Blow Up of Solutions of the Unsteady Prandtl's Equation

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Abstract

We prove that for certain class of compactly supported C^∞ initial data, smooth solutions of the unsteady Prandtl's equation blow up in finite time.

1 Introduction

Consider the unsteady Prandtl's equation in half space $R \times R^+ = \{(x,y), y > 0\}$:

$$u_t + uu_x + vu_y = u_{yy} + P_x, \quad u_x + v_y = 0 \tag{1.1}$$

with initial and boundary condition

$$u(x,y,0) = u_0(x,y),\ u(x,0,t) = v(x,0,t) = 0,\ \lim_{y \to +\infty} u(x,y,t) = U(x,t) \tag{1.2}$$

where U and P satisfy Bernoulli's law

$$\frac{U^2(x,t)}{2} + P(x,t) = \text{Constant}$$

In this paper we will restrict ourselves to the case when $U = P = 0$.

The main purpose of this paper is to prove the following:

THEOREM 1.1 *Assume that u_0 takes the form $u_0(x,y) = -xb_0(x,y)$ where $a_0(\cdot) = -u_{0x}(0,\cdot)$ satisfies the conditions of Lemma 2.1, then smooth solutions of (1.1-1.2) do not exist globally in time.*

Here by smooth solutions, we mean solutions in $C([0,T], H^s(R \times R^+))$ for large enough s, and the blow up is in the maximum norm of u_x or u_{xy}.

At the present time, there are no local existence results for (1.1-1.2) with compactly supported initial data (Existence of solutions is proved for monotone initial data in [4] and for analytic initial data in [5], see also

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[1]). Therefore the above theorem asserts that either smooth solutions do not exist even locally in time, or smooth solutions blow up in finite time. In general, the blow-up will not occur at the boundary. This is in sharp contrast with the steady case where the blow-up is caused by an adverse pressure gradient from far field, and always occurs first at the boundary [3]. It is consistent with the numerical results of Van Dommelen, Shen et.al. [6].

2 Proof of the Theorem

We will study solutions of (1.1-1.2) with initial data of the form assumed in the theorem. Suppose that smooth solutions of (1.1-1.2) exist locally and let u be such a solution. If we restrict (1.1) to the half line $L = \{y > 0, x = 0\}$, and let $w(y,t) = u(0,y,t)$, we see that w satisfies an equation of the form

$$w_t + f(y,t)w + g(y,t)w_y = w_{yy}$$

where $f(y,t) = u_x(0,y,t), g(y,t) = v(0,y,t)$ are smooth, with initial data $w(y,0) = 0$ and boundary data $w(0,t) = 0$. Since the solution of this problem is unique, we conclude that $u(0,y,t) = w(y,t) \equiv 0$ as long as u stays smooth. Therefore we can always write the solution in the form (since any smooth function that vanishes at $x = 0$ can be written in this form):

$$u(x,y,t) = -xb(x,y,t), v(x,y,t) = \int_0^y (b(x,z,t) + b_x(x,z,t)x)dz. \tag{2.1}$$

Substituting (2.1) into (1.1), we get

$$b_t = b_{yy} + b(b + b_x x) - vb_y \tag{2.2}$$

Let $a(y,t) = b(0,y,t), a_0(y) = b(0,y)$, then a satisfies (2.2) restricted to $x = 0$:

$$a_t = a_{yy} + a^2 - \left(\int_0^y a(z,t)dz\right) a_y \tag{2.3}$$

with the boundary condition

$$a(0,t) = 0, \quad \lim_{y\to+\infty} a(y,t) = 0, \quad a(y,0) = a_0(y) \tag{2.4}$$

Lemma 2.1 *Define*

$$F(a) = \int_0^\infty a^2 dy, \quad E(a) = \int_0^\infty (\frac{1}{2}a_y^2 - \frac{1}{4}a^3)dy.$$

