Recent Results on Mathematical & Statistical Hydrodynamics

Weinan E*, **Yakov Sinai**†

*Mathematics Department and the Program in Applied and Computational Mathematics, Princeton University & Courant Institute of Mathematical Sciences, NYU

†Mathematics Department of Princeton University & Landau Institute of Theoretical Physics

This paper is a review of the recent work by the authors and their collaborators on stochastic partial differential equations in hydrodynamics. We will discuss the stochastic Burgers equation, stochastic Navier-Stokes equation and the stochastic passive scalar transport equation. In constrast to previsous work (see for example [FV]) on this subject which concentrates on issues related to existence of the stochastic dynamics, the works reviewed here emphasize qualitative properties of the solutions, such as the existence and uniqueness of an invariant measure under physical assumptions, asymptotic behavior of their statistics, etc. We will also discuss some recent work on the deterministic Navier-Stokes equation.

Part I
Burgers System

1 Definition and General Properties of Burgers System

Burgers System is a simplification of Navier-Stokes system. In the d-dimensional case it is written for d unknown functions $u(x,t) = \{u_j(x,t),\ 1 \le j \le d\}$ of d space variables $x = \{x_j,\ 1 \le j \le d\}$ and time and has the form

$$\frac{\partial u_j}{\partial t} + \sum_{\ell=1}^{d} u_\ell \frac{\partial u_j}{\partial x_\ell} = \nu \sum_{\ell=1}^{d} \frac{\partial^2 u_j}{\partial x_\ell} + f_j(x,t) \tag{1}$$

It differs from Navier-Stokes system by the absence of pressure and incompressibility condition. By this reason the set of problems for Burgers system is sometimes called pressure-less hydrodynamics. The coefficient ν is the viscosity and the case $\nu = 0$ is called inviscid Burgers system. Other names are Hopf system, Riemann equation for $d = 1$. In the last case it is the simplest quasi-linear equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f(x,t).$$

In general the vector $f = \{f_j, 1 \le j \le d\}$ in (1) is the external forcing. Ya. B. Zeldovitch proposed to consider the inviscid free system ($f = 0$) as an equation describing the evolution of the rarefied gas of non-interacting particles (see [Z]). According to his idea pure kinematics of the underlying particles can lead to singularities in the distribution of mass and is responsible for the non-uniformity of matter in the universe.

The system (1) is simpler than the full Navier-Stokes system on the space of gradient-like solutions. Namely, if $u(x,0) = -2\nu \frac{\nabla\varphi(x,0)}{\varphi(x,0)}$ for some positive function $\varphi(x,0)$ and $f_j = \frac{\partial F(x,t)}{\partial x_j}$ then for all $t > 0$ the vector $u(x,t) = -2\nu \frac{\nabla\varphi(x,t)}{\varphi(x,t)}$ where the function $\varphi(x,t)$ satisfies the heat equation

$$\frac{\partial \varphi(x,t)}{\partial t} = \nu \, \Delta \, \varphi(x,t) + \frac{1}{\nu} F(x,t) \, \varphi \tag{2}$$

The transition from the equation for u to the equation for φ is called the Hopf-Cole substitution (see [C], [Ho1]). However, it was known probably much earlier (see [W]).

The case $\nu > 0$ is in many respects simpler than the case $\nu = 0$ which is strictly speaking the limit $\nu \to 0$. In discussing the theory for $\nu = 0$ we shall only mention related results for $\nu > 0$. However, the questions of continuity at $\nu = 0$ are as usual hard.

We shall discuss basically two sets of problems for the Burgers system.

A. The inviscid system (1) with random initial conditions. Burgers himself considered in his monograph (see [Bu]) the case $d = 1$, $f = 0$ and the initial condition $u(x, 0) = B(x)$ where B is the white noise, i.e. a generalized random process. He showed that a solution $u(x, t)$ for $t > 0$ is a piece-wise linear function of x. The discontinuities in solutions of quasi-linear equations are called shocks. One can say that the irregularities of the initial conditions produce the irregularities of solutions, i.e. the shocks. We will discuss the work of Avellaneda and E on the asymptotics for the statistics of the shocks (see [AE], [A], [ARE]). Frisch was the first who understood that an interesting situation arises if $B(x)$ is replaced by an homogeneous random process with independent increments like Brownian motion or Levy stable process. In the joint paper with Z. She, E. Aurell, (see [SAF]) the authors constructed numerically solutions and discovered the appearance for any $t > 0$ of infinitely many shocks. In this case we have an interesting example how dynamics transforms a continuous distribution of mass into a discrete one. Mathematical theory related to the case of [SAF] was constructed in [Si1]. We shall discuss some of the related problems below.

B. Statistically homogeneous regimes in the inviscid Burgers equation with random forcing. We shall discuss mainly the results of the papers by W. E, K. Khanin, A. Mazel and Ya. Sinai (see [EKMS1], [EKMS2]) proven for $d = 1$ and some related problems. We will also discuss the master equation approach of W.E and E. Vanden Eijnden (see [EV1], [EV2]) for studying the statistical properties of the Burgers system.

