elementary operations still work the same way. Before:

$$\begin{cases} 2x & +y & +z & = 5 & (A) \\ 4x & +(-6)y & +(0z) & = -2 & (B) \\ (-2)x & +7y & +2z & = 9 & (C) \end{cases}$$

eliminate $x$ from the second and third equations

$$\Rightarrow \begin{cases} 2x & +y & +z & = 5 & (A) \\ & (-8)y & +(-2)z & = -12 & (B') = (B) + (-2)(A) \\ & 8y & +3z & = 14 & (C') = (C) + (A) \end{cases}$$

eliminate $y$ from the first and last equations

$$\Rightarrow \begin{cases} 2x & & +\frac{3}{4}z & = \frac{7}{2} & (A') = (A) + \frac{1}{8}(B') \\ & (-8)y & +(-2)z & = -12 & (B') \\ & & z & = 2 & (C'') = (C') + (B') \end{cases}$$

in the end, eliminate $z$ from the first and second equations

$$\Rightarrow \begin{cases} 2x & & & = 2 & (A'') = (A') + \frac{-3}{4}(C'') \\ & (-8)y & & = -8 & (B'') = (B') + 2(C'') \\ & & z & = 2 & (C'') \end{cases}$$

So the solution is $(1, 1, 2)$.

- In matrix notation: we first make the first column $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ using the elementary operations

$$\begin{bmatrix} 2 & 1 & 1 & \vdots & 5 \\ 4 & -6 & 0 & \vdots & -2 \\ -2 & 7 & 2 & \vdots & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \vdots & \frac{5}{2} \\ 0 & -8 & -2 & \vdots & -12 \\ 0 & 8 & 3 & \vdots & 14 \end{bmatrix}$$

Then we make the second column $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and third column $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{8} & \vdots & \frac{7}{4} \\ 0 & 1 & \frac{1}{4} & \vdots & \frac{3}{2} \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

From this we can directly read off the solution.

- The "coefficient matrix" in the above is put into the so-called "reduced row echelon form" (**rref**).

- The transformation of the augmented matrix of the system into one where the coefficient matrix is in rref is by the process known as *Gauss-Jordan elimination*.

- The allowed operations for G-J elimination is precisely the three elementary operations (for more details, see pages 13-18 of your book):

  - Permutation of rows
  - Multiplication of a row by a non-zero number
  - Replacement of a row by the sum of that row and a non-zero multiple of another row.

  And the goal is to get the coefficients in rref: A matrix is in rref if

  - All nonzero rows are above any rows of all zeros
  - The first (as in, to the farthest left) nonzero entry of a row is 1
  - The first nonzero entry of a row is always to the *right* of the first nonzero entry of the row above
  - All other entries in the column occupied by a "first nonzero entry" are zeros

  Are the following matrices in rref?

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 2 & 3 \\ 1 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 & 0 & 3 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Given a matrix $A$. The notation rref($A$) means a matrix in reduced row echelon form given by Gauss Jordan elimination. One might think that the swapping of rows by the first elementary operation might give several different rref matrices for each matrix $A$: in fact, the rref matrix rref($A$) is unique.

- The *rank* of a coefficient matrix $A$ is the number of leading ones/pivots in rref($A$). If the rank of a coefficient matrix is the same as the number of columns, then the system of equation has exactly 1 solution. If the rank of a coefficient matrix is smaller than the number of columns, then the system of equation either has no solutions or infinitely many. Can the rank of a coefficient matrix be bigger than the number of columns?

- Example: find the solutions to the system

$$\begin{cases} 2x_1 + 3x_3 - 5x_4 + 4x_6 & = 0 \\ x_1 + 2x_4 - 3x_5 - 2x_6 & = -2 \\ x_2 + x_3 - x_4 - x_5 & = 3 \\ x_2 - 2x_3 + 8x_4 - 3x_5 + 6x_6 & = 5 \end{cases}$$

First, we write the augmented matrix of the system

$$\begin{bmatrix} 2 & 0 & 3 & -5 & 0 & 4 & \vdots & 0 \\ 1 & 0 & 0 & 2 & -3 & -2 & \vdots & -2 \\ 0 & 1 & 1 & -1 & -1 & 0 & \vdots & 3 \\ 0 & 1 & -2 & 8 & -3 & 6 & \vdots & 5 \end{bmatrix}$$