Let a_0 be a non-negative compactly supported data such that $E(a_0) < 0$. Then there exists a finite time T such that either

$$\lim_{t \to T} max_y |a| = +\infty.$$

or

$$\lim_{t \to T} a_y(0,t) = +\infty.$$

PROOF: Let us first make a comment about decay of a as $y \to \infty$. Assume that $\max_y a$ stays bounded. Since

$$\frac{dF}{dt} \leq 3 \int_0^\infty a^3 dy \leq 3(\max_y |a|) F$$

we must have that $F(a)$ stays bounded. Integrating (2.3) with respect to y over $(0,\infty)$ we obtain

$$\frac{d}{dt} \int_0^\infty a(y,t) dy = -a_y(0,t) + 2F(a) \leq 2F(a)$$

This implies that $\int_0^\infty a(y,t)dy$ stays bounded. Since a is non-negative, we have that $\int_0^y a(z,t)dz$ is uniformly bounded. Assuming that $\tilde{v}, \tilde{h}$ are smooth and uniformly bounded, it is a standard exercise to show that bounded solutions of

$$c_t + \tilde{v}(y,t) c_y = c_{yy} + h(y,t) c$$

with compactly supported initial data decays exponentially fast at infinity. Since a satisfies an equation of this form, this argument showes that a decays exponentially fast at infinity as long as its maximum norm stays bounded.

In the following we will show that $F(a)$ blows up at finite time assuming that $a_y(0,t)$ stays finite. We will use the following integral identities valid for smooth solutions of (2.3):

$$\frac{d}{dt} F(a) = -2 \int_0^\infty a_y^2 dy + 3 \int_0^\infty a^3 dy \tag{2.5}$$

$$\frac{d}{dt} \int_0^\infty \frac{1}{2} a_y^2 dy = \int_0^\infty (-a_{yy}^2 + \frac{3}{2} a_y^2 a) dy \tag{2.6}$$

$$\frac{d}{dt} \int_0^\infty \frac{1}{3} a^3 dy = \int_0^\infty (-2a_y^2 a + \frac{4}{3} a^4) dy \tag{2.7}$$

These identities follow from simple integration by parts. It is useful to note

$$\int_0^\infty a_y^2 a dy = -\frac{1}{2}\int_0^\infty a^2 a_{yy} dy$$

Combining (2.6) and (2.7), we get

$$\frac{dE}{dt} = -\int_0^\infty \{(a_{yy} + \frac{3}{4}a^2)^2 + \frac{7}{16}a^4\} dy$$

Thus, if $E(a_0) < 0$, we have $E(a) < 0$ for $t > 0$.

Next we use an idea in [2] to compute the time derivative of $G(a) = -\frac{E(a)}{F(a)^\beta}$. First we have

$$\begin{aligned}
-F(a)\frac{dE(a)}{dt} &= \int_0^\infty a^2 dy \int_0^\infty (a_{yy} + \frac{3}{4}a^2)^2 dy + \frac{7}{16}\int_0^\infty a^2 dy \int_0^\infty a^4 dy \\
&= \int_0^\infty a^2 dy \int_0^\infty (a_t - \frac{1}{4}a^2 + a_y \int_0^y a dz)^2 dy + \frac{7}{16}\int_0^\infty a^2 dy \int_0^\infty a^4 dy \\
&\geq \left(\int_0^\infty a\{a_t - \frac{1}{4}a^2 + a_y \int_0^y a dz\} dy\right)^2 + \frac{7}{16}\left(\int_0^\infty a^3 dy\right)^2 \\
&= \frac{1}{4}\left(\frac{dF}{dt} - \frac{3}{2}\int_0^\infty a^3 dy\right)^2 + \frac{7}{16}\left(\int_0^\infty a^3 dy\right)^2
\end{aligned}$$

Since

$$\frac{dF}{dt} - \frac{3}{2}\int_0^\infty a^3 dy = \int_0^\infty a_y^2 dy - 6E \geq -6E$$

$$\int_0^\infty a^3 dy = 2\int_0^\infty a_y^2 - 4E \geq -4E$$

we have,

$$-F\frac{dE}{dt} \geq -\frac{3}{2}E\left(\frac{dF}{dt} - \frac{3}{2}\int_0^\infty a^3 dy\right) - \frac{7}{4}E\int_0^\infty a^3 dy = -E\left(\frac{3}{2}\frac{dF}{dt} - \frac{1}{2}\int_0^\infty a^3 dy\right)$$

Since

$$\frac{dF}{dt} = -4E + 2\int_0^\infty a^3 dy \geq 2\int_0^\infty a^3 dy,$$

we get if we choose $\beta < \frac{5}{4}$.