2 Inviscid Burgers Equation and Its Modifications with Random Initial Conditions

In the one-dimensional case the problem is a pure classics. We have the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

and we want to construct solutions for irregular initial conditions which are typical realizations of some random process. The usual method of characteristics does not work due to irregularities since the characteristics intersect each other for arbitrary small $t > 0$. We have to use the so-called Lax-Oleinik variational principle (see [Lax], [O], [VDFN], [W]). According to it take the function $W(x, t) = \int_0^x [u(y, 0) + \frac{y}{t}]dy = \int_0^x u(y, 0)dy + \frac{x^2}{2t}$. If $u(y, 0)$

is a realization of a random process such that $|u(y,0)|$ grows more slowly than $|y|$ then W is a parabola-like function perturbed by a random process. The next step is to consider the convex *minoranta* C of W, i.e. the largest convex function bounded from above by W. The graph of $C(x,t)$ consists of straight line intervals and a closed set outside them. The derivative

$$F(x,t) \;=\; \frac{\partial C(x,t)}{\partial x}$$

is constant where $C(x,t)$ is linear. By this reason F is a devil's staircase-type function. Then the solution

$$u(x,t) \;=\; \frac{x}{t} \;-\; F^{-1}(x,t), \tag{3}$$

where $F^{-1}(x,t)$ is the inverse function. The easiest way to understand (3) is to consider F as a continuous curve on the plane and F^{-1} as a continuous curve which appears if we interchange the axes (see Figure 1).

The solution $u(x,t)$ can also be considered as a continuous curve on the plane with vertical segments corresponding to the shocks.

From the mathematical point of view the main problem can be formulated as the analysis of convex minorantas of random processes. Let us give an example of a simple general statement which can be proven here. Take for simplicity $t = 1$ and consider the random process

$$W(x) \;=\; W(x,1) \;=\; \int_0^x [u(y,0) \;+\; y]\,dy \;=\; \int_0^x u(y,0)\,dy \;+\; \frac{x^2}{2}.$$

Assume that $u(\cdot,0)$ is continuous with probability 1. Then W is differentiable a.e. and $W'(x) = u(x,0) + x$. Let us call a point $\bar{x}$ special (for $W(x)$ or $u(\cdot,0)$) if there exists a neighborhood $(\bar{x}-\delta, \bar{x}+\delta)$ such that either

$$W(x) \;\geq\; W(\bar{x}) \;+\; (x-\bar{x})\,(u(\bar{x},0) \;+\; \bar{x}) \tag{4'}$$

or

$$W(x) \;\leq\; W(\bar{x}) \;+\; (x-\bar{x})\,(u(\bar{x},0) \;+\; \bar{x}) \tag{4''}$$

for all $x \in (\bar{x}-\delta, \bar{x}+\delta)$, i.e. either the graph of W lies above or below its tangent line passing through $\bar{x}$.

The event which is described by the last inequalities depends on the behavior of the process $u(x,0)$ in an arbitrary small neighborhood of $\bar{x}$, i.e. belongs to the local σ-algebra of the process u. In many cases (Brownian motion, fractional Brownian motion, Levy stable processes, diffusion processes) it is known that this σ-algebra is trivial, i.e. has only events of probability 0 or 1. From this one can derive that our event ($4'$) or ($4''$) has probability 0. This can be done in all cases mentioned above.

Let us return to the graph of F depicted on the Figure 1. Recall the following definition.

Definition 1 *A devil's staircase is called complete if the union of intervals where it is constant is a set of full measure.*

Assume that for every $\bar{x}$ the probability that it is special is zero. Using Fubini Theorem one can easily conclude that with probability 1 the graph of F is a complete devil's staircase, i.e. the set of special points has measure zero a.e. u. Frisch raised the problem of estimating the Hausdorff dimension of this set (see [SAF]). The answer for the case of Brownian motion was obtained in [Si1] where it was shown that it is $\frac{1}{2}$. The proof is based on the estimate

$$P\left\{\int_0^t b(s)ds > -a_0 + a_1 t \text{ for all } 0 \le t \le T\right\} \sim \frac{C(a_0, a_1)}{T^{1/4}}$$

as $T \to \infty$. Its derivation can be found in several publications, see e.g. Lachal [La], [Si2] Handa in [H] obtained a simple proof of the estimate of the Hausdorff estimation from below valid in a more general situation.

The case of Levy stable process was studied numerically and qualitatively by Janicki and Woyczynski (see [JW]). A complete answer was obtained recently by J. Bertoin (see [Be]).

Next we consider the situation with white noise initial data. This is the problem Burgers himself was interested in and devoted most of his book [Bu] on. In his book Burgers calculated many interesting statistical quantities for this problem. His solutions were often expressed in the form of series. By this reason, it was hard to extract explicit information on the asymptotics of statistical quantities such as the distribution of shock strength. However, recently Frachebourg and Martin [FrM] gave simpler formulas for many of these statistical quantities from which the asymptotics can be readily extracted. This work provides very valuable exact results for this classical problem.

