

# Gauge Finite Element Method for Incompressible Flows

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**Abstract:** We present a finite element method for computing viscous incompressible flows based on the gauge formulation introduced in [6]. This formulation replaces the pressure by a gauge variable. This new gauge variable is a numerical tool and differs from standard gauge variable that arises from decomposing a compressible velocity field. It has the advantage that an additional boundary condition can be assigned to the gauge variable, thus eliminating the issue of pressure boundary condition associated with the original primitive variable formulation. The computational task is then reduced to solving standard heat and Poisson equations which are approximated by straightforward piecewise linear (or higher order) finite elements. This method can achieve high order accuracy at a cost comparable to that of solving standard heat and Poisson equations. It is naturally adapted to complex geometry, and it is much simpler than traditional finite element methods for incompressible flows. We present several numerical examples on both structured and unstructured grids.

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## 1. The gauge formulation of the Navier-Stokes equation

We start with the incompressible Navier-Stokes equation

$$(1.1) \quad \begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \frac{1}{\text{Re}}\Delta\mathbf{u} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

on  $\Omega$ , where  $\mathbf{u} = (u, v)$  is the velocity and  $p$  is the pressure, with the simplest physical boundary condition:

$$(1.2) \quad \mathbf{u} = 0$$

at  $\Gamma = \partial\Omega$ . The well-known difficulty with the numerical computation of viscous incompressible flows is the lack of an evolution equation for pressure. This is reflected in the additional boundary condition needed either explicitly or implicitly in various numerical methods for solving (1.1-1.2). Much discussion has been devoted to this issue. We refer to [8] for a review of various proposals. So far, it is generally agreed that projection method provides the best approach in tackling the issue of pressure boundary condition.

A new approach was introduced in [6] in which the Navier-Stokes equation was written in a form using a new variable  $\phi$ . Instead of advecting  $\mathbf{u}$ , [6] proposes to advect an auxiliary variable  $\mathbf{a} = \mathbf{u} - \nabla\phi$ . Indeed the Navier-Stokes equation can be rewritten as :

$$(1.3) \quad \begin{cases} \mathbf{a}_t - \frac{1}{\text{Re}}\Delta\mathbf{a} + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0 \\ -\Delta\phi = \nabla \cdot \mathbf{a} \\ \mathbf{u} = \mathbf{a} + \nabla\phi \end{cases}$$

$\phi$  is referred to as the gauge variable since writing  $\mathbf{u} = \mathbf{a} + \nabla\phi$  is equivalent to choosing another gauge if we view vorticity  $\omega = \nabla \times \mathbf{u}$  as the analog of the magnetic field, and  $\mathbf{u}$  as the analog of vector potential. Pressure has disappeared from the equations and can be recovered through:

$$(1.4) \quad \phi_t - \frac{1}{\text{Re}}\Delta\phi = p$$

Replacing  $p$  by  $\phi$  has the advantage that an additional boundary condition (as well as initial condition) can be assigned to  $\phi$  since  $\phi$  solves a parabolic equation with  $p$  as the right hand side. Corresponding to (1.2), we can either prescribe:

$$(1.5) \quad \frac{\partial\phi}{\partial\mathbf{n}} = 0, \quad \mathbf{a} \cdot \mathbf{n} = 0, \quad \mathbf{a} \cdot \boldsymbol{\tau} = -\frac{\partial\phi}{\partial\boldsymbol{\tau}}$$

or

$$(1.6) \quad \phi = 0, \quad \mathbf{a} \cdot \mathbf{n} = -\frac{\partial\phi}{\partial\mathbf{n}}, \quad \mathbf{a} \cdot \boldsymbol{\tau} = 0$$

on  $\Gamma$ . Here  $\boldsymbol{\tau}$  is the unit vector in the tangential direction. We will call (1.5) the Neumann formulation and (1.6) the Dirichlet formulation. The boundary conditions for  $\mathbf{a}$  come from (1.2) and the relation  $\mathbf{u} = \mathbf{a} + \nabla\phi$ .