$$-F\frac{dE}{dt} + \beta E\frac{dF}{dt} \geq -E\left((\frac{3}{2} - \beta)\frac{dF}{dt} - \frac{1}{2}\int_0^\infty a^3 dy\right) \geq -E(\frac{5}{2} - 2\beta)\int_0^\infty a^3 dy \geq 0.$$

Now we fix β in $(1, \frac{5}{4})$, we have

$$\frac{d}{dt}G(a) \geq 0 \tag{2.8}$$

Therefore

$$\frac{dF}{dt} \geq -6E \geq 6G(a_0)F^{\beta} \tag{2.9}$$

Hence there exists a finite time T, such that

$$\lim_{t\to T} F(a) = +\infty \tag{2.10}$$

Therefore we must have

$$\lim_{t\to T} \max_y |a| = +\infty. \tag{2.11}$$

This completes the proof of Lemma 2.1. ∎

To complete the proof of the theorem, we note that smooth solutions of (2.3-2.4) are unique. Therefore we have, at $x = 0$,

$$\sup_{y>0} |\frac{u(x,y,t)}{x}| \to +\infty$$

as $t \to T$. This implies that

$$\sup_{x,y} |u_x| \to +\infty$$

as $t \to T$.

Remark. 2.2 It is worth emphasizing that since we showed in the begining of this section that smooth solutions with the assumed initial data have to be of the form (2.1), and hence $a(y,t) = u_x(0,y,t)$ has to satisfy (2.3), in the final part of the proof we were using the uniqueness result for (2.3-2.4) instead of a uniqueness result for (1.1-1.2) which is not available yet.

Remark. 2.3 For $b \geq 0$, (2.1) corresponds to flows impinging from the left and right at $x = 0$. The origin is a stagnation point.

3 An Example: The Inviscid Prandtl's Equation

In the absence of the viscous term, (2.3) becomes

$$a_t = a^2 - \left(\int_0^y a(z,t)dz\right) a_y \tag{3.1}$$

This can be rewritten as

$$D_t a = a_t + v a_y = a^2 \tag{3.2}$$

if we define $v(y,t) = \int_0^y a(z,t)dz$. (3.2) can be solved explicitly using Lagrangian coordinates which we denote by α. Let the $Y(\alpha,\cdot)$ be the Eulerian path of a particle with Lagrangian coordinate α. Y solves

$$\frac{dY(\alpha,t)}{dt} = \int_0^{Y(\alpha,t)} a(y,t)dy$$

The solution of (3.2) is given by

$$a(Y(\alpha,t),t) = \left(\frac{1}{a_0(\alpha)} - t\right)^{-1}$$

The Jacobian of the Eulerian-Lagrangian coordinate transformation $J = \frac{\partial Y}{\partial \alpha}$ satisfies

$$\frac{dJ}{dt} = a(Y(\alpha,t),t)J$$

and $J(\alpha,0) = 1$. Therefore we have

$$J(\alpha,t) = \exp\int_0^t a(Y(\alpha,s),s)ds = \exp\int_0^t \left(\frac{1}{a_0(\alpha)} - s\right)^{-1} ds = \frac{1}{a_0(\alpha)}\left(\frac{1}{a_0(\alpha)} - t\right)^{-1}$$

We can now compute the particle path from

$$\frac{dY}{dt} = \int_0^Y a(y,t)dy = \int_0^\alpha a(Y(\beta,t),t)\frac{\partial Y}{\partial \beta}d\beta$$

$$= \int_0^\alpha \frac{1}{a_0(\beta)}\left(\frac{1}{a_0(\beta)} - t\right)^{-2} d\beta = \int_0^\alpha \frac{a_0(\beta)}{(1 - ta_0(\beta))^2}d\beta$$

This gives

$$Y(\alpha,t) = \int_0^\alpha (1 - ta_0(\beta))^{-1}d\beta$$

At the critical time T we have $a_0(\alpha^*)T = 1$ for some α^*. If a_0 is smooth, we will have $Y(\alpha,T) = +\infty$ for $\alpha > \alpha^*$. This means that the semi-infinite interval $(\alpha^*, +\infty)$ is transported to infinity at the critical time T.

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