The crudest question that can be asked about this problem is the dynamic space-time scaling relations. More precisely, since the distribution of white noise is invariant under the scaling $\delta^{\frac{1}{2}} u_0(\delta x) \stackrel{D}{=} u_0(x)$, we can consider a rescaling of the variables

$$\delta x = x', \delta^{\frac{1}{2}} u = u', \delta^{\gamma} t = t'$$

and ask: for which value of γ the equation in the rescaled variables (x', t', u') has non-trivial dynamics as $\delta \to 0$. This amounts to asking, at large scales, whether the dynamics is dominated by the convective term or diffusive term. For the generalized Burgers equation

$$u_t + \left(\frac{1}{p}|u|^p\right)_x = u_{xx}$$

the answer is given by

Theorem 1
$$\gamma = \begin{cases} 1 + \frac{1}{2}(p-1) & \text{for } p \le 3 \\ 2 & \text{for } p \ge 3 \end{cases}$$
and the effective equations in the rescaled variables are (omitting the primes):

$$\begin{aligned} &\text{For } p < 3 \quad u_t + \left(\tfrac{1}{p}|u|^p\right)_x = 0 \\ &\text{For } p = 3 \quad u_t + \left(\tfrac{1}{p}|u|^p\right)_x = u_{xx} \\ &\text{For } p > 3 \quad u_t = u_{xx} \end{aligned}$$

This result says that the problem has diffusive behavior at large scales for $p > 3$ and convective behavior at large scales for $p < 3$. Similar questions can also be asked for initial data with different statistics. See [GMS, W]. Through the variational characterization of the solutions, the question is often reduced to studying the statistics of extreme values of the initial data. Such issues are dealt with in the book [L] and the result depends on the behavior of the tail of the one-point statistics.

Going back to the (inviscid) Burgers equation with white noise initial data. The solution has the scaling property
$$u(x,t) \stackrel{D}{=} t^{-2/3}\overline{u}\left(\frac{x}{t^{2/3}}\right)$$
where $\overline{u}(x) = u(x,1)$. It can be shown that $\overline{u}$ consists of straight lines (the ramps) of slope 1, separated by shocks. The study of $\overline{u}$ amounts to studying the convexification of the function $F(y) = \frac{y^2}{2} + B(y)$ where $B(y)$ is two-sided Wiener process.

The next question we address is the statistics of the strength of the shocks. The following result is proved by Avellaneda and E [AE].

Theorem 2 *There exists constants C_1 and C_2 such that*
$$C_1\sqrt{s} \le P\{S < s\} \le C_2\sqrt{s}$$
for $s << 1$.

Here P is the conditional probability that there is a shock at x_0 and S is the strength of that shock. Since the process is spatially homogenous, P does not depend on x_0.

This result was proved by adapting Pitman and Groenboom's method for studying the convex hull of the Wiener process using time reversal and conditional diffusion process techniques. See [G, Pi]. More precise result is obtained in the recent work of Frachebourg and Martin [FrM].

Finally we consider the statistics of large shocks. It turns out that the leading order asymptotics for the distribution of large shocks is given by the tail behavior of the primitive function of the initial data which is the two-sided Wiener process for the present case. The following argument captures the heart of the matter. To have a shock of amplitude λ at

$x = 0$, we must have $F(y) = \frac{y^2}{2} + B(y) \leq 0$ for $0 \leq y \leq \lambda$. The probability of such an event in smaller than

$$P\left\{\frac{\lambda^2}{2} + B(\lambda) \leq 0\right\} \leq e^{-\frac{1}{2\lambda}\left(\frac{\lambda^2}{2}\right)^2} = e^{-\frac{\lambda^3}{8}}$$

It turns out that aside from algebraic factors, this also provides a lower bound. This was first recognized in [AE, Av, ARE]

Theorem 3 *There exist constants C_1 and C_2 such that*

$$C_1 \leq s^{-3} \ln P\{S > s\} \leq C_2$$

as $s \to \infty$

The result in this form was proved by Molchan and Ryan independently [Mo, R]. They also proved various generalizations of this statement. To prove the rigorous bounds amounts to estimating certain large deviation probabilities associated with the Wiener process. See [Mo, R]. This argument can be generalized to initial data with more general statistics. For example, with Brownian motion initial data considered earlier, leading order asymptotics for the distribution for large shocks is given by

$$P\left\{\frac{\lambda^2}{2} + \int^{\lambda} B(z)dz < 0\right\} \asymp e^{-\frac{3}{8}\lambda}$$

for $\lambda >> 1$. To the best of the authors' knowledge, this is not yet rigorously proved.

Another interesting question that has been studied recently is the fluctuation of the shock position under the white noise perturbation [Fa, WX]. For Burgers equation, it is proved by Wehr and J. Xin [WX] that the statistics of the fluctuation is Gaussian.

3 Adhesion Dynamics and Generalizations of the Burgers Equation

Burgers equation is connected with the dynamics of the so-called sticky particles. Assume that at $t = 0$ we have a set of particles (x_i, v_i, m_i), $i = 1, \ldots, I$. Each particle moves with its velocity until it collides with another particle. At the collision a new particle is created whose mass and momentum are equal to the sum of masses and momenta of colliding particles. The whole process preserves the total mass and momentum. It can be described in the one-dimensional case by the system of two conservation laws

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2)_x &= 0. \end{aligned} \tag{5}$$

We shall consider also a more general case of particles interacting with the help of gravitational forces. In the one-dimensional case the gravitational force acting on a particle is proportional to the difference between the mass on the right and on the left of the particle. Formally it can be described by the system of equations

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2)_x &= -\rho \mathcal{R}_x \\ \mathcal{R}_{xx} &= \rho. \end{aligned} \tag{6}$$

However, the form of (5) and (6) is not general enough because in principle we have to consider arbitrary distributions of masses having no densities. General solution of (5) or (6) should be a family of finite measures P_t describing the distribution of mass. The distribution of momentum is given by a signed measure I_t absolutely continuous with respect to P_t so that the Radon-Nikodym derivative $\frac{dI_t(x)}{dP_t(x)} = u(x,t)$ is the velocity at x, t.

