

Projection Method with Spatial Discretizations

Weinan E

School of Mathematics

Institute for Advanced Study

Princeton, NJ 08540

and

Jian-Guo Liu

Courant Institute of Mathematical Sciences

New York, NY 10012

Abstract:

In [5], we studied convergence and the structure of the error for several projection methods when the spatial variable was kept continuous¹. In this paper, we address similar problems for the fully discrete case when the spatial variable is discretized using the staggered grid. We prove that the numerical solution has full accuracy upto the boundary, despite the fact that there are numerical boundary layers present in the semi-discrete solutions.

§1. Introduction

In this paper, we continue our study of convergence and error structure for the projection method. In previous work [5], we studied semi-discrete situations when the temporal variable is discretized, but the spatial variable is kept continuous. We proved that as far as velocity is concerned, the method of Chorin and Temam [?, ?] is uniformly first order accurate upto the boundary, and the second order method of [?, ?] is uniformly second order accurate upto the boundary. However, due to the presence of numerical boundary layers, pressure has only half order of accuracy near the boundary. We characterized explicitly the structure of the numerical boundary layer, and proved that the full accuracy for pressure

¹We will refer to this situation as semi-discrete

can be recovered if we subtract the numerical boundary layer modes from the solution. We also showed using normal mode analysis that the boundary layer modes turn into oscillatory modes for the pressure-increment formulation of [?, 2]. For a summary of these results, we refer to [?].

The presence of singular modes in the semi-discrete solutions raise serious doubts on the convergence of the fully discrete method. Indeed it is known for some time that accuracy and even convergence can be lost if care is not exercised at the projection step. This is documented carefully in [?]. It is also known for a long time that full accuracy is kept if the spatial discretization is done on a staggered grid. This paper is devoted to a proof of this empirical fact.

We work still leaves open the very important issue of characterizing the minimum condition that the spatial discretization has to satisfy in order to guarantee accuracy. The working assumption seems to be that as long as the projection step is truly a projection, i.e. the numerical projection operator P_h satisfies: (1) P_h is self-adjoint; (2) $P_h^2 = P_h$ (we will refer to this as the projection condition), accuracy in velocity will be kept. It remains to be seen whether this is true in general. On the other hand, there are many situations in which enforcing the projection condition is difficult. Therefore we are motivated to seek numerical methods which does not require the projection condition. The gauge method [?] has so far proved to be a very attractive alternative in this regard.

This paper is organized as follows. In the next section, we review the projection method, with emphasis on spatial discretization on the staggered grid (also known as the MAC grid [?]). In Section 3, we summarize our main results. Then in Sections 4 and 5 we present the proof for the first and second order methods respectively.

§2. Review of the Projection Methods

In primitive variables, Navier-Stokes equation (NSE) takes the following form

$$(2.1) \quad \begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \Delta \mathbf{u}, \\ \nabla \cdot \mathbf{u} = 0. \end{cases}$$

Here $\mathbf{u} = (u, v)$ is the velocity, and p is the pressure. For simplicity, we will only consider the case when the no-slip boundary condition is supplemented to (2.1):

$$(2.2) \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega$$

where Ω is an open domain in \mathbf{R}^2 with smooth or piecewise smooth boundary.

§2.1. Time discretization

The first order projection method of [?, ?] proceeds in two steps.

Step 1: Computing the intermediate velocity field \mathbf{u}^* .

$$(2.6) \quad \begin{cases} \frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} + (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n = \Delta \mathbf{u}^*, \\ \mathbf{u}^* = 0, \quad \text{on } \partial\Omega. \end{cases}$$

Step 2: Projecting to the space of divergence-free vector fields to obtain \mathbf{u}^{n+1} .

$$(2.7) \quad \begin{cases} \mathbf{u}^* = \mathbf{u}^{n+1} + \Delta t \nabla p^{n+1}, \\ \nabla \cdot \mathbf{u}^{n+1} = 0. \end{cases}$$

The projection step enforces the boundary condition:

$$(2.10) \quad \mathbf{u}^{n+1} \cdot \mathbf{n} = 0, \quad \text{or} \quad \frac{\partial p^{n+1}}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega$$

Second order schemes:

We will concentrate on the formulation in [?, ?], and refer to [?] for a summary of other second order methods.

$$(2.12) \quad \begin{cases} \frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} + (\mathbf{u}^{n+1/2} \cdot \nabla) \mathbf{u}^{n+1/2} = \Delta \frac{\mathbf{u}^* + \mathbf{u}^n}{2}, \\ \mathbf{u}^* + \mathbf{u}^n = \Delta t \nabla p^{n-1/2}, \quad \text{on } \partial\Omega, \\ \mathbf{u}^* = \mathbf{u}^{n+1} + \Delta t \nabla p^{n+1/2}, \\ \nabla \cdot \mathbf{u}^{n+1} = 0, \\ \frac{\partial p^{n+1/2}}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega. \end{cases}$$

In this formulation, homogeneous Neumann BC for pressure is retained. An inhomogeneous BC for \mathbf{u}^* is introduced so that the slip velocity of \mathbf{u}^{n+1} at the boundary is of order Δt^2 .

Remark The nonlinear convection term $(\mathbf{u}^{n+1/2} \cdot \nabla) \mathbf{u}^{n+1/2}$ can be treated in many ways. In the Theorems 1 and 2, we use the Adams-Bashforth formula: $\frac{3}{2}(\mathbf{u}^n \cdot \nabla) \mathbf{u}^n - \frac{1}{2}(\mathbf{u}^{n-1} \cdot \nabla) \mathbf{u}^{n-1}$.

It is readily seen that the projection step enforces

$$(2.13) \quad \frac{\partial p^{n+1}}{\partial \mathbf{n}} = \frac{\partial p^n}{\partial \mathbf{n}} = \dots = \frac{\partial p^0}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega$$

for the numerical solution. In general this is not satisfied by the exact solution of (2.1). Therefore we expect that $\frac{\partial p^n}{\partial \mathbf{n}}$ has $O(1)$ error at the boundary. As will be seen below, this causes \mathbf{u}^* and p^n to have numerical boundary layers.

§2.2. Spatial discretization

We will consider the case when the MAC scheme is used to discretize in \mathbf{x} .

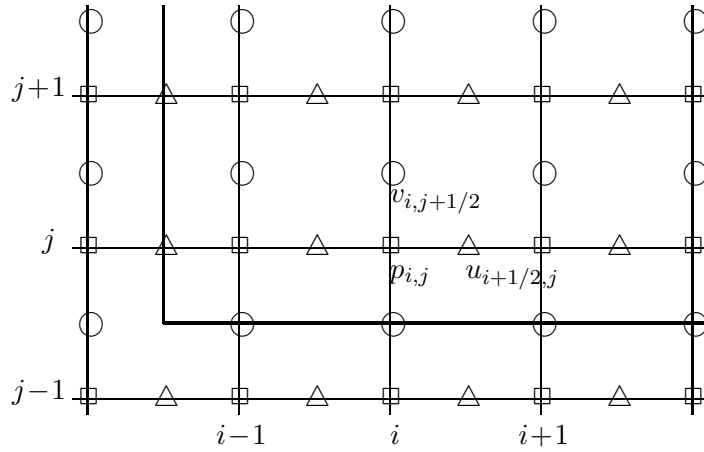


Figure 1: The MAC mesh

An illustration of the MAC mesh near the boundary is given in Figure 1. Here pressure is evaluated at the square points (i, j) , the u velocity at the triangle points $(i \pm 1/2, j)$, and the v velocity at the circle points $(i, j \pm 1/2)$. The discrete divergence is computed at the square points:

$$(\nabla \cdot \mathbf{u})_{i,j} = \frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta y}.$$

Other differential operators are discretized as:

$$(\Delta u)_{i+1/2,j} = \frac{u_{i+3/2,j} - 2u_{i+1/2,j} + u_{i-1/2,j}}{\Delta x^2} + \frac{u_{i+1/2,j+1} - 2u_{i+1/2,j} + u_{i+1/2,j-1}}{\Delta y^2},$$

$$\begin{aligned}
(\Delta v)_{i,j+1/2} &= \frac{v_{i+1,j+1/2} - 2u_{i,j+1/2} + u_{i-1,j+1/2}}{\Delta x^2} + \frac{v_{i,j+3/2} - 2v_{i,j+1/2} + v_{i,j-1/2}}{\Delta y^2}, \\
(p_x)_{i+1/2,j} &= \frac{p_{i+1,j} - p_{i,j}}{\Delta x}, \\
(p_y)_{i,j+1/2} &= \frac{p_{i,j+1} - p_{i,j}}{\Delta y}, \\
\bar{u}_{i,j+1/2} &= \frac{1}{4}(u_{i+1/2,j} + u_{i-1/2,j} + u_{i+1/2,j+1} + u_{i-1/2,j+1}), \\
\bar{v}_{i+1/2,j} &= \frac{1}{4}(v_{i+1,j+1/2} + v_{i+1,j-1/2} + v_{i,j+1/2} + v_{i,j-1/2}). \\
\mathcal{N}_h(\mathbf{u}, a)_{i+1/2,j} &= u_{i+1/2,j} \frac{a_{i+3/2,j} - a_{i-1/2,j}}{2\Delta x} + \bar{v}_{i+1/2,j} \frac{a_{i+1/2,j+1} - a_{i+1/2,j-1}}{2\Delta y} \\
\mathcal{N}_h(\mathbf{u}, b)_{i,j+1/2} &= \bar{u}_{i,j+1/2} \frac{b_{i+1,j+1/2} - b_{i-1,j+1/2}}{2\Delta x} + v_{i,j+1/2} \frac{b_{i,j+3/2} - b_{i,j-1/2}}{2\Delta y}
\end{aligned}$$

Clearly the truncation errors of these approximations are of second order.