And now we try to perform Gauss-Jordan elimination to make the coefficient matrix into rref. First switch the first and second rows (since the second row already has a "leading one") and eliminate the first column from what is now the second row.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & -3 & -2 & \vdots & -2 \\ 0 & 0 & 3 & -9 & 6 & 8 & \vdots & 4 \\ 0 & 1 & 1 & -1 & -1 & 0 & \vdots & 3 \\ 0 & 1 & -2 & 8 & -3 & 6 & \vdots & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & -3 & -2 & \vdots & -2 \\ 0 & 1 & 1 & -1 & -1 & 0 & \vdots & 3 \\ 0 & 0 & 3 & -9 & 6 & 8 & \vdots & 4 \\ 0 & 0 & -3 & 9 & -2 & 6 & \vdots & 2 \end{bmatrix}$$

Next we had to swap the second and third row, since the third row has a leading one in the second column and the second row has a 0 there. After doing that, we eliminated the second column from the fourth row. Next we multiply the third row by $1/3$ and use it to eliminate the third column from the various rows.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & -3 & -2 & \vdots & -2 \\ 0 & 1 & 0 & 2 & -3 & -\frac{8}{3} & \vdots & \frac{5}{3} \\ 0 & 0 & 1 & -3 & 2 & \frac{8}{3} & \vdots & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 4 & 14 & \vdots & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & \frac{17}{2} & \vdots & \frac{5}{2} \\ 0 & 1 & 0 & 2 & 0 & \frac{47}{6} & \vdots & \frac{37}{6} \\ 0 & 0 & 1 & -3 & 0 & -\frac{13}{3} & \vdots & -\frac{5}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{7}{2} & \vdots & \frac{3}{2} \end{bmatrix}$$

Finally we divide the remaining row by 4 and use it to eliminate the fifth column of all the rows. The rref corresponds to the following system of linear equations:

$$\begin{cases} x_1 + 2x_4 + \frac{17}{2}x_6 & = \frac{5}{2} \\ x_2 + 2x_4 + \frac{47}{6}x_6 & = \frac{37}{6} \\ x_3 - 3x_4 - \frac{13}{3}x_6 & = -\frac{5}{3} \\ x_5 + \frac{7}{2}x_6 & = \frac{3}{2} \end{cases}$$

- Once the augmented matrix has been reduced to rref, we see that there are some columns with leading ones and some columns without. In the previous example, the first, second, third, and fifth columns have leading ones, while the second and sixth do not. The columns with leading ones corresponds to "dependent variables" while the columns without leading ones corresponds to "independent variables" or "free parameters". For any value we choose the "independent variables" to take, we can directly read off 'a' solution from the rref. Looking back to our example, the system can be rearranged to look like

$$\begin{cases} x_1 & = \frac{5}{2} - 2x_4 - \frac{17}{2}x_6 \\ x_2 & = \frac{37}{6} - 2x_4 - \frac{47}{6}x_6 \\ x_3 & = -\frac{5}{3} + 3x_4 + \frac{13}{3}x_6 \\ x_5 & = \frac{3}{2} - \frac{7}{2}x_6 \end{cases}$$

- In this example, we have two "independent variables" or "free parameters", so we say that our system of equations has a 2-parameter family of solutions. In particular, we can choose the parameters

  $$x_4 = s\ , \quad x_6 = t$$

  then the solutions can be written in vector form (a vector is a one column matrix)

  $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{37}{6} \\ -\frac{5}{3} \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -\frac{17}{2} \\ -\frac{47}{6} \\ \frac{13}{3} \\ 0 \\ -\frac{7}{2} \\ 1 \end{bmatrix} t$$

FEB 8, 2008

## i. Number of solutions to a linear system

- As mentioned last time, the rank of a matrix (recall, the rank of the coefficient matrix is the number of "leading one"s in the matrix) tells us something about the number of solutions.

- Consistent vs. Inconsistent systems: a linear system is inconsistent if it contains a statement that cannot be logically true. Writing a linear system as an augmented matrix, and taking the rref of the matrix, an inconsistency occurs when and only when one of the rows looks like

  $$\begin{bmatrix} 0 & 0 & \dots & 0 & \vdots & 1 \end{bmatrix}$$

  In other words, a linear system is inconsistent if, using the three elementary operations, we arrive at an equivalent system containing the statement $0 = 1$.