Definition 2 *(See [ERS]). A family $\{P_t,\, I_t,\, t>0\}$ is called weak solution of (5) if*

(i) *it is continuous in t in weak topology;*

(ii) *for any $f, g \in C_0^1(R^1)$, the space of C^1-functions with compact support, and $t_2 > t_1$.*

$$\int f(x)\, dP_{t_2}(x) - \int f(x)\, dP_{t_2}(x) = \int_{t_1}^{t_2} d\tau \int F'(y)\, dI_t(y) \tag{7'}$$

$$\int g(x)\, dI_{t_2}(x) - \int g(x)\, dI_{t_1}(x) = \int_{t_1}^{t_2} d\tau \int g'(y)\, u(y,t)\, dI_t(y) \tag{7''}$$

In the case of (6) the definition is analogous if we notice that the third equation can be replaced by

$$R_x(x,t) = P_t(x, \infty) - P_t(-\infty, x).$$

Then instead of (7'') we have

$$\int g(y)\, dI_{t_2}(y) - \int g(y)\, dI_{t_1}(y) =$$

$$= \int_{t_1}^{t_2} \cdot\, d\tau \int g'(y)\, u(y,t)\, dI_t(y) + \int_{t_1}^{t_2} d\tau \int R_y(y,t)\, dP_t(y).$$

A remarkable property of the equations (5), (6) is their integrability (in some sense). It was discovered by Martin and Piasecki (see [MP]). In [ERS] we rediscovered this not knowing about the paper [MP].

The idea is the following. Since the dynamics has an adhesive character for any $t > 0$ the set of particles which are sticked together is a closed segment on R^1. Denote the partition of R^1 onto these segments (which can be also points) by ξ_t.

The crucial remark is that the position of all particles $x \in C_t$ at time t where C_{ξ_t} is an element of partition of ξ_t is the position $x(t)$ of the center of mass of C_{ξ_t} moving in the case of (5) uniformly with the velocity $\bar{v}$ such that $X(t) = X(0) + I(C_{\xi_t})t$, $I(C_{\xi_t}) = \int_{C_{\xi_t}} u(x,0)\,dP_0$. In the case of particles with gravitational forces the rule is the same except that the trajectory of the center of mass is a quadratic function of t describing the motion with the constant acceleration.

Using this picture we can introduce the Lagrangian map $\mathcal{L}_t$ which maps x to the position of the center of mass of the element $C_{\xi_t}(x)$ containing x at time t. The generalized variational principle for the systems (5), (6) proven in [MP] and [ERS] says that $\mathcal{L}_t(x)$ is the trajectory of x.

The partition ξ_t can be constructed explicitly from the initial condition and in this sense (5) and (6) are integrable. We shall explain this for the system (5), in the case of (6) it is quite similar. It turns out that it is possible to characterize points x which are left end-points of the partition ξ_t. Namely, x is such a point if for every $y_1 < x < y_2$ $\bar{y}_1 + t\, I_t((y_1, x)) < \bar{y}_2 + t\, I_t([x, y_2])$ where $\bar{y}_1, \bar{y}_2$ are centers of mass of the sets (y_1, x) and $[x, y_2]$ respectively.

In [MP] Martin and Piasecki considered an ensemble of n gravitating points with the coordinates $\frac{ka}{n}$, $k = 1, \ldots, n$, masses $\frac{1}{n}$ and random initial velocities v_i having symmetric Gaussian distribution. They showed that there exists a non-random moment of time τ^* such that with probability tending to 1 as $n \to \infty$ before this moment all masses are macroscopically small while after τ^* the almost collapse takes place in the sense that one particle appears whose mass tends to 1 as $n \to \infty$. Recently T. Suidan proved a similar statement for a much wider class of distributions of the velocities.

4 1$\mathcal{D}$-Burgers Equation with Random Forcing

Consider the equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + \sum_k f_k'(x)\, B_k(t). \tag{7}$$

Here the last sum is finite, $B_k(t)$ are independent standard white noises, $f_k \in C^2$ are deterministic linearly independent functions. We consider (7) with periodic boundary condition on the interval $[0, 1)$ which requires periodicity of f_k with period 1. The extension to the whole line of the theory described below is apparently a very difficult problem. Using Hopf-Cole substitution we can reduce it to a problem about statistics of directed polymers which is known to be difficult.

The equation (7) is in fact a stochastic differential equation in a functional space.

The basic problem is the existence and uniqueness of the stationary measure for the arising Markov process. The case $\nu > 0$ was considered in [Si3] where the existence and uniqueness of the stationary measure were proven. The argument used some methods from statistical mechanics.

The case $\nu = 0$ which should be considered as the limiting for $\nu \to +0$ is much harder. Formally we have again the equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = \sum_k f_k'(x)\, B_k(t) \tag{8}$$

We shall restrict ourselves to the case $\int_0^1 u(x,t)\,dx = 0$. The general case can be considered in a similar way (see [EKMS1]).