The boundary condition $u = 0$ is imposed at the vertical physical boundary, whereas $v = 0$ is imposed at the “ghost” circle points which are $\frac{\Delta x}{2}$ to the left or right of the physical boundary. Similarly the boundary condition $v = 0$ is imposed at the horizontal physical boundary, but $u = 0$ is imposed at the “ghost” triangle points with a distance of $\frac{\Delta y}{2}$ away from the physical boundary.

One shortcoming of the MAC scheme is the serious constraint on geometry. Although slightly more general situations can be studied, in the present paper we will on the situation when $\Omega = [-1, 1] \times [0, 2\pi]$ with periodic boundary condition in the y direction and no-slip boundary condition in the x -direction: $\mathbf{u}(x, 0, t) = \mathbf{u}(x, 2\pi, t)$, $\mathbf{u}(-1, y, t) = 0$, $\mathbf{u}(1, y, t) = 0$. We will use $\partial' \Omega$ to denote the part of the boundary at $x = \pm 1$. We will always assume that $\Delta x \sim \Delta y$ and $h = \min(\Delta x, \Delta y)$.

Notations: We will use C to denote generic constants which may depend on the norms of the exact solutions. Norms will be taken over the entire domain Ω .

§3. Summary of Results and Outline of Proofs

The main results of this paper are the following (the constants are independent of Δt and h):

Theorem 1. *Let (\mathbf{u}, p) be a solution of the Navier-Stokes equation (2.1) with smooth initial data $\mathbf{u}^0(\mathbf{x})$ satisfying the compatibility condition*

$$(3.13) \quad \mathbf{u}^0(\mathbf{x}) = 0, \quad \partial_y p(\mathbf{x}, 0) = \partial_{xy}^2 p(\mathbf{x}, 0) = 0, \quad \text{on } \partial' \Omega,$$

Let (\mathbf{u}_h, p_h) be the numerical solution of the projection method (2.6), (2.7) and (2.10) coupled with the MAC spatial discretization. Assume that $\Delta t \ll h$. Then we have

$$(3.14) \quad \|\mathbf{u} - \mathbf{u}_h\|_{L^\infty} + \Delta t^{1/2} \|p - p_h\|_{L^\infty} \leq C(\Delta t + h^2),$$

$$(3.15) \quad \|p - p_h - p_c\|_{L^\infty} \leq C(\Delta t + h^2),$$

where

$$(3.16) \quad \begin{aligned} p_c(\mathbf{x}, t) &\equiv \Delta t^{1/2} \beta \frac{e^\alpha}{e^\alpha - 1} e^{-\alpha|x-1|/\Delta t^{1/2}} D_+^x p_h(x - \Delta t^{1/2}, y, t) \\ &\quad + \Delta t^{1/2} \beta \frac{e^\alpha}{e^\alpha - 1} e^{-\alpha|x+1|/\Delta t^{1/2}} D_+^x p_h(x + \Delta t^{1/2}, y, t), \\ \alpha &= \frac{\Delta t^{1/2}}{\Delta x} \operatorname{arccosh} \left(1 + \frac{\Delta x^2}{2\Delta t} \right), \quad \beta = \frac{\Delta x}{\Delta t^{1/2}} (1 - e^{-\alpha \Delta x / \Delta t^{1/2}})^{-1}. \end{aligned}$$

Theorem 2. *Let (\mathbf{u}, p) be a smooth solution of the Navier-Stokes equation (2.1) with smooth initial data $\mathbf{u}^0(\mathbf{x})$ satisfying the compatibility condition*

$$(3.17) \quad \partial_x^{\alpha_1} \partial_y^{\alpha_2} \mathbf{u}^0(\mathbf{x}) = 0, \quad \text{on } \partial \Omega, \text{ for } \alpha_1 + \alpha_2 \leq 6,$$

Let (\mathbf{u}_h, p_h) be the numerical solution of the projection method (2.12) coupled with the MAC spatial discretization. Assume that $\Delta t^2 \ll h$. Then we have

$$(3.18) \quad \|\mathbf{u} - \mathbf{u}_h\|_{L^\infty} + \Delta t^{3/2} \|p - p_h\|_{L^\infty} + \Delta t \|p - p_h\|_{L^\infty(0,T,L^2)} \leq C(\Delta t^2 + h^2).$$

$$(3.19) \quad \|p - p_h - p_c\|_{L^\infty} \leq C(\Delta t + h^2),$$

where

$$(3.20) \quad \begin{aligned} p_c &\equiv \Delta t^{1/2} \beta \frac{e^\alpha}{e^\alpha - 1} e^{-\alpha|x-1|/\Delta t^{1/2}} D_+^x p_h(x - \Delta t^{1/2}, y, t) \\ &\quad + \Delta t^{1/2} \beta \frac{e^\alpha}{e^\alpha - 1} e^{-\alpha|x+1|/\Delta t^{1/2}} D_+^x p_h(x + \Delta t^{1/2}, y, t), \end{aligned}$$

$$\alpha = \frac{\Delta t^{1/2}}{\Delta x} \operatorname{arccosh} \left(1 + \frac{\Delta x^2}{\Delta t} \right), \quad \beta = \frac{\Delta x}{\Delta t^{1/2}} (1 - e^{-\alpha \Delta x / \Delta t^{1/2}})^{-1}.$$

Remark. The constraints on the size of Δt is only technical and is different from the standard stability condition.

The proof of these results follow the general strategy outlined in Section 3 of [5]. Step 3 is made simpler since we can now use the inverse inequality.

§4. First Order Schemes with Spatial Discretization

We will concentrate on the following version of the first order projection method with the standard MAC spatial discretization:

$$(4.1) \quad \left\{ \begin{array}{l} \frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} + \mathcal{N}_h(\mathbf{u}^n, \mathbf{u}^n) = \Delta_h \mathbf{u}^*, \\ \mathbf{u}^* = 0, \quad \text{on } \partial' \Omega, \\ \mathbf{u}^* = \mathbf{u}^{n+1} + \Delta t \nabla_h p^n, \\ \nabla_h \cdot \mathbf{u}^{n+1} = 0, \\ \mathbf{n} \cdot \mathbf{u}^{n+1} = 0, \quad \text{on } \partial' \Omega. \end{array} \right.$$

For $\mathbf{a} = (a, b)$, $\mathbf{c} = (c, d)$, $\mathbf{u} = (u, v)$, we define the following discrete inner products on the grid:

$$(4.2) \quad \begin{aligned} ((\mathbf{a}, \mathbf{c})) &= \Delta x \Delta y \sum_{i=1}^{N-1} \sum_{j=1}^N a_{i+1/2, j} c_{i+1/2, j} + \Delta x \Delta y \sum_{i=1}^N \sum_{j=1}^N b_{i, j+1/2} d_{i, j+1/2} \\ ((\mathbf{u}, \nabla_h p)) &= \Delta y \sum_{i=1}^{N-1} \sum_{j=1}^N u_{i+1/2, j} (p_{i+1, j} - p_{i, j}) + \Delta x \sum_{i=1}^N \sum_{j=1}^N v_{i, j+1/2} (p_{i, j+1} - p_{i, j}) \\ ((\nabla_h \cdot \mathbf{u}, p)) &= \Delta y \sum_{i=1}^{N-1} \sum_{j=1}^N (u_{i+1/2, j} - u_{i-1/2, j}) p_{i, j} + \Delta x \sum_{i=1}^N \sum_{j=1}^N (v_{i, j+1/2} - v_{i, j-1/2}) p_{i, j} \end{aligned}$$

and discrete norms

$$(4.3) \quad \|\mathbf{u}\| = ((\mathbf{u}, \mathbf{u}))^{1/2}, \quad \|\mathbf{u}\|_\infty = \max_{i, j} |\mathbf{u}_{i, j}|$$

Denote $h = \min(\Delta x, \Delta y)$.