In principle, (1.3) together with (1.5) or (1.6) can be solved by any reasonable numerical method for the Poisson equation and heat equations. In [6], we concentrated on finite difference methods. In this paper, we present the most simple-minded finite element method. We will discretize  $\phi$  and  $\mathbf{a}$  using the standard  $C^0$  elements. As we will see later, the complexity of the resulting method is not very different from that of solving standard heat and Poisson equations.

**Remark 1.** The idea of reformulating the Navier-Stokes equation by introducing a gauge variable appeared in [13] and was used in [2] in the context of the vortex method. Their formulation was shown to be numerically unstable in [5]. We also refer to [15, 11] and [14] for other related work.

## 2. Description of the finite element method

There are many different ways of formulating finite element methods for (1.3). Instead of presenting general procedures, we concentrate on a special scheme which is quite likely the simplest second order method. We first discretize time using Crank-Nicolson:

$$(2.1) \quad \frac{\mathbf{a}^{n+1} - \mathbf{a}^n}{\Delta t} + \nabla \cdot (\mathbf{u}^{n+\frac{1}{2}} \otimes \mathbf{u}^{n+\frac{1}{2}}) = \frac{1}{\text{Re}} \Delta \frac{\mathbf{a}^{n+1} + \mathbf{a}^n}{2}$$

The convective term can be approximated using Adams-Bashforth:

$$(2.2) \quad \mathbf{u}^{n+\frac{1}{2}} \otimes \mathbf{u}^{n+\frac{1}{2}} = \frac{3}{2} \mathbf{u}^n \otimes \mathbf{u}^n - \frac{1}{2} \mathbf{u}^{n-1} \otimes \mathbf{u}^{n-1}$$

Next we describe the spatial discretization. Here we work with the Dirichlet formulation (1.6) for two reasons. The first is that the Neumann formulation was studied quite extensively in [6] in the context of finite difference methods. The second is that numerically solving the Dirichlet problem of the Poisson equation is much easier than solving the Neumann problem.

At each time step, we have:

$$(2.3) \quad \left(I - \frac{\Delta t}{2\text{Re}} \Delta\right) \mathbf{a}^{n+1} = \left(I + \frac{\Delta t}{2\text{Re}} \Delta\right) \mathbf{a}^n - \Delta t \nabla \cdot (\mathbf{u}^{n+\frac{1}{2}} \otimes \mathbf{u}^{n+\frac{1}{2}})$$

with boundary condition

$$(2.4) \quad \mathbf{a}^{n+1} \cdot \mathbf{n} = - \left( 2 \frac{\partial \phi^n}{\partial \mathbf{n}} - \frac{\partial \phi^{n-1}}{\partial \mathbf{n}} \right), \quad \mathbf{a}^{n+1} \cdot \boldsymbol{\tau} = 0$$

and

$$(2.5) \quad -\Delta \phi^{n+1} = \nabla \cdot \mathbf{a}^{n+1}$$

$$\phi^{n+1}|_{\partial\Omega} = 0$$

These equations are discretized in space using the standard piecewise linear finite element method.

Having  $\mathbf{a}^{n+1}$  and  $\phi^{n+1}$ , we compute  $\mathbf{u}^{n+1}$  by  $\mathbf{u}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^{n+1}$ . In general,  $\mathbf{u}^{n+1}$  is discontinuous across cell boundaries. We therefore compute the convective term using a conservative form:

$$(2.6) \quad ((\mathbf{u}^n \cdot \nabla)\mathbf{u}^n, \psi) = - \int_{\Omega} (\nabla\psi)(\mathbf{u}^n \otimes \mathbf{u}^n) dx$$

for all  $\psi \in V_h \times V_h$  where  $V_h$  is the space of continuous piecewise linear function that vanishes at the boundary.