The equation (8) has a hidden dissipation. To see this, consider (7) with $\nu > 0$ and write using Ito's formula

$$d\int_0^1 \frac{u^2(x,t)}{2}\,dx = \int_0^1 \frac{u^2(x,t+dt) - u^2(x,t)}{2}\,dx =$$

$$= \int_0^1 u(x,t)\,(u(x,t+dt) - u(x,t))\,dx + \frac{1}{2}\int_0^1 (u(x,t+dt) - u(x,t))^2\,dx =$$

$$= -\frac{1}{2}\int_0^1 u(x,t)\frac{\partial}{\partial x}u^2(x,t)\,dxdt - \nu\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dxdt - \int_0^1 u'(x,t)\sum_k f_k(x)\,dxB_k(t)\,dtdx$$

$$+\frac{1}{2}\sum_k\int_0^1 f_k^2(x)\,dx\,dt = -\nu\int_0^1\left(\frac{\partial u}{\partial x}\right)^2 dx\,dt$$

$$-\sum_k\int_0^1 u_k'(x,t)\,f_k(x)\,dx\,B_k(t)\,dt + \frac{1}{2}\sum_k\int_0^1 f_k^2(x)\,dx\,dt.$$

Taking the expectation of both sides we have

$$\frac{d}{dt}E\int_0^1\frac{u^2(x,t)}{2}\,dx \geq -8\pi^2\nu\, E\int_0^1\frac{u^2(x,t)}{2}\,dx + \frac{1}{2}\sum_k\int_0^1 f_k^2(x)\,dx.$$

This shows that $E\int \frac{u^2(x,t)}{2}\,dx$ remains bounded. As $\nu \to 0$ the main contribution to $\nu\int_0^1\left(\frac{\partial u)}{\partial x}\right)^2 dx$ comes from small neighborhoods of shocks. The theory which is discussed below demonstrates

$$\lim_{\nu\to 0}\nu\int_0^1\left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{1}{12}\sum_i\,[u(x_i - 0,t) - u(x_i + 0,t)]^3$$

where the last sum goes over all shocks. It follows from the so-called entropy condition (see [O], [Sm]) that each expression in the square brackets is positive.

We construct in [EKMS1] the stationary measure for (8) explicitly. To describe our approach we need some notations. Denote by $\mathcal{F}_{t_1}^{t_2}$, $t_1 < t_2$, the least σ-algebra generated by all $B_k(t)$, $t_1 \leq t \leq t_2$. If $\{S^\tau\}$ is the one-parameter group of shifts, $S^\tau B_k(t) = B_k(t+\tau)$, then $\{S^\tau\}$ preserves the natural measure P defined on all $\mathcal{F}_{t_1}^{t_2}$ and is ergodic and mixing *wrt* to P.

Assume now that we succeeded in constructing a special functional $M(B)$ which is measurable *wrt* $\mathcal{F}_\infty^0$ and takes values in the Skorokhod space $\mathcal{D}_0$ of functions with discontinuities of the first kind whose integral is zero. This space is a natural space for solutions of (8). The main property of this functional is that $u(x,t) = M\, S^t(B)$ is a weak solution of (8). Then due to the invariance of P under the group of shifts $\{S^\tau\}$ the probability distribution of $M(B)$ gives rise to a stationary measure for the Markov process (8).

The functional M has a special form which is described below.

Consider an arbitrary piece-wise C^1-function $x(t) : R^- \to S^1$ and the formal integral

$$A(\{x(t)\}) = \int_{-\infty}^{0} \left[\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + \sum_k f_k(x)\, B_k(t) \right] dt$$

We shall call it action. The last part is a stochastic Ito integral. However, the well-known difficulties of stochastic calculus do not appear here because we can make the integration by parts and reduce everything to the integration of piece-wise continuous functions.

Definition 3 *A function $\bar{x}(t)$ is called a one-sided minimizer if $A\{x(t)\}$ is minimum wrt all local continuous perturbation, i.e. for any $x(t)$ such that $x(t) = \bar{x}(t)$ for all $t \leq t'$*

$$A(x(t)) - A(\bar{x}(t)) = \int_{t'}^{0} \left[\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + \sum_k f_k(x)\, B_k(t) \right.$$
$$\left. - \frac{1}{2} \left(\frac{d\bar{x}}{dt} \right)^2 - \sum_k f_k(\bar{x})\, B_k(t) \right] dt \geq 0.$$

It is easy to show that each one-sided minimizer satisfies Euler-Lagrange equation which in our case takes the form

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = \sum f_k'(x)\, B_k(t) \tag{9}$$

It must be also understood as a stochastic differential equation.

Not every solution of (9) is a one-sided minimizer. We describe below some special properties of minimizers.

Theorem 4 *With probability 1 for every point $x \in S^1$ there exists at least one one-sided minimizer $\bar{x}(t)$ for which $\bar{x}(0) = x$.*

It is a general fact of variational calculus that any two one-sided minimizers cannot have two intersections. In our probabilistic setting a stronger statement holds.