Lemma 4.1 *We have the following*

(i) *Inverse inequality:*

$$(4.4) \quad \|f\|_\infty \leq \frac{1}{h} \|f\|,$$

(ii) *Poincare inequality: suppose $f|_{x=\pm 1} = 0$, then*

$$(4.5) \quad \|f\| \leq \|\nabla_h f\|,$$

(iii) *Suppose $\mathbf{n} \cdot \mathbf{u}|_{x=\pm 1} = 0$, then we have*

$$(4.6) \quad ((\mathbf{u}, \nabla_h p)) = ((\nabla_h \cdot \mathbf{u}, p))$$

(iv) *Suppose $\mathbf{u}|_{x=\pm 1} = 0$, then we have*

$$(4.7) \quad 2((\mathbf{u}, \Delta_h \mathbf{u})) \leq -\|\nabla_h \mathbf{u}\|^2 - \|\nabla_h \cdot \mathbf{u}\|^2$$

(v) *Suppose $\mathbf{a}|_{x=\pm 1} = 0$ and $\mathbf{c} \cdot \mathbf{n}|_{x=\pm 1} = 0$, then we have*

$$(4.8) \quad |((\mathbf{a}, \mathcal{N}_h(\mathbf{u}, \mathbf{c}))| \leq \|\mathbf{c}\| \|\nabla_h \mathbf{a}\| \|\mathbf{u}\|_{W^{1,\infty}}$$

Proof : The proof of (i-iii) is standard. We first show (iv). Summation by parts gives

$$(4.9) \quad ((\mathbf{u}, \Delta_h \mathbf{u})) = -\|\nabla \mathbf{u}\|^2 + \sum_j [v_{0,j+1/2}(v_{1,j+1/2} - v_{0,j+1/2}) - v_{N,j+1/2}(v_{N+1,j+1/2} - v_{N,j+1/2})]$$

Since $v|_{x=\pm 1} = 0$, we have

$$(4.10) \quad v_{1,j+1/2} = -v_{0,j+1/2}, \quad v_{N,j+1/2} = -v_{N+1,j+1/2}$$

Hence

$$(4.11) \quad ((\mathbf{u}, \Delta_h \mathbf{u})) = -\|\nabla_h \mathbf{u}\|^2 + 2 \sum_j (v_{0,j+1/2}^2 - v_{N,j+1/2}^2)$$

But

$$(4.12) \quad \begin{aligned} \|\nabla_h \mathbf{u}\|^2 &\geq \|\nabla_h \cdot \mathbf{u}\|^2 + \sum_j [(v_{1,j+1/2} - v_{0,j+1/2})^2 + (v_{N+1,j+1/2} - v_{N,j+1/2})^2] \\ &= \|\nabla_h \cdot \mathbf{u}\|^2 + 4 \sum_j [(v_{0,j+1/2})^2 + (v_{N+1,j+1/2})^2] \end{aligned}$$

Combination of (4.11) and (4.12) gives (4.7).

To show (v), denote $I = ((\mathbf{a}, \mathcal{N}_h(\mathbf{u}, \mathbf{c}))$). We have

$$(4.13) \quad \begin{aligned} I &= \Delta x \Delta y \sum_{i,j} a_{i+1/2,j} (u_{i+1/2,j} D_0^x c_{i+1/2,j} + \bar{v}_{i+1/2,j} D_0^y c_{i+1/2,j}) \\ &\quad + \Delta x \Delta y \sum_{i,j} b_{i,j+1/2} (\bar{u}_{i,j+1/2} D_0^x d_{i,j+1/2} + v_{i,j+1/2} D_0^y d_{i,j+1/2}) \end{aligned}$$

Summation by parts gives

$$(4.14) \quad \begin{aligned} I &= -\Delta x \Delta y \sum_{i,j} c_{i+1/2,j} [D_0^x (u_{i+1/2,j} a_{i+1/2,j}) + D_0^y (\bar{v}_{i+1/2,j} a_{i+1/2,j})] \\ &\quad - \Delta x \Delta y \sum_{i,j} d_{i,j+1/2} [D_0^x (\bar{u}_{i,j+1/2} b_{i,j+1/2}) + D_0^y (v_{i,j+1/2} b_{i,j+1/2})] \\ &\quad + \frac{1}{4} \Delta x \Delta y \sum_j (\bar{u}_{N+1,j+1/2} d_{N,j+1/2} - \bar{u}_{N,j+1/2} d_{N+1,j+1/2}) D_-^x b_{N+1,j+1/2} \\ &\quad - \frac{1}{4} \Delta x \Delta y \sum_j (\bar{u}_{1,j+1/2} d_{0,j+1/2} - \bar{u}_{0,j+1/2} d_{1,j+1/2}) D_+^x b_{0,j+1/2} \end{aligned}$$

Here we have used the fact that

$$(4.15) \quad b_{1,j+1/2} = -b_{0,j+1/2}, \quad b_{N,j+1/2} = -b_{N+1,j+1/2}$$

Now, (4.8) follows directly. This completes the proof of the lemma.

Again we set $\varepsilon = \Delta t^{1/2}$, $\xi = (x+1)/\varepsilon$, $x_i = -1 + i\Delta x$, $\xi_i = i\Delta\xi$, $\Delta\xi = \Delta x/\varepsilon$, $t^n = n\Delta t$, $t^{n-1/2} = (n-1/2)\Delta t$, $n = 1, 2, \dots$. Clearly, we have

$$(4.16) \quad \begin{aligned} D_+^\xi \mathbf{a}(\xi_i, y_j, t) &= \frac{\mathbf{a}(\xi_{i+1}, y_j, t) - \mathbf{a}(\xi_i, y_j, t)}{\Delta\xi} \\ &= \varepsilon \frac{\mathbf{a}(x_{i+1}/\varepsilon, y_j, t) - \mathbf{a}(x_i/\varepsilon, y_j, t)}{\Delta x} = \varepsilon D_+^x \mathbf{a}(x_i/\varepsilon, y_j, t) \end{aligned}$$

This shows that $D_+^\xi = \varepsilon D_+^x$. We will use the notation

$$(4.17) \quad D_\xi^2 = D_-^\xi D_+^\xi, \quad D_y^2 = D_-^y D_+^y,$$

and

$$(4.18) \quad \nabla_\xi = (D_+^\xi, 0), \quad \nabla_y = (0, D_+^y),$$

Denote the solutions of (4.1) as $(\mathbf{u}_h, \mathbf{u}_h^*, p_h)$. Motivated by the discussions in §4, we make the following ansatz, valid at $t^n = n\Delta t$, $n = 1, 2, \dots$

$$(4.19) \quad \begin{cases} \mathbf{u}_h^*(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{x}, t) + \sum_{j=2} \varepsilon^j [\mathbf{u}_j^*(\mathbf{x}, t) + \mathbf{a}_j^*(\xi, y, t)], \\ \mathbf{u}_h(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{x}, t) + \sum_{j=2} \varepsilon^j \mathbf{u}_j(\mathbf{x}, t), \\ p_h(\mathbf{x}, t) = p_0(\mathbf{x}, t) + \sum_{j=1} \varepsilon^j [p_j(\mathbf{x}, t) + \varphi_j(\xi, y, t)]. \end{cases}$$

Note that the functions involved are defined only on the numerical grid. So these formulas and the following ones should be understood as being valid on the grid. We have

$$(4.20) \quad \Delta_h \mathbf{u}_h^* = \Delta_h \mathbf{u}_0 + \sum_{j=2} \varepsilon^j (\Delta_h \mathbf{u}_j^* + \varepsilon^{-2} D_\xi^2 \mathbf{a}_j^* + D_y^2 \mathbf{a}_j^*),$$

$$(4.21) \quad \nabla_h \cdot \mathbf{u}_h = \nabla_h \cdot \mathbf{u}_0 + \sum_{j=2} \varepsilon^j \nabla_h \cdot \mathbf{u}_j,$$

$$(4.22) \quad \nabla_h p_h = \nabla_h p_0 + \varepsilon^{-1} \nabla_\xi \varphi_0 + \nabla_y \varphi_0 + \sum_{j=1} \varepsilon^j (\nabla_h p_j + \varepsilon^{-1} \nabla_\xi \varphi_j + \nabla_y \varphi_j),$$

$$(4.23) \quad \begin{aligned} \mathbf{u}_h^{n+1}(\mathbf{x}) &= \mathbf{u}_0(\mathbf{x}, t^{n+1}) + \sum_{j=2} \varepsilon^j \mathbf{u}_j(\mathbf{x}, t^{n+1}) \\ &= \sum_{k=0} \frac{1}{k!} \varepsilon^{2k} \mathbf{u}_0^{(k)}(\mathbf{x}, t^n) + \sum_{j=2} \varepsilon^j \sum_{k=0} \frac{1}{k!} \varepsilon^{2k} \mathbf{u}_j^{(k)}(\mathbf{x}, t^n). \end{aligned}$$

Next we substitute these relations into (4.1) in order to determine the coefficients of ε^j in (4.19). We get hierarchies of equations by collecting equal powers of ε .

The first equation in (4.1) gives:

$$(4.24) \quad \mathbf{u}_2^* + \mathbf{a}_2^* - \mathbf{u}_2 + \mathcal{N}_h(\mathbf{u}_0, \mathbf{u}_0) = \Delta_h^2 \mathbf{u}_0^* + D_\xi^2 \mathbf{a}_2^*.$$

For $j \geq 1$,

$$(4.25) \quad \mathbf{u}_{j+2}^* + \mathbf{a}_{j+2}^* - \mathbf{u}_{j+2} + \sum_{k=0}^j \mathcal{N}_h(\mathbf{u}_k, \mathbf{u}_{j-k}) = \Delta_h \mathbf{u}_j^* + D_\xi^2 \mathbf{a}_{j+2}^* + D_y^2 \mathbf{a}_j^*.$$

The second equation in (4.1) implies

$$(4.26) \quad \mathbf{u}_2^* + \mathbf{a}_2^* = \mathbf{u}_2 + \partial_t \mathbf{u}_0 + \nabla_h p_0 + \nabla_\xi \varphi_1 + \nabla_y \varphi_0.$$

For $j = 2\ell - 1$, $\ell \geq 1$,

$$(4.27) \quad \mathbf{u}_{j+2}^* + \mathbf{a}_{j+2}^* = \mathbf{u}_{j+2} + \partial_t \mathbf{u}_j + \nabla_h p_j + \nabla_\xi \varphi_{j+1} + \nabla_y \varphi_j + \sum_{k=2}^{\ell} \frac{1}{k!} \mathbf{u}_{j-2k+2}^{(k)}.$$

For $j = 2\ell$, $\ell \geq 1$,

$$(4.28) \quad \mathbf{u}_{j+2}^* + \mathbf{a}_{j+2}^* = \mathbf{u}_{j+2} + \partial_t \mathbf{u}_j + \nabla_h p_j + \nabla_\xi \varphi_{j+1} + \nabla_y \varphi_j + \frac{1}{(\ell+1)!} \mathbf{u}_0^{(\ell+1)} + \sum_{k=2}^{\ell} \frac{1}{k!} \mathbf{u}_{j-2k+2}^{(k)}.$$

From the third equation in (4.1), we obtain

$$(4.29) \quad \nabla_h \cdot \mathbf{u}_j = 0, \quad j = 0, 1, \dots.$$

The boundary conditions become

$$(4.30) \quad \mathbf{u}_j^* + \mathbf{a}_j^* = 0, \quad D_+^x p_{j-1} + D_+^\xi \varphi_j = 0, \quad \text{at } x = -1, \quad \xi = 0,$$

for $j > 0$.