- A inconsistent system has no solutions.

- A consistent system (one that is not inconsistent) has either exactly one solution (if all the variables are leading, i.e. if every column of the rref coefficient matrix has a leading one, or if all the variables are "dependent" variables) or it has infinitely many solutions (if there is at least one free parameter/independent variable).

- Suppose we have a system of $N$ equations in $M$ variables. The coefficient matrix, which we call $A$, has $N$ rows and $M$ columns. Now we look at rref($A$). By construction, there is at most one leading one per row, and there is at most one leading one per column. So the rank of $A$ (which we write rank($A$)) cannot be more than the smaller of $N$ and $M$.

$$\Rightarrow rank(A) \leq N\ , \quad rank(A) \leq M$$

- Furthermore, if rank($A$) $= N$, there must be exactly one leading one in every row. Therefore, there cannot be a row (in the augmented matrix) that looks like $\begin{bmatrix} 0 & 0 & \dots & 0 & \vdots & 1 \end{bmatrix}$. So the system is consistent.

- If rank($A$) $= M$, there must be exactly one leading one in every column of rref($A$). Which means that there are $M$ dependent variables. In this case, if the system is consistent, we will have exactly one solution. If the system is inconsistent, we will have no solutions. So all in all, we can say that when rank($A$) $= M$, there is at most one solution to the system.

- It is nice to note that the value of rank($A$) is the number of dependent variables. And since every variable is either dependent or independent, we have

$$\text{the number of dependent variables} \quad + \quad \text{the number of independent variables} = M$$

and so

$$\text{the number of free parameters} = M - rank(A)$$

A consequence of this is that

- If rank($A$) $< M$, then either the system is inconsistent, or the system has infinitely many solutions.

- A consequence is the following: suppose $N < M$. Since rank($A$)$\leq N < M$, we have that *a system with fewer equations than variables has either no solutions or infinitely many*.

## ii. Matrix algebra

- Addition: you can only add two matrices of the same size. The addition is term by term.

$$\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \dots & a_{1m}+b_{1m} \\ \vdots & & \vdots \\ a_{n1}+b_{n1} & \dots & a_{nm}+b_{nm} \end{bmatrix}$$

- Scalar multiplication: you can multiply a matrix by a number. You multiply "in" to every term

$$k \times \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} = \begin{bmatrix} ka_{11} & \dots & ka_{1m} \\ \vdots & & \vdots \\ ka_{n1} & \dots & ka_{nm} \end{bmatrix}$$

- Notation: $a_{ij}$ is the entry in the $i$'th row and $j$'th column of the matrix $A$. An $n \times m$ matrix has $n$ rows and $m$ columns.

- Matrix-vector products: you can multiply an $N \times M$ matrix *on the right* by a column vector of $M$ rows. To do so, we write the $N \times M$ matrix, call it $A$, as $M$ column vectors of $N$ rows stacked next to each other:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1M} \\ \vdots & & \vdots \\ a_{N1} & \dots & a_{NM} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_M \end{bmatrix}$$

  where

$$\boldsymbol{v}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{Ni} \end{bmatrix}$$

  Then the multiplication of $A$ by a column vector $\boldsymbol{X}$ is given by

$$AX = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_M \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_M\boldsymbol{v}_M$$

  Treating the vectors $\boldsymbol{v}_i$ as $N \times 1$ matrices, the multiplication by $x_i$ is scalar multiplication above, and the addition is the matrix addition above. So the product of an $N \times M$ matrix by a $M$-vector is an $N$-vector.

- Some more notations:

  – $\mathbb{R}^N$ is the set of all ordered $N$-tuples of real numbers. You can think of the space as the set of all column vectors with $N$ rows.

  – $\in$ is the notation for "belong in" or "is an element of". So we can say

$$\begin{bmatrix} 1 \\ 19/2 \\ \pi \\ 2.589 \end{bmatrix} \in \mathbb{R}^4$$

  – The dot product of two elements of $\mathbb{R}^M$ (or the dot product between an $M$-element row vector $\boldsymbol{w}$ with an $M$-element column vector $\boldsymbol{v}$) is

$$\boldsymbol{w} \cdot \boldsymbol{v} = w_1v_1 + w_2v_2 + \cdots w_Mv_M$$

  – A scalar is a pure number: i.e. a number that is not a matrix or a vector.