Theorem 5 *With probability* 1 *any two one-sided minimizers cannot intersect. More precisely, if two one-sided minimizers intersect at some moment of time then their continuations as solutions of (9) are no longer one-sided minimizers.*

It follows easily from Theorem 2 that the set of $x \in S^1$ which are end-points of several minimizers is at most countable. Now we describe the functional M. Put $M(B)(x) = \frac{d\bar{x}}{dt}$ if the one-sided minimizer $\bar{x}(t)$ for which $\bar{x}(0) = x$ is unique. For all other points $\lim_{x' \to x-0} M(B)(x')$, $\lim_{x'' \to x+0} M(B)(x'')$ exist and $\lim_{x' \to x-0} M(B)(x') > \lim_{x'' \to x+0} M(B)(x'')$. In other words the points of non-unicity of one-sided minimizers are shocks. We shall denote $M(B) = \bar{u}(x, 0)$.

It follows from the Lax-Oleinik variational principle (see [Lax], [O]) that $\bar{u}(x,t) = M(S^t B)$, $-\infty < t < \infty$, is a weak solution of the inviscid Burgers equation (8) (see also [EKMS1]). This gives the existence of at least one stationary measure for (8). The uniqueness will follow from other properties of (8).

The following theorem gives the existence of the so-called main shock.

Theorem 6 *There exists a unique continuous curve $\bar{y}(t)$, $-\infty < t < \infty$, such that $\bar{u}(x,t)$ is discontinuous at $x = \bar{y}(t)$.*

This curve is called the main shock of our solution $\bar{u}(x,t)$. It is possible to show that all other shocks eventualy merge with the main shock. In this sense the main shock serves as the attractor of all one-sided minimizers.

Definition 4 *A curve $\bar{\bar{x}}(t) : R^1 \to S^1$ is called two-sided minimizer if it gives minimum to the two-sided action*

$$A(x) = \int_{-\infty}^{\infty} \left[\frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 + \sum f_k(x)\, B_k(\tau) \right] d\tau$$

wrt to all local continuous perturbations.

Theorem 7 *With probability* 1 *the two-sided minimizer $\bar{\bar{x}}(t)$ exists and is unique.*

It plays a very special role in our theory. It turns out that it is a hyperbolic trajectory of the system (9) in the sense that it has stable and unstable manifolds which in our situation are one-dimensional curves. The main conclusion is the following theorem (see [EKMS1]).

Theorem 8 *The solution $\{\bar{u}(x,0)\} = M(B)$ is a subset of the unstable manifold of the two-sided minimizer. More precisely, if $\gamma^{(u)}$ is this unstable manifold then $(x, \bar{u}(x,0)) \in \gamma^{(u)}$. With probability* 1 *the number of shocks of $\bar{u}(x,0)$ is finite.*

5 Several Asymptotics for the Stationary Distribution

Denote by μ_0 the stationary probability measure corresponding to (8) which is the distribution of $\bar{u}(x, 0)$. It has several properties which follow more or less directly from the construction.

1°. If μ_ν is the analogous measure for the viscous Burgers equation then

$$\lim_{\nu \to 0} \mu_\nu = \mu_0$$

The convergence here is in the weak topology (see [EKMS1]).

2°. The Markov process (8) has a unique stationary measure (see [EKMS1]).

The whole theory outlined in §4 allows to get also several quantitative results (see[EKMS2]).

6 Asymptotic Behavior of the Probability Density Functions (PDFs)

Next we consider the asymptotic behavior of the probability distribution function (PDF) of u_x and $\delta u = u(x + r, t) - u(x)$ for our stationary measure.

6.1 PDF of velocity gradient

This question has attracted a lot of attention recently. Let $Q(\xi)$ be the PDF of the regular part of $\xi = u_x$. This is well-defined since it is proved in [EKMS1] that shocks are isolated and away from the shocks the solution is smooth. It was suggested for the first time by Polyakov (see [Po]) that Q has the following asymptotics:

$$Q(\xi) \sim \begin{cases} C_-|\xi|^{-\alpha}, & \text{as } \xi \to -\infty \\ C_+\xi^\beta e^{-\xi^3/3B_1} & \text{as } \xi \to +\infty \end{cases} \tag{10}$$

where B_1 is a constant associated with the forcing. However, a variety of values for the exponents α and β were predicted in various papers as a consequence of the variety of techniques used. By invoking operator product expansion, Polyakov [Po] suggested that $\alpha = \frac{5}{2}$ and $\beta = \frac{1}{2}$. Boldyrev extended Polyakov's argument and suggested that $\alpha \in [2, 3]$ [Bo]. E et.al. [EKMS2] suggested that the main contribution for large negative values of ξ comes from the pre-shocks, i.e. the small regions near the points of shock creation. This argument gives the value $\alpha = \frac{7}{2}$. Gotoh and Kraichnan argued that the viscous term can be neglected to leading order, giving rise to the values $\alpha = 3$, and $\beta = 1$ [GK].