Next we go through all these equations, order by order, to see if they are solvable. Since this is very similar to what we did in §4.1, we will only give a summary of results.

The coefficients in the expansions (4.19) can be obtained successively in the following order:

$$(4.31) \quad \begin{cases} \partial_t \mathbf{u}_0 + \nabla_h p_0 + \mathcal{N}_h(\mathbf{u}_0, \mathbf{u}_0) = \Delta_h \mathbf{u}_0, \\ \nabla_h \cdot \mathbf{u}_0 = 0 \\ \mathbf{u}_0 = 0, \quad \text{at } x = \pm 1, \\ \mathbf{u}_0(\cdot, 0) = \mathbf{u}^0(\cdot) \end{cases}$$

Using the following lemma, we know that (4.31) has a smooth solution in the sense that the divided difference of various orders are bounded. The lemma itself, as well as Lemma 4.3, belongs to the folklore of classical numerical analysis.

Lemma 4.2. *Let (\mathbf{u}, p) be a solution of the Navier-Stokes equation (2.1) with smooth initial data $\mathbf{u}^0(x)$ satisfying some compatibility conditions. Let (\mathbf{u}_0, p_0) be a solution of (4.31).*

Then (\mathbf{u}_0, p_0) is smooth in the sense that its discrete derivatives are bounded. Moreover, we have

$$(4.32) \quad \|\mathbf{u} - \mathbf{u}_0\|_{L^\infty} + \|p - p_0\|_{L^\infty} \leq Ch^2$$

We next have

$$(4.33) \quad \mathbf{u}_2^* = \mathbf{u}_2 + \partial_t \mathbf{u}_0 + \nabla_h p_0,$$

$$(4.34) \quad \begin{cases} \varphi_1 = D_\xi^2 \varphi_1, \\ D_+^\xi \varphi_1|_{\xi=0} = -D_+^x p_0|_{x=-1}, \end{cases}$$

This gives

$$(4.35) \quad \varphi_1(\xi, y, t) = \beta D_+^x p_0(-1, y, t) e^{-\alpha \xi}.$$

where

$$(4.36) \quad \alpha = \frac{1}{\Delta \xi} \operatorname{arccosh}(1 + \Delta \xi^2/2), \quad \beta = \Delta \xi (1 - e^{-\alpha \Delta \xi})^{-1}.$$

$$(4.37) \quad a_2^* = D_+^\xi \varphi_1, \quad b_2^* = 0,$$

$$(4.38) \quad \mathbf{u}_3^* = \mathbf{u}_3,$$

$$(4.39) \quad \varphi_2 = 0, \quad a_3^* = 0, \quad b_3^* = D_+^y \varphi_1,$$

$$(4.40) \quad \begin{cases} \partial_t \mathbf{u}_2 + \nabla_h p_2 + \mathcal{N}_h(\mathbf{u}_0, \mathbf{u}_2) + \mathcal{N}_h(\mathbf{u}_2, \mathbf{u}_0) \\ \quad = \Delta_h \mathbf{u}_2 + \Delta_h(\partial_t \mathbf{u}_0 + \nabla_h p_0) - \frac{1}{2} \partial_t^2 \mathbf{u}_0, \\ \nabla_h \cdot \mathbf{u}_2 = 0, \\ \mathbf{u}_2|_{x=-1} = -\nabla_h p_0|_{x=-1} - \nabla_\xi \varphi_1|_{\xi=0}, \quad \text{on } \partial\Omega \end{cases}$$

With a suitable initial data, we know from the following lemma that (4.40) has a smooth solution.

Lemma 4.3. *Let (\mathbf{u}, p) be a solution of the linear ODE*

$$(4.41) \quad \begin{cases} \partial_t \mathbf{u} + \nabla_h p + \mathcal{N}_h(\mathbf{u}_0, \mathbf{u}) + \mathcal{N}_h(\mathbf{u}, \mathbf{u}_0) = \Delta_h \mathbf{u} + \mathbf{f}, \\ \nabla_h \cdot \mathbf{u} = 0, \\ \mathbf{u} = \mathbf{g}, \quad \text{at } x = \pm 1, \\ \mathbf{u}(\cdot, 0) = \mathbf{u}^0(\cdot) \end{cases}$$

where \mathbf{f} , \mathbf{g} and \mathbf{u}^0 smooth and satisfies some compatibility conditions. Then (\mathbf{u}, p) is smooth in the sense that its divided differences of various order are bounded.

Continue in this fashion, we get

$$(4.42) \quad \begin{cases} \varphi_3 = D_\xi^2 \varphi_3 + D_y^2 \varphi_1, \\ D_+^\xi \varphi_3 |_{\xi=0} = -D_+^x p_2 |_{x=-1}, \end{cases}$$

The solution for (4.42) is

$$(4.43) \quad \varphi_3(\xi, y, t) = \beta D_+^x p_2 e^{-\alpha \xi} + \beta_1 (\xi + \gamma) D_+^x D_y^2 p_0 |_{x=-1} e^{-\alpha \xi}.$$

where

$$(4.44) \quad \beta_1 = \frac{1}{(1 - e^{-\alpha \Delta \xi})(e^{-\alpha \Delta \xi} - e^{\alpha \Delta \xi})}, \quad \gamma = \Delta \xi \frac{e^{-\alpha \Delta \xi}}{1 - e^{-\alpha \Delta \xi}}$$

$$(4.45) \quad b_4^* = 0, \quad a_4^* = D_+^\xi \varphi_3.$$

$$(4.46) \quad \begin{cases} \partial_t \mathbf{u}_3 + \nabla_h p_3 + \mathcal{N}_h(\mathbf{u}_0, \mathbf{u}_3) + \mathcal{N}_h(\mathbf{u}_3, \mathbf{u}_0) = \Delta_h \mathbf{u}_3 + \Delta_h \nabla_h p_2, \\ \nabla_h \cdot \mathbf{u}_3 = 0, \\ \mathbf{u}_3 |_{x=-1} = -\nabla_y \varphi_1 |_{\xi=0} \end{cases}$$

$$(4.47) \quad \begin{cases} \varphi_4 = D_\xi^2 \varphi_4, \\ D_+^\xi \varphi_4 |_{\xi=0} = -D_+^x p_3 |_{x=-1}, \end{cases}$$

$$(4.48) \quad a_5^* = D_+^\xi \varphi_4, \quad b_5^* = D_+^y \varphi_3.$$

Obviously this procedure can be continued and we obtain

$$(4.49) \quad \begin{cases} \varphi_j = D_\xi^2 \varphi_j + D_y^2 \varphi_{j-2}, \\ D_+^\xi \varphi_j |_{\xi=0} = -D_+^x p_{j-1} |_{x=-1}, \end{cases}$$

$$(4.50) \quad \varphi_j = \sum_{k=0}^{[j/2]} F_{j,k}(y) \xi^k e^{-\alpha \xi},$$

$$(4.51) \quad a_j^* = D_+^\xi \varphi_{j-1}, \quad b_j^* = D_+^y \varphi_{j-2},$$

Now if we let

$$(4.52) \quad \begin{cases} U^* = \mathbf{u}_0^* + \sum_{j=1}^{2N} \varepsilon^j (\mathbf{u}_j^* + \mathbf{a}_j^*), \\ U^n = \mathbf{u}_0 + \sum_{j=1}^{2N} \varepsilon^j \mathbf{u}_j, \\ P^n = p_0 + \sum_{j=1}^{2N} \varepsilon^j (p_j + \varphi_j) + \varepsilon^{2N+1} \varphi_{2N+1}, \end{cases}$$

then we have

$$(4.53) \quad \begin{cases} \frac{U^* - U^n}{\Delta t} + \mathcal{N}_h(U^n, U^n) = \Delta_h U^* + \Delta t^\alpha \mathbf{f}, \\ U^* = 0, \quad \text{at } x = \pm 1, \\ U^* = U^{n+1} + \Delta t \nabla_h P^n + \Delta t^{\alpha+1} \mathbf{g}, \\ \nabla_h \cdot U^{n+1} = 0, \\ D_+^x P^n = \mathbf{n} \cdot U^{n+1} = 0, \quad \text{at } x = \pm 1, \end{cases}$$

where $\alpha = N - 1/2$, f and \mathbf{g} are bounded and smooth if (\mathbf{u}_0, p_0) is sufficiently smooth. It is easy to see that

$$(4.54) \quad \max_{0 \leq t \leq T} \|U^n(\cdot)\|_{W^{1,\infty}} \leq C^*.$$

For the initial data, we have

$$(4.55) \quad U^0(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \Delta t \mathbf{w}^0(\mathbf{x})$$

where \mathbf{w}^0 is a bounded function. Furthermore under the compatibility condition (3.13), we can construct a better approximate initial data

$$(4.57) \quad U^0(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \Delta t^2 \mathbf{w}^0(\mathbf{x}).$$