The method described above is second order accurate on regular grids, as shown in Table 1, but degenerates to first order accuracy on irregular grids. This is due to the involvement of gradient terms in the relation  $\mathbf{u} = \mathbf{a} + \nabla\phi$ . To achieve second order accuracy on general irregular grids, we use piecewise quadratic polynomials to approximate  $\phi$ . We call this the  $P_1 - P_2$  method, and the method described earlier the  $P_1 - P_1$  method. There is an obvious generalization to  $P_k - P_{k+1}$  methods for  $k \geq 1$ .

**Remark 2.** The main difference between the gauge method and the projection method [3, 16, 7, 1, 10, 17, 4] is that the gauge method requires solving **standard** Poisson equations for the gauge, whereas the projection method requires numerically performing the Helmholtz decomposition at the projection step. This amounts to solving the pressure Poisson equation via a mixed formulation. We demonstrated in [6] that one cannot use standard Poisson solvers for the pressure Poisson equation because of difficulties at the boundary.

**Remark 3.** The time lag (also called vertical extrapolation) in (2.4) is necessary to decouple the computation of  $\mathbf{a}^{n+1}$  and  $\phi^{n+1}$ . We showed in [6] that this does not affect the stability and accuracy of the overall method.

**Remark 4.** Rigorous error estimates of this method were proved in [18]. Basically for smooth solutions, this method achieves the expected accuracy. In principle the regularity requirement on the solution can be relaxed, especially at  $t = 0$ , as in [9]. But this is a much more involved analysis and it is not yet done.

### 3. Numerical results

#### 3.1. Flow between concentric circles: an accuracy check

The method described above is expected to be second order accurate. To check that this is actually the case, we performed a detailed accuracy check on a simple model problem: flow between two concentric circles. The radius of the inner and outer circle are respectively 1 and 2. The outer

circle remains stationary and the inner circle rotates in the counter-clockwise direction with unit speed. A standard regular polar coordinate grid is used. Shown in Table I are the numerical results using the  $P_1 - P_1$  method. We obtain second order accuracy as a result of the regularity of the grid.

$h$	$\frac{\ \mathbf{u} - \mathbf{u}_h\ _{L_1}}{h^2}$	$\frac{\ \mathbf{u} - \mathbf{u}_h\ _{L_2}}{h^2}$	$\frac{\ \mathbf{u} - \mathbf{u}_h\ _{L_\infty}}{h^2}$
1/9	5.71	5.84	7.62
1/18	5.84	5.92	7.73
1/36	5.90	5.98	8.01

**Table I**

### 3.2. Other examples

We present here several examples involving increasingly complex geometry and demonstrate the flexibility of gauge finite element method for such geometries. In all these examples, the flow is driven by the impulsively started rotation of the inner circle. In the first example the computational domain is an annulus with the inner circle displaced. The computational grid is displayed in Figure 1. The computed velocity field using the  $P_1 - P_1$  method is presented in Figure 2. In the second example, the computational domain is a rectangle with a circular hole at the lower left corner. The computed velocity field is shown in Figure 3. Finally the numerical grid for the third example is shown in Figure 4, with the computed velocity field in Figure 5.

### 4. Concluding remarks

As we demonstrated in [6] and here also, the main advantage of the gauge method is its simplicity and flexibility for computing low Reynolds number flows. The gauge finite element method has the additional advantage of handling complicated geometries easily.

The method we presented here can be generalized in many different ways. Higher order methods can be obtained by using higher order backward differentiation formulas in time, and higher order  $C^0$  finite element in space. Extension to three space dimension is straightforward. At the present time, we have developed a program for computing two-dimensional viscous flows with any specified

geometry, and work on extending it to three dimensions is underway.

Because of its simplicity, the gauge method also opens up new ways of handling more complicated physical problems such as visco-elastic flows. Work in this direction is also in progress.