Let Q^ν be the PDF for the statistical stationary state of the viscous problem. Assuming statistical homogeneity, Q^ν satisfies the following master equation

$$B_1 Q^\nu_{\xi\xi} + \xi Q^\nu + (\xi^2 Q^\nu)_\xi - \nu(< \xi_{xx}|\xi > Q^\nu)_\xi = 0 \tag{11}$$

where $< \xi_{xx}|\xi >$ is the average of ξ_{xx} conditional on ξ. B_1 is the same parameter occurred earlier. This equation is not closed because of the viscous term and the crux of the matter is how to evaluate this term. In [EV1, EV2], E and Vanden Einjden made an evaluation of this term using formal matched asymptotics and in the limit as $\nu \to 0$, they obtain

$$F(\xi) = -\lim_{\nu\to 0} \nu(< \xi_{xx}|\xi > Q^\nu)_\xi = \varrho \int_{-\infty}^{0} sV(s,\xi)ds \tag{12}$$

where ϱ is the mean number of shocks, $V(s,\xi) = \frac{1}{2}(V_+(s,\xi) + V_-(s,\xi))$ and $V_\pm(s,\xi)$ are the PDFs of $(s(y_0,t), \xi_\pm(y_0,t) = u_x(y_0\pm,t))$ conditional on y_0 being a shock location. The equation for $Q(\xi) = \lim_{\nu\to 0} Q^\nu(\xi)$ is then

$$B_1 Q_{\xi\xi} + \xi Q + (\xi^2 Q)_\xi + F(\xi) = 0. \tag{13}$$

An alternative definition of Q is $Q(\xi) = \lim_{\delta\to 0} Q^\delta(\xi)$ where Q^δ is the PDF of the divided difference $(u(x+\delta,t) - u(x,t))/\delta$. It is not yet rigorously proved that these two definitions give the same answer although the calculation in [EV2] using matched asymptotics strongly suggests that they do. Nevertheless, for the second definition of Q, it can be proved, using BV-calculus developed by Volpert [Vo] and more generally in geometric measure theory, that Q satisifes the equation (13).

To proceed further, E and Vanden Einjden made the following assumptions:

1. Solutions of (7) are piecewise smooth in the (x,t) plane.

2. Shocks are created at zero amplitude and shock strength adds up at collision.

These statements are slightly stronger than the ones proved in [EKMS1]. It is believed that the techniques developed in [EKMS1], with some improvement, are enough to prove these statements, but this has not been done yet. Nevertheless under these assumptions, E and Vanden Einjden proved that Q has the following representation

$$Q(\xi) = |\xi|^{-3} \int_{\infty}^{\xi} \xi' F(\xi') d\xi' + O(|\xi|^{-6}) \tag{14}$$

Furthermore, $G(\xi) = \xi F(\xi)$ is absolutely integrable on R^1. This implies immediately that

$$\lim_{\xi\to-\infty} |\xi|^3 Q(\xi) = 0 \tag{15}$$

Assuming that Q has the form (10), this eliminates the possibility that $\alpha \le 3$, leaving out the only candidate $\alpha = \frac{7}{2}$.

In order to proceed further, E and Vanden Einjden derived master equations for $V_\pm(s,\xi)$. Based on that, a self-consistent asymptotic argument suggests that the main contributions to $F(\xi)$ comes from shock creation. This confirms the main assumption made in [EKMS2] and their prediction $\alpha = \frac{7}{2}$. This circle of ideas are not fully rigorous. But they are strongly suggestive that the correct answer is $\alpha = \frac{7}{2}$ and $\beta = 1$. Furthermore, this is among the very few cases where specific answers are obtained from the master equation without making uncontrolled closure assumptions. It must also be remarked that even though Polyakov's predictions for the exponents α and β were incorrect [Po], by writing down equations of the type (11) for the first time, he made it clear that the difficulty with closing the master equations is associated with the so-called "dissipative anomaly", not the nonlinear term. This observation has led to other very interesting predictions.

6.2 PDF for the velocity difference

Now let us consider $\lambda = \delta u(r) = u(x+r,t) - u(x,t)$. It is fairly clear that the PDF for the right tail of λ, $Q(\lambda, r)$ should behave in the same way as Q, see (10). It is also clear that the extreme left tail of $Q(\lambda, r)$ is dominated by large shocks, which should be $e^{-C|\lambda|^3}$. In the middle, there are at least two more regimes. Near the origin, for small values of λ, $\delta u(r) \approx r u_x = r\xi$. Therefore this regime should scale as $|\lambda|^{-\frac{7}{2}}$. Moving to the left, between this regime and the far left tail, there should be a regime dominated by the statistics of small shocks. Very little is known about this regime. Any insight, including careful numerical results, is very welcome!

7 Concluding Remarks

It is important to understand in which sense the theory of inviscid Burgers equation (8) is special.

In the theory of Dynamical systems a beautiful theory of twist maps known also as Aubry-Mather theory was developed twenty years ago (see [A], [Ma], [Si5]). It can be described as related to two-dimensional maps of the cylinder $R^1 \times S^1$ which in the simplest case have the form

$$T(x, \varphi) = (z', \varphi'),$$

$z' = z + V'(\varphi)$, $\varphi' = \varphi + z' \pmod 1$, and V is a smooth periodic function. The discrete version of the Burgers equation with random forcing can be described as the sequence (z_n, φ_n) where

$$z_{n+1} = z_n + \xi_n, \varphi_{n+1} = \varphi_n + z_{n+1}$$

and ξ_n is a sequence of independent identically distributed random variables. However, it should be stressed that the transition from (8) to (10) is not straight-forward and the analogue of Aubry-Mather theory for (10) is not done so far.

Bec, Frisch, Khanin (see [BFK]) considered the case which they called kicked Burgers equation, where the forcing acts at the discrete moments of time and derived results similar to the ones in [EKMS2]. See also [FBV].