Proof of Theorem 1: Assume a priori that

$$(4.58) \quad \max_{0 \leq t^n \leq T} \|\mathbf{u}^n\|_{W^{1,\infty}} \leq \tilde{C}.$$

In the following estimates, the constant will sometimes depend on C^* and \tilde{C} . Later on we will estimate \tilde{C} . Let

$$(4.59) \quad \mathbf{e}^n = U^n - \mathbf{u}^n, \quad \mathbf{e}^* = U^* - \mathbf{u}^*, \quad q^n = P^n - p^n.$$

Subtracting (4.53) from (4.1) we get the following error equation

$$(4.60) \quad \left\{ \begin{array}{l} \frac{\mathbf{e}^* - \mathbf{e}^n}{\Delta t} + \mathcal{N}_h(\mathbf{e}^n, U^n) + \mathcal{N}_h(\mathbf{u}^n, \mathbf{e}^n) = \Delta_h \mathbf{e}^* + \Delta t^\alpha \mathbf{f}^n, \\ \mathbf{e}^* = 0, \quad \text{at } x = \pm 1, \\ \frac{\mathbf{e}^{n+1} - \mathbf{e}^*}{\Delta t} + \nabla_h q^n = \Delta t^\alpha \mathbf{g}^n, \\ \nabla_h \cdot \mathbf{e}^{n+1} = 0, \\ D_+^x q^n = \mathbf{e}^{n+1} \cdot \mathbf{n} = 0, \quad \text{at } x = \pm 1, \\ \mathbf{e}^0 = \Delta t^\alpha \mathbf{w}^0. \end{array} \right.$$

Taking the scalar product of the first equation of (4.60) with $2\mathbf{e}^*$ and integrating by parts, we obtain

$$(4.61) \quad \begin{aligned} & \|\mathbf{e}^*\|^2 - \|\mathbf{e}^n\|^2 + \|\mathbf{e}^* - \mathbf{e}^n\|^2 + \Delta t \|\nabla_h \mathbf{e}^*\|^2 \\ & \leq \Delta t^{2\alpha+1} \|\mathbf{f}^n\|^2 + \Delta t \|\mathbf{e}^*\|^2 - 2\Delta t ((\mathbf{e}^*, \mathcal{N}_h(\mathbf{e}^n, U^n))) \\ & \quad - 2\Delta t ((\mathbf{e}^*, \mathcal{N}_h(\mathbf{u}^n, \mathbf{e}^n))) \\ & \leq \Delta t^{2\alpha+1} \|\mathbf{f}^n\|^2 + C\Delta t (\|\mathbf{e}^*\|^2 + \|\mathbf{e}^n\|^2) + \frac{1}{2}\Delta t \|\nabla_h \mathbf{e}^*\|^2. \end{aligned}$$

Here we have used Lemma 4.1. Taking the scalar product of the second equation of (4.60) with $2\mathbf{e}^{n+1}$ yields

$$(4.62) \quad \|\mathbf{e}^{n+1}\|^2 - \|\mathbf{e}^*\|^2 + \|\mathbf{e}^{n+1} - \mathbf{e}^*\|^2 \leq \Delta t \|\mathbf{e}^{n+1}\|^2 + \Delta t^{2\alpha+1} \|\mathbf{g}^n\|^2.$$

Combining (4.61) and (4.62), we get

$$(4.63) \quad \begin{aligned} & \|e^{n+1}\|^2 - \|e^n\|^2 + \|e^* - e^n\|^2 + \|e^{n+1} - e^*\|^2 + \Delta t \|\nabla_h e^*\|^2 \\ & \leq C \Delta t (\|e^{n+1}\|^2 + \|e^n\|^2) + \Delta t^{2\alpha+1} (\|f^n\|^2 + \|g^n\|^2). \end{aligned}$$

Applying the discrete Gronwall lemma to the last inequality, we arrive at

$$(4.64) \quad \|e^n\| + \|e^* - e^n\| + \|e^{n+1} - e^*\| + \Delta t^{1/2} \|\nabla_h e^*\| \leq C_1 \Delta t^\alpha.$$

Using the second equation of (4.60) we have

$$(4.65) \quad \|e^n\| + \Delta t \|\nabla_h q^n\| \leq C_1 \Delta t^\alpha$$

Now by inverse inequality (4.4) we have

$$(4.66) \quad \|e^n\|_{L^\infty} + \Delta t \|\nabla_h q^n\|_{L^\infty} + h \|e^n\|_{W^{1,\infty}} \leq C_1 \frac{\Delta t^\alpha}{h}.$$

Chose $N = 3$ and $\Delta t^\alpha \ll h^2$, if we choose Δt small enough, we will always have

$$(4.67) \quad \|e^{n+1}\|_{W^{1,\infty}} \leq 1.$$

Therefore in (4.58) we can choose

$$(4.68) \quad \tilde{C} = 1 + \max_{n \leq [\frac{T}{\Delta t}] + 1} \|U^n(\cdot)\|_{W^{1,\infty}}$$

which depends only on the exact solution (\mathbf{u}, p) . This proves

$$(4.69) \quad \|\mathbf{u}_0 - \mathbf{u}_h\|_{L^\infty} + \|p_0 - p_h\|_{L^2} + \Delta t^{1/2} \|p_0 - p_h\|_{L^\infty} + \|p_0 - p_h - p_c\|_{L^\infty} \leq C \Delta t$$

But we also have from Lemma 4.2

$$(4.70) \quad \|\mathbf{u} - \mathbf{u}_0\|_{L^\infty} + \|p - p_0\|_{L^\infty} \leq Ch^2$$

Thus

$$(4.71) \quad \|\mathbf{u} - \mathbf{u}_h\|_{L^\infty} + \|p - p_h\|_{L^2} + \Delta t^{1/2} \|p - p_h\|_{L^\infty} + \|p - p_h - p_c\|_{L^\infty} \leq C(\Delta t + h^2)$$

This completes the proof of Theorem 1.

§5. Second Order Schemes with Spatial Discretization

In this section we carry out the same program as in §4 for the second order method (2.12) with the standard MAC spatial discretization:

$$(5.1) \quad \left\{ \begin{array}{l} \frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \Delta_h \frac{\mathbf{u}^* + \mathbf{u}^n}{2}, \\ \mathbf{u}^* + \mathbf{u}^n = \Delta t \nabla_h p^{n-1/2}, \quad \text{at } x = \pm 1, \\ \mathbf{u}^* = \mathbf{u}^{n+1} + \Delta t \nabla_h p^{n+1/2}, \\ \nabla_h \cdot \mathbf{u}^{n+1} = 0, \\ \mathbf{n} \cdot \mathbf{u}^{n+1} = 0, \quad \text{at } x = \pm 1. \end{array} \right.$$

Here we leave out the nonlinear term since it does not affect the major steps but complicates substantially the presentation.

We begin with the following ansatz:

$$(5.2) \quad \left\{ \begin{array}{l} \mathbf{u}^*(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}, t^n) + \sum_{j=2} \varepsilon^j [\mathbf{u}_j^*(\mathbf{x}, t^n) + \mathbf{a}_j^*(\xi, y, t^n)], \\ \mathbf{u}^n(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}, t^n) + \sum_{j=4} \varepsilon^j \mathbf{u}_j(\mathbf{x}, t^n), \\ p^{n-1/2}(\mathbf{x}) = p_0(\mathbf{x}, t^{n-1/2}) + \varepsilon \varphi_1(\xi, y, t^{n-1/2}) + \varepsilon^3 \varphi_3(\xi, y, t^{n-1/2}) \\ \quad + \sum_{j=4} \varepsilon^j [p_j(\mathbf{x}, t^{n-1/2}) + \varphi_j(\xi, y, t^{n-1/2})]. \end{array} \right.$$

Here again we set $\varepsilon = \Delta t^{1/2}$, $\xi = (x+1)/\varepsilon$, $t^n = n\Delta t$, $t^{n-1/2} = (n-1/2)\Delta t$, $n = 1, 2, \dots$.