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## References

- [1] J.B. Bell, P. Colella, and H.M. Glaz, *A second-order projection method for the incompressible Navier Stokes equations*, J. Comput. Phys. **85** (1989), 257–283.
- [2] T. Buttke, *Velocity methods: Lagrangian numerical methods which preserve the Hamiltonian structure of incompressible fluid flow*, in *Vortex Flows and Related Numerical Methods*, J. T. Beale, G. H. Cottet and S. Huberson ed., Kluwer, 1993.
- [3] A.J. Chorin, *Numerical solution of the Navier-Stokes equations*, Math. Comp. **22** (1968), 745–762.
- [4] Weinan E and J.-G. Liu, *Projection method I: convergence and numerical boundary layers*, SIAM J. Numer. Anal. **32** (1995), 1017–1057; *Projection method II: Godunov-Ryabenki analysis*, SIAM J. Numer. Anal. **33** (1996), 1597–1621.
- [5] Weinan E and J.-G. Liu, *Finite difference schemes for incompressible flows in the velocity-impulse density formulation*, J. Comput. Phys., **130** (1997) 67–76.
- [6] Weinan E and J.-G. Liu, *Gauge method for viscous incompressible flows*, submitted to J. Comput. Phys.
- [7] M. Fortin, R. Peyret, and R. Temam, J. Mecanique, **10** (1971) 357–390.
- [8] P.M. Gresho and R.L. Sani, *On pressure boundary conditions for the incompressible Navier-Stokes equations*, Int. J. Numer. Methods Fluids, **7** (1987) 1111–1145.
- [9] J. Heywood and R. Rannacher, *Finite element approximations of the nonstationary Navier-Stokes problem, part III. Smoothing property and higher order error estimates for spatial discretizations*, SIAM J. Numer. Anal., **25**, 490–512 (1988).

- [10] J. Kim and P. Moin, *Application of a fractional-step method to incompressible Navier-Stokes equations*, J. Comp. Phys. **59** (1985), 308–323.
- [11] J. H. Maddocks and R. L. Pego, *An unconstrained Hamiltonian formulation for incompressible fluid flow*, Comm. Math. Phys., **170**, (1995) 207.
- [12] S.A. Orszag, M. Israeli, and M.O. Deville, *Boundary conditions for incompressible flows*, J. Scientific Computing **1** (1986), 75–111.
- [13] V. I. Oseledets, *On a new way of writing the Navier-Stokes equation: The Hamiltonian formalism*, Russ. Math. Surveys, **44** (1989) 210.
- [14] G. Russo and P. Smereka, *Impulse formulation of the Euler equations: general properties and numerical methods*, preprint, 1997.
- [15] D. Summers and A. J. Chorin, *Numerical vorticity creation based on impulse conservation*, Proc. Nat. Acad. Sci. **93**, 1881-1885 (1996).
- [16] R. Temam, *Sur l'approximation de la solution des equations de Navier-Stokes par la méthode des fractionnaires II*, Arch. Rational Mech. Anal. **33** (1969), 377–385.
- [17] J. van Kan, *A second-order accurate pressure-correction scheme for viscous incompressible flow*, SIAM J. Sci. Stat. Comp. **7** (1986), 870–891.
- [18] C. Wang and J. G. Liu, *Convergence of the gauge method for incompressible flow*, submitted to Math. Comp., 1998.

Figure 1: Numerical grid for the asymmetric annulus. Number of vertices = 500. Number of elements = 275.

Figure 2: Computed steady state velocity field on the grid shown in Figure 1. Reynolds number=100.

Figure 3: Computed velocity field at  $t = 2.8$ , with impulsive start-up of the circle. Other parameters are: Number of elements=1193, Reynolds number=1.

Figure 4: Numerical grid for Example 3. Number of vertices = 1021. Number of elements = 568.

Figure 5: Computed velocity field at  $t = 11$  on the grid shown in Figure 4. Reynolds number = 10.