Our theory described in §4, §5 is related also to the Baxendale theory of stochastic dynamical systems (see [Ba]). It follows from the general theory that such systems have several positive, i.e. unstable, Lyapunov exponents and several negative, i.e. stable, Lyapunov exponents. Due to the hidden dissipation which was explained at the beginning of §4 Burgers system (8) has actually only negative Lyapunov exponents. One can hope that our approach works for the whole class of such systems. For example, there are some reasons to expect that it will work for the multi-dimensional Burgers equation. At the same time Navier-Stokes systems with arbitrary viscosity can have presumably any number of positive Lyapunov exponents and the situation there is quite different. We discuss Navier-Stokes system in the second part of this text.

Part II
Navier-Stokes Systems with Periodic Boundary Conditions

8 General Remarks Concerning NSS

The general d-dimensional Navier-Stokes system (NSS) is written for d unknown functions $u(x,t) = (u_1(x,t), \ldots, u_d(x,t))$ of d variables $x = (x_1, \ldots, x_d)$ and time t and the pressure $p(x,t)$.

$$\frac{\partial u_i}{\partial t} + \sum u_k \frac{\partial u_i}{\partial x_k} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f^{(i)}. \tag{16}$$

We consider incompressible liquides with the density $\rho \equiv 1$. In this case we have an additional equation

$$\operatorname{div} u = \sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i} \equiv 0 \tag{17}$$

The functions $f^{(i)}$ are the components of the external forcing, $\nu > 0$ is the viscosity.

The equations (16), (17) play the central role in hydrodynamics, especially in connection with turbulence. In mathematical theory the basic question is how complicate can be solutions of (16), (17).

The integral $E(u) = \frac{1}{2} \int u^2(x,t)dx = \frac{1}{2} \int \sum_{i=1}^{d} u_i^2(x,t)dx$ is the energy. Usually people are interested in solutions of (16) (17) with finite $E(u)$.

The first simplification is to consider (16), (17) with periodic boundary conditions which allows us to use Fourier series. We write

$$u(x,t) = \sum_{k \in \mathbb{Z}^d} u_k(t) e^{2\pi i (k,x)}.$$

Here $u_k \in R^d$ for each k, the incompressibility condition is reduced to the requirement $u_k \perp k$, $k \in \mathbb{Z}^d$. Instead of (16), (17) we have

$$\frac{du_k}{dt} = -2\pi i \sum_{k_1 \in \mathbb{Z}^d} (u_{k_1}, k) \sqcap_k u_{k-k_1} - \nu |k|^2 u_k + f_k \tag{18}$$

Here $\sqcap_k$ is the operator of orthogonal projection to the $(d-1)$-dimensional plane orthogonal to k, and f_k are components of the external forcing. For simplicity we assume that only finitely many f_k are non-zero. In the finite energy case $\sum_k |u_k|^2 < \infty$. Sometimes the system (18) is called Galerkin system for the initial NSS. Finite-dimensional Galerkin approximations for (18) appear if in (18) $k_1, k, k - k_1 \in O$ where O is a finite convex symmetric domain (i.e. k and $-k$ belong to O simultaneously). We shall use the notation (18_O) for the corresponding system. Let us formulate the following simple lemma.

Lemma 1 *If $E(u) < \infty$ then for each k the right-hand side of (18) is finite.*

This lemma shows that at least formally the vector field corresponding to (18) is well-defined on the whole Hilbert space $L^2(\mathbb{Z}^d)$ of u with finite $E(u)$. Certainly, the lemma does not imply that the rhs also belongs to $L^2(\mathbb{Z}^d)$. Introduce the following definition.

Definition 5 *Strong solution of (18) on the interval $[0, T]$ is a one-parameter family $\{u_k(t)\} \in L^2(\mathbb{Z}^d)$ such that for each k the function $u_k(t)$ is differentiable in t and (18) hold for each $k \in \mathbb{Z}^d$.*

Local existence of strong solutions means that for a given $\{u_k(0), k \in \mathbb{Z}^d\}$ one can find a closed interval $[0, t_0]$ such that a strong solution $\{u_k(t), k \in \mathbb{Z}^d\}$, $t \in [0, t_0]$ exists. In other words, there is a finite piece of trajectory of NSS which goes out of $\{u_k(0)\}$. Global existence means that the whole trajectory $\{u_k(t), t \geq 0\}$ going out of $\{u_k(0)\}$ exists. The local existence statement is certainly much simpler (see below). One can introduce analogous definitions for any finite Galerkin approximation (18_O).

The problem of existence and uniqueness of solutions for NSS was considered for the first time by J. Leray in his classical paper [Le] in 1934. He proved the global existence of weak solutions for the Cauchy problem in R^3. Later Hopf [Ho2] extended these results to bounded domains with zero boundary conditions. The existence of strong solutions was proven in the two-dimensional case by Ladyzenskaya in the sixties (see [La]). Other methods were developed in the works of Foias and Temam (see [FT] and the review paper [T]), Yudovich [Y], Constantin-Fioas [CF] and others. However, the main problem of global existence of strong solutions in the three-dimensional case remains open. Some people including one of the authors of the present paper even do not believe in a positive answer.

9 Existence Theorems of Global Solutions of NSS

The methods of the proof of global existence theorem are based on the energy and enstrophy estimates