The formulas are to be understood as being valid at the grid points. Substituting (5.2) into (5.1) and collecting equal powers of ε , we get the following equations:

From the first equation in (5.1), we get

$$(5.3) \quad \mathbf{u}_2^* + \mathbf{a}_2^* - \mathbf{u}_2 = \frac{1}{2}(\Delta_h \mathbf{u}_0^* + D_\xi^2 \mathbf{a}_2^* + \Delta_h \mathbf{u}_0).$$

For $j \geq 1$,

$$(5.4) \quad \mathbf{u}_{j+2}^* + \mathbf{a}_{j+2}^* - \mathbf{u}_{j+2} = \frac{1}{2}(\Delta_h \mathbf{u}_j^* + D_\xi^2 \mathbf{a}_{j+2}^* + D_y^2 \mathbf{a}_j^* + \Delta_h \mathbf{u}_j).$$

From the third equation in (5.1), we get

$$(5.5) \quad \mathbf{u}_2^* + \mathbf{a}_2^* = \mathbf{u}_2 + \partial_t \mathbf{u}_0 + \nabla_h p_0 + \nabla_\xi \varphi_1,$$

$$(5.6) \quad \mathbf{u}_3^* + \mathbf{a}_3^* = \mathbf{u}_3 + \partial_t \mathbf{u}_1 + \nabla_h p_1 + \nabla_\xi \varphi_2 + \nabla_y \varphi_1.$$

For $j = 2\ell$,

$$(5.7) \quad \begin{aligned} \mathbf{u}_{j+2}^* + \mathbf{a}_{j+2}^* &= \mathbf{u}_{j+2} + \partial_t \mathbf{u}_j + \nabla_h p_j + \nabla_\xi \varphi_{j+1} + \nabla_y \varphi_j \\ &+ \frac{1}{(\ell+1)!} \mathbf{u}_0^{(\ell+1)} + \sum_{k=2}^{\ell} \frac{1}{k!} \mathbf{u}_{j-2k+2}^{(k)} + \frac{1}{2^\ell \ell!} \nabla_h p_0^{(\ell)} + \sum_{k=1}^{\ell-1} \frac{1}{2^k k!} \nabla_h p_{j-2k}^{(k)} \\ &+ \sum_{k=1}^{\ell} \frac{1}{2^k k!} (\nabla_\xi \varphi_{j-2k+1}^{(k)} + \nabla_y \varphi_{j-2k}^{(k)}). \end{aligned}$$

For $j = 2\ell + 1$,

$$(5.8) \quad \begin{aligned} \mathbf{u}_{j+2}^* + \mathbf{a}_{j+2}^* &= \mathbf{u}_{j+2} + \partial_t \mathbf{u}_j + \nabla_h p_j + \nabla_\xi \varphi_{j+1} + \nabla_y \varphi_j \\ &+ \sum_{k=2}^{\ell+1} \frac{1}{k!} \mathbf{u}_{j-2k+2}^{(k)} + \sum_{k=1}^{\ell} \frac{1}{2^k k!} (\nabla_h p_{j-2k}^{(k)} + \nabla_\xi \varphi_{j-2k+1}^{(k)} + \nabla_y \varphi_{j-2k}^{(k)}). \end{aligned}$$

From the incompressibility condition, we get

$$(5.9) \quad \nabla_h \cdot \mathbf{u}_j = 0, \quad \text{for } j \geq 0.$$

The boundary conditions imply that for $x = -1$, $\xi = 0$,

$$(5.10) \quad \mathbf{u}_0 = 0,$$

$$(5.11) \quad \mathbf{u}_2 + \mathbf{u}_2^* + \mathbf{a}_2^* = \nabla_h p_0 + \nabla_\xi \varphi_1,$$

$$(5.12) \quad \mathbf{u}_3 + \mathbf{u}_3^* + \mathbf{a}_3^* = \nabla_h p_1 + \nabla_\xi \varphi_2 + \nabla_y \varphi_1;$$

for $j = 2\ell$, $\ell \geq 1$

$$(5.13) \quad \begin{aligned} \mathbf{u}_j + \mathbf{u}_j^* + \mathbf{a}_j^* &= \nabla_h p_{j-2} + \nabla_\xi \varphi_{j-1} + \nabla_y \varphi_{j-2} + \frac{(-1)^{\ell-1}}{2^{\ell-1}} \frac{1}{(\ell-1)!} \nabla_h p_0^{(\ell-1)} \\ &+ \sum_{k=1}^{\ell-2} \frac{(-1)^k}{2^k k!} \nabla_h p_{j-2k-2}^{(k)} + \sum_{k=1}^{\ell-1} \frac{(-1)^k}{2^k k!} (\nabla_\xi \varphi_{j-2k-1}^{(k)} + \nabla_y \varphi_{j-2k-2}^{(k)}) \end{aligned}$$

for $j = 2\ell + 1$, $\ell \geq 1$

$$(5.14) \quad \begin{aligned} \mathbf{u}_j + \mathbf{u}_j^* + \mathbf{a}_j^* &= \nabla_h p_{j-2} + \nabla_\xi \varphi_{j-1} + \nabla_y \varphi_{j-2} \\ &+ \sum_{k=1}^{\ell-1} \frac{(-1)^k}{2^k k!} (\nabla_h p_{j-2k-2}^{(k)} + \nabla_\xi \varphi_{j-2k-1}^{(k)} + \nabla_y \varphi_{j-2k-2}^{(k)}); \end{aligned}$$

and for $j \geq 0$

$$(5.15) \quad D_+^x p_j + D_+^\xi \varphi_{j+1} = 0.$$

Next we go through all these equations, order by order, to see if they are solvable. It can be checked that the coefficients in the expansions (5.2) can be obtained successively in the following order:

$$(5.16) \quad \begin{cases} \partial_t \mathbf{u}_0 + \nabla_h p_0 = \Delta_h \mathbf{u}_0, \\ \nabla_h \cdot \mathbf{u}_0 = 0, \\ \mathbf{u}_0 = 0, \quad \text{at } x = \pm 1. \end{cases}$$

$$(5.17) \quad \mathbf{u}_2^* = \mathbf{u}_2 + \partial_t \mathbf{u}_0 + \nabla_h p_0,$$

$$(5.18) \quad \begin{cases} \varphi_1 = \frac{1}{2} D_\xi^2 \varphi_1, \\ D_+^\xi \varphi_1 |_{\xi=0} = -D_+^x p_0 |_{x=-1}, \end{cases}$$

$$(5.19) \quad \varphi_1 = \beta D_+^x p_0 |_{x=-1} e^{-\alpha \xi},$$

where

$$(5.20) \quad \alpha = \frac{1}{\Delta \xi} \operatorname{arccosh}(1 + \Delta \xi^2), \quad \beta = \Delta \xi (1 - e^{-\alpha \Delta \xi})^{-1}.$$

$$(5.21) \quad a_2^* = D_+^\xi \varphi_1, \quad b_2^* = 0,$$

We next have:

$$(5.22) \quad \mathbf{u}_3^* = \mathbf{u}_3,$$

$$(5.23) \quad \varphi_2 = 0, \quad a_3^* = 0, \quad b_3^* = D_+^y \varphi_1,$$

$$(5.24) \quad \begin{cases} \varphi_3 = \frac{1}{2}(D_\xi^2 \varphi_3 + D_y^2 \varphi_1), \\ D_+^\xi \varphi_3 |_{\xi=0} = 0, \end{cases}$$

The solution for (5.24) is

$$(5.25) \quad \varphi_3(y, \xi, t) = \beta_1(\xi + \gamma) D_+^x D_y^2 p_0 |_{x=-1} e^{-\alpha \xi}.$$

where

$$(5.26) \quad \beta_1 = \frac{1}{2(1 - e^{-\alpha \Delta \xi})(e^{-\alpha \Delta \xi} - e^{\alpha \Delta \xi})}, \quad \gamma = \Delta \xi \frac{e^{-\alpha \Delta \xi}}{1 - e^{-\alpha \Delta \xi}}$$

$$(5.27) \quad a_4^* = \frac{1}{2} D_+^\xi \partial_t \varphi_1 + D_+^\xi \varphi_3, \quad b_4^* = 0,$$

$$(5.28) \quad \mathbf{u}_4^* = \mathbf{u}_4 + \frac{1}{2} \partial_t^2 \mathbf{u}_0 + \frac{1}{2} \partial_t \nabla_h p_0,$$

$$(5.29) \quad \varphi_4 = 0, \quad a_5^* = 0, \quad b_5^* = \frac{1}{2} D_+^y \partial_t \varphi_1 + D_+^y \varphi_3,$$

$$(5.30) \quad \mathbf{u}_5^* = \mathbf{u}_5,$$

$$(5.31) \quad \begin{cases} \partial_t \mathbf{u}_4 + \nabla_h p_4 = \Delta_h \mathbf{u}_4 + \frac{1}{4} \Delta_h (\partial_t^2 \mathbf{u}_0 + \partial_t \nabla_h p_0) - \frac{1}{6} \partial_t^3 \mathbf{u}_0 - \frac{1}{8} \partial_t^2 \nabla_h p_0, \\ \nabla_h \cdot \mathbf{u}_4 = 0, \\ \mathbf{u}_4 |_{x=-1} = -\frac{1}{2} (\partial_t \nabla_h p_0 + \frac{1}{2} \partial_t \nabla_\xi \varphi_1) |_{x=-1, \xi=0}. \end{cases}$$

Now if we let

$$(5.32) \quad \begin{cases} U^* = \mathbf{u}_0^* + \sum_{j=1}^{2N} \varepsilon^j (\mathbf{u}_j^* + \mathbf{a}_j^*), \\ U^n = \mathbf{u}_0 + \sum_{j=1}^{2N} \varepsilon^j \mathbf{u}_j, \\ P^{n-1/2} = p_0 + \sum_{j=1}^{2N} \varepsilon^j (p_j + \varphi_j) + \varepsilon^{2N+1} \varphi_{2N+1}, \end{cases}$$

then we have

$$(5.33) \quad \left\{ \begin{array}{l} \frac{U^* - U^n}{\Delta t} = \Delta_h \frac{U^* + U^n}{2} + \Delta t^\alpha \mathbf{f}, \\ U^* + U^n = \Delta t \nabla_h P^{n-1/2}, \quad \text{at } x = \pm 1, \\ U^* = U^{n+1} + \Delta t \nabla_h P^{n+1/2} + \Delta t^{\alpha+1} \mathbf{g}, \\ \nabla_h \cdot U^{n+1} = 0, \\ D_+^x P^{n+1/2} = \mathbf{n} \cdot U^{n+1} = 0, \quad \text{at } x = \pm 1, \end{array} \right.$$

where $\alpha = N - 1/2$, \mathbf{f} and \mathbf{g} are bounded and smooth if (\mathbf{u}_0, p_0) is sufficiently smooth. It is easy to see that

$$(5.34) \quad \max_{0 \leq t \leq T} \|U^n(\cdot)\|_{W^{1,\infty}} \leq C^*,$$

For the initial approximation, we have

$$(5.35) \quad U^0(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \Delta t^2 \mathbf{w}^0(\mathbf{x})$$

without the extra compatibility condition, and

$$(5.36) \quad U^0(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \Delta t^4 \mathbf{w}^0(\mathbf{x})$$

with the compatibility condition (3.17).