- Back to matrix-vector products: we could also have decomposed the matrix $A$ as a bunch of $M$-column row vectors stacked on top of each other

$$A = \begin{bmatrix} a_{11} & \dots & a_{1M} \\ \vdots & & \vdots \\ a_{N1} & \dots & a_{NM} \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \vdots \\ \boldsymbol{w}_N \end{bmatrix}$$

where

$$\boldsymbol{w}_i = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{iM} \end{bmatrix}$$

Then we write the multiplication as

$$AX = \begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \vdots \\ \boldsymbol{w}_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_1 \cdot X \\ \vdots \\ \boldsymbol{w}_N \cdot X \end{bmatrix}$$

with the dot product defined as before. This gives us a easy way to compute the product of a matrix against a vector by "turning the column vector horizontal" and then "overlaying it against each row of the matrix". (Demonstrate.)

- Matrix-vector multiplication is linear: let $A$ be an $N \times M$ matrix and $X, Y$ two column vectors of size $M$ and $k$ a scalar. Then

$$A(X + Y) = AX + AY$$

and

$$k(AX) = A(kX)$$

- Linear combination: a vector $\boldsymbol{u}$ is said to be a linear combination of vectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_M\}$ we can find scalars $x_1, \dots, x_M$ such that

$$\boldsymbol{u} = x_1\boldsymbol{v}_1 + \dots + x_M\boldsymbol{v}_M$$

In particular, writing an $N \times M$ matrix $A$ as the vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_M$ stacked side by side (see first definition of matrix-vector product), the product $AX$ against a vector $X \in \mathbb{R}^M$ is a linear combination of the vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_M$.

### iii. Linear systems in terms of matrices

- The following systems are equivalent:

$$\begin{bmatrix} 2 & 3 & \vdots & 7 \\ 1 & -4 & \vdots & 3 \end{bmatrix} \iff \begin{cases} 2x & +3y & = 7 \\ x & -4y & = 3 \end{cases} \iff \begin{bmatrix} 2x + 3y \\ x - 4y \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

which

$$\iff \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \iff x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

- The first arrow is the equivalence between a linear system and its augmented matrix. The second arrow is casting a linear system of $N$ equations as the equivalence of two $N$-entry vectors (vectors are equal if and only if all the entries are equal). The third arrow decomposes the vector using the second definition of matrix-vector products. The fourth arrow uses the equivalence of a matrix-vector product and a linear-combination statement.

- So we see that asking whether a linear system has a solution is the same as asking whether the vector formed by the entries on the RHS is a linear combination of the column vectors formed by the coefficients.

- Example: Can $\begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$ be a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$? To see that, we write down the equivalent augmented matrix of the system and put it in rref:

$$\begin{bmatrix} 1 & 0 & \vdots & 1 \\ 0 & 0 & \vdots & 5 \\ 0 & 1 & \vdots & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \vdots & 1 \\ 0 & 1 & \vdots & 0 \\ 0 & 0 & \vdots & 1 \end{bmatrix}$$

  The last row shows that the system has no solution, and so there's no solution to the original question.

- A linear system of $N$ equations in $M$ variables can be written as $A\boldsymbol{x} = \boldsymbol{b}$, where $A$ is an $N \times M$ matrix, $\boldsymbol{x}$ a vector of $M$ entries and $\boldsymbol{b}$ a vector of $N$ entries.
  - If $\boldsymbol{b}$ is identically 0, we say the system is *homogeneous*.
  - The solution to a homogeneous system are the vectors $\boldsymbol{x}$ that are perpendicular to all the ROW vectors of $A$ (seen by the dot-product decomposition of matrix-vector product).
  - In the future, we want to be able to make sense of "dividing by a matrix", i.e. solve $A\boldsymbol{x} = \boldsymbol{b}$ by setting

$$\boldsymbol{x} = \boldsymbol{b}/A$$

## Homework for this week

1.1: 10, 14, 20, 26, 28

1.2: 6, 11, 18, 24, 30, 34, 42

1.3: 4, 7, 14, 23, 28, 36, 47, 55