Proof of Theorem 2. Assume a priori that

$$(5.37) \quad \max_{0 \leq t^n \leq T} \|\mathbf{u}^n\|_{W^{1,\infty}} \leq \tilde{C}.$$

As in the proof of Theorem 1, we let

$$(5.38) \quad \mathbf{e}^n = U^n - \mathbf{u}^n, \quad \mathbf{e}^* = \hat{U}^* - \hat{\mathbf{u}}^*, \quad q^n = P^{n-1/2} - p^{n-1/2}.$$

where

$$(5.39) \quad \begin{aligned} 2\hat{\mathbf{u}}^* &= \mathbf{u}^* + \mathbf{u}^n - \Delta t \nabla_h p^{n-1/2}, \\ 2\hat{U}^* &= U^* + U^n - \Delta t \nabla_h P^{n-1/2}. \end{aligned}$$

From (5.1) and (5.34), we get

$$(5.40) \quad \left\{ \begin{array}{l} \frac{2(\mathbf{e}^* - \mathbf{e}^n)}{\Delta t} + \nabla_h \left(q^n - \frac{1}{2} \Delta t \Delta_h q^n \right) = \Delta_h \mathbf{e}^* + \frac{1}{2} \mathcal{N}_h(\mathbf{e}^{n-1}, U^{n-1}) \\ \quad + \frac{1}{2} \mathcal{N}_h(\mathbf{u}^{n-1}, \mathbf{e}^{n-1}) - \frac{3}{2} \mathcal{N}_h(\mathbf{e}^n, U^n) - \frac{3}{2} \mathcal{N}_h(\mathbf{u}^n, \mathbf{e}^n) + \Delta t^\alpha \mathbf{f}^n, \\ \mathbf{e}^* = 0, \quad \text{at } x = \pm 1, \\ \frac{\mathbf{e}^{n+1} + \mathbf{e}^n - 2\mathbf{e}^*}{\Delta t} + \nabla_h(q^{n+1} - q^n) = \Delta t^\alpha \mathbf{g}^n, \\ \nabla_h \cdot \mathbf{e}^{n+1} = 0, \\ D_+^x q^{n+1/2} = \mathbf{e}^{n+1} \cdot \mathbf{n} = 0, \quad \text{at } x = \pm 1, \\ \mathbf{e}^0 = \Delta t^\alpha \mathbf{w}^0. \end{array} \right.$$

Taking the scalar product of the first equation of (5.40) with \mathbf{e}^* and integrating by parts, we get

$$(5.41) \quad \begin{aligned} & \|\mathbf{e}^*\|^2 - \|\mathbf{e}^n\|^2 + \|\mathbf{e}^* - \mathbf{e}^n\|^2 + \frac{1}{2} \Delta t \|\nabla \mathbf{e}^*\|^2 + \frac{1}{2} \Delta t \|\nabla \cdot \mathbf{e}^*\|^2 \\ & \leq -\Delta t \int_{\Omega} \mathbf{e}^* \cdot \nabla (q^n - \frac{1}{2} \Delta t \Delta_h q^n) d\mathbf{x} + C \Delta t^{2\alpha+1} \|\mathbf{f}^n\|^2 \\ & \quad + C \Delta t (\|\mathbf{e}^n\|^2 + \|\mathbf{e}^{n-1}\|^2 + \|\mathbf{e}^*\|^2) + \frac{1}{2} \Delta t \|\nabla \mathbf{e}^*\|^2. \end{aligned}$$

Taking the scalar product of the second equation of (5.42) with \mathbf{e}^{n+1} , we obtain

$$(5.42) \quad \begin{aligned} & \|\mathbf{e}^{n+1}\|^2 - \|\mathbf{e}^*\|^2 + \|\mathbf{e}^{n+1} - \mathbf{e}^*\|^2 - \frac{1}{2} (\|\mathbf{e}^{n+1}\|^2 - \|\mathbf{e}^n\|^2) - \frac{1}{2} \|\mathbf{e}^{n+1} - \mathbf{e}^n\|^2 \\ & \leq C \Delta t^{2\alpha+1} \|\mathbf{g}^n\|^2 + C \Delta t \|\mathbf{e}^{n+1}\|^2. \end{aligned}$$

Combining the these two estimates, we get

$$(5.43) \quad \begin{aligned} & \|\mathbf{e}^{n+1}\|^2 - \|\mathbf{e}^n\|^2 + \|\mathbf{e}^{n+1} + \mathbf{e}^n - 2\mathbf{e}^*\|^2 + \Delta t \|\nabla_h \mathbf{e}^*\|^2 + \Delta t \|\nabla_h \cdot \mathbf{e}^*\|^2 \\ & \leq -2\Delta t (\langle \mathbf{e}^*, \nabla_h (q^n - \frac{1}{2} \Delta t \Delta_h q^n) \rangle) + C \Delta t (\|\mathbf{e}^n\|^2 + \|\mathbf{e}^{n-1}\|^2 + \|\mathbf{e}^{n+1}\|^2) \\ & \quad + C \Delta t^{2\alpha+1} (\|\mathbf{f}^n\|^2 + \|\mathbf{g}^n\|^2). \end{aligned}$$

To estimate the first term on the right hand of (5.45), we let

$$(5.44) \quad \begin{aligned} I & \equiv -2\Delta t (\langle \mathbf{e}^*, \nabla_h (q^n - \frac{1}{2} \Delta t \Delta_h q^n) \rangle) \\ & = -2\Delta t (\langle \mathbf{e}^*, \nabla_h q^n \rangle) - \Delta t^2 (\langle \nabla_h \cdot \mathbf{e}^*, \Delta_h q^n \rangle) \equiv I_1 + I_2. \end{aligned}$$

Using the second equation and integrating by parts, we can write the first term as

$$\begin{aligned}
I_1 &= -2\Delta t((\mathbf{e}^*, \nabla_h q^n)) \\
&= -\Delta t^2((\nabla_h(q^{n+1} - q^n), \nabla_h q^n)) - \Delta t^{\alpha+2}((\mathbf{g}^n, \nabla_h q^n)) \\
(5.45) \quad &= -\frac{1}{2}\Delta t^2(\|\nabla_h q^{n+1}\|^2 - \|\nabla_h q^n\|^2) \\
&\quad + \frac{1}{2}\Delta t^2\|\nabla_h(q^{n+1} - q^n)\|^2 - \Delta t^{\alpha+2}((\mathbf{g}^n, \nabla_h q^n)).
\end{aligned}$$

Since

$$\begin{aligned}
(5.46) \quad \frac{1}{2}\Delta t^2\|\nabla_h(q^{n+1} - q^n)\|^2 &= \frac{1}{2}\|\mathbf{e}^{n+1} + \mathbf{e}^n - 2\mathbf{e}^*\|^2 \\
&\quad + \frac{1}{2}\|\Delta t^{\alpha+1}\mathbf{g}^n\|^2 + \Delta t^{\alpha+1}((\mathbf{g}^n, \mathbf{e}^{n+1} + \mathbf{e}^n - 2\mathbf{e}^*)).
\end{aligned}$$

We have

$$\begin{aligned}
(5.47) \quad I_1 &= -\frac{1}{2}\Delta t^2(\|\nabla_h q^{n+1}\|^2 - \|\nabla_h q^n\|^2) + \frac{1}{2}\|\mathbf{e}^{n+1} + \mathbf{e}^n - 2\mathbf{e}^*\|^2 \\
&\quad + \frac{1}{2}\|\Delta t^{\alpha+1}\mathbf{g}^n\|^2 + \Delta t^{\alpha+1}((\mathbf{g}^n, \mathbf{e}^{n+1} + \mathbf{e}^n - 2\mathbf{e}^*)) - \Delta t^{\alpha+2}((\mathbf{g}^n, \nabla_h q^n)).
\end{aligned}$$

Next we rewrite the second term as

$$\begin{aligned}
(5.48) \quad I_2 &= -\Delta t^2((\nabla_h \cdot \mathbf{e}^*, \Delta_h q^n)) \\
&= -\frac{1}{2}\Delta t^3((\Delta_h(q^{n+1} - q^n), \Delta_h q^n)) - \frac{1}{2}\Delta t^{\alpha+3}((\nabla_h \cdot \mathbf{g}^n, \Delta_h q^n)) \\
&= -\frac{1}{4}\Delta t^3(\|\Delta_h q^{n+1}\|^2 - \|\Delta_h q^n\|^2) + \frac{1}{4}\Delta t^3\|\Delta_h(q^{n+1} - q^n)\|^2 \\
&\quad - \frac{1}{2}\Delta t^{\alpha+3}((\nabla_h \cdot \mathbf{g}^n, \Delta_h q^n)) \\
&= -\frac{1}{4}\Delta t^3(\|\Delta_h q^{n+1}\|^2 - \|\Delta_h q^n\|^2) + \Delta t\|\nabla_h \cdot \mathbf{e}^*\|^2 + \frac{1}{4}\Delta t^{2\alpha+3}\|\nabla_h \cdot \mathbf{g}^n\|^2 \\
&\quad - \Delta t^{\alpha+2}((\nabla_h \cdot \mathbf{g}^n, \nabla_h \cdot \mathbf{e}^*)) - \frac{1}{2}\Delta t^{\alpha+3}((\nabla_h \cdot \mathbf{g}^n, \Delta_h q^n)).
\end{aligned}$$

Combining these two terms we arrive at

$$\begin{aligned}
(5.49) \quad I &= -\frac{1}{2}\Delta t^2(\|\nabla_h q^{n+1}\|^2 - \|\nabla_h q^n\|^2) - \frac{1}{4}\Delta t^3(\|\Delta_h q^{n+1}\|^2 - \|\Delta_h q^n\|^2) \\
&\quad + \frac{1}{2}\|\mathbf{e}^{n+1} + \mathbf{e}^n - 2\mathbf{e}^*\|^2 + \Delta t \|\nabla_h \cdot \mathbf{e}^*\|^2 + \Delta t^{\alpha+1}((\mathbf{g}^n, \mathbf{e}^{n+1} + \mathbf{e}^n - 2\mathbf{e}^*)) \\
&\quad - \Delta t^{\alpha+2}((\mathbf{g}^n, \nabla_h q^n)) - \Delta t^{\alpha+2}((\nabla_h \cdot \mathbf{g}^n, \nabla_h \cdot \mathbf{e}^*)) \\
&\quad - \frac{1}{2}\Delta t^{\alpha+3}((\nabla_h \cdot \mathbf{g}^n, \Delta_h q^n)) + \frac{1}{4}\Delta t^{2\alpha+3}\|\nabla_h \cdot \mathbf{g}^n\|^2 + \frac{1}{2}\|\Delta t^{\alpha+1}\mathbf{g}^n\|^2.
\end{aligned}$$

This gives

$$\begin{aligned}
(5.50) \quad I &\leq -\frac{1}{2}\Delta t^2(\|\nabla_h q^{n+1}\|^2 - \|\nabla_h q^n\|^2) - \frac{1}{4}\Delta t^3(\|\Delta_h q^{n+1}\|^2 - \|\Delta_h q^n\|^2) \\
&\quad + \frac{1}{2}\|\mathbf{e}^{n+1} + \mathbf{e}^n - 2\mathbf{e}^*\|^2 + \Delta t \|\nabla_h \cdot \mathbf{e}^*\|^2 + \Delta t \|\mathbf{e}^{n+1} + \mathbf{e}^n - 2\mathbf{e}^*\|^2 \\
&\quad + 2\Delta t^3\|\nabla_h q^n\|^2 + 2\Delta t^4\|\Delta_h q^n\|^2 + 2\Delta t^{2\alpha+1}(\|\mathbf{g}^n\|^2 + \Delta t\|\mathbf{g}^n\|_{H^1}^2).
\end{aligned}$$

Going back to (5.45) we obtain

$$\begin{aligned}
(5.51) \quad &\|\mathbf{e}^{n+1}\|^2 - \|\mathbf{e}^n\|^2 + \frac{1}{2}\|\mathbf{e}^{n+1} + \mathbf{e}^n - 2\mathbf{e}^*\|^2 + \Delta t \|\nabla_h \mathbf{e}^*\|^2 \\
&\quad + \frac{1}{2}\Delta t^2(\|\nabla_h q^{n+1}\|^2 - \|\nabla_h q^n\|^2) + \frac{1}{4}\Delta t^3(\|\Delta_h q^{n+1}\|^2 - \|\Delta_h q^n\|^2) \\
&\leq \Delta t^3\|\nabla_h q^n\|^2 + \Delta t^4\|\Delta_h q^n\|^2 + C \Delta t (\|\mathbf{e}^n\|^2 + \|\mathbf{e}^{n-1}\|^2 + \|\mathbf{e}^{n+1}\|^2) \\
&\quad + C \Delta t^{2\alpha+1}(\|\mathbf{f}^n\|^2 + \Delta t\|\mathbf{g}^n\|_{H^1}^2).
\end{aligned}$$

Gronwall lemma gives

$$(5.52) \quad \|\mathbf{e}^n\| + \|\mathbf{e}^*\| + \Delta t \|\nabla_h q^n\| + \Delta t^{3/2} \|\Delta_h q^n\| + \Delta t^{1/2}\|\nabla_h \mathbf{e}^*\| \leq C_1 \Delta t^\alpha.$$

Now by inverse inequality (4.4) we have

$$(5.53) \quad \|\mathbf{e}^n\|_{L^\infty} + h\|\mathbf{e}^n\|_{W^{1,\infty}} + \Delta t\|\nabla_h q^n\|_{L^\infty} \leq C_1 \frac{\Delta t^\alpha}{h}.$$

Chose $N = 5$ and $\Delta t^\alpha \ll h^2$, if we choose Δt small enough, we will always have

$$(5.54) \quad \|\mathbf{e}^{n+1}\|_{L^\infty} \leq 1.$$

Therefore in (5.39) we can choose

$$(5.55) \quad \tilde{C} = 1 + \max_{n \leq \lceil \frac{T}{\Delta t} \rceil} \|U^n(\cdot)\|_{W^{1,\infty}}$$

which depends only on the exact solution (\mathbf{u}, p) . This proves

$$(5.56) \quad \|\mathbf{u}_0 - \mathbf{u}_h\|_{L^\infty} + \|p_0 - p_h\|_{L^2} + \Delta t^{1/2} \|p_0 - p_h\|_{L^\infty} + \|p_0 - p_h - p_c\|_{L^\infty} \leq C \Delta t^2$$

From Lemma 4.2, we have

$$(5.57) \quad \|\mathbf{u} - \mathbf{u}_h\|_{L^\infty} + \|p - p_h\|_{L^2} + \Delta t^{1/2} \|p - p_h\|_{L^\infty} + \|p - p_h - p_c\|_{L^\infty} \leq C(\Delta t^2 + h^2)$$

This completes the proof of Theorem 2.

References

- [1] C.R. Anderson, *Derivation and solution of the discrete pressure equations for the incompressible Navier-Stokes equations*, preprint.
- [2] J.B. Bell, P. Colella, and H.M. Glaz, *A second-order projection method for the incompressible Navier Stokes equations*, J. Comput. Phys. **85** (1989), 257–283.
- [3] A.J. Chorin, *Numerical solution of the Navier-Stokes equations*, Math. Comp. **22** (1968), 745–762.
- [4] A.J. Chorin, *On the convergence of discrete approximations to the Navier-Stokes equations*, Math. Comp. **23** (1969), 341–353.
- [5] Weinan E and J.-G. Liu, *Projection method I: convergence and numerical boundary layers*, SIAM J. Numer. Anal. **32** (1995), 1017–1057; *Projection method II: Godunov-Ryabenki analysis*, SIAM J. Numer. Anal. **33** (1996), 1597–1621.
- [6] P.M. Gresho, *Incompressible fluid dynamics: some fundamental formulation issues*, Rev. Fluid Mech. **23** (1991), 413–453.

- [7] P.M. Gresho, *Some interesting issues in incompressible fluid dynamics, both in the continuum and in numerical simulation*, Adv. in Appl. Mechanics **28** (1992), 45–139.
- [8] J. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem, part III. Smoothing property and higher order error estimates for spatial discretization*, SIAM J. Numer. Anal., **25**, 490-512, 1988.
- [9] G. Karniadakis, M. Israeli and S. A. Orszag, *High-order splitting methods for the incompressible Navier-Stokes Equations*, J. Comput. Phys., **97**, No. 2, 414-443, 1991.
- [10] J. Kim and P. Moin, *Application of a fractional-step method to incompressible Navier-Stokes equations*, J. Comp. Phys. **59** (1985), 308–323.
- [11] D. Michelson, *Convergence theorems for finite difference approximations for hyperbolic quasilinear initial-boundary value problems*, Math. Comp. **49** (1987), 445–459.
- [12] H. Okamoto, *On the semi-discrete finite element approximation for the nonstationary Navier-Stokes equation*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **29** (1982), 612–652.
- [13] S.A. Orszag, M. Israel, and M.O. Deville, *Boundary conditions for incompressible flows*, J. Scientific Computing **1** (1986), 75–111.
- [14] Jie Shen, *On error estimates of projection methods for Navier-Stokes equations: first order schemes*, SIAM J. Numer. Anal. **29** 1992, 57–77.
- [15] Jie Shen, *On error estimates of some higher order projection and penalty-projection methods for Navier-Stokes equations*, Numer. Math. **62** (1992), 49–73.
- [16] G. Strang, *Accurate partial differential methods II. non-linear problems*, Numerische Mathematik **6** (1964), 37–46.
- [17] R. Temam, *Sur l'approximation de la solution des equations de Navier-Stokes par la méthode des fractionnaires II*, Arch. Rational Mech. Anal. **33** (1969), 377–385.
- [18] R. Temam, *Navier Stokes Equations*, North Holland, New York, 1977.

- [19] R. Temam, *Remark on the pressure Boundary condition for the projection method*, Theoretical and Computational Fluid Dynamics **3** (1991), 181–184.
- [20] J. van Kan , *A second-order accurate pressure-correction scheme for viscous incompressible flow*, SIAM J. Sci. Stat. Comp. **7** (1986), 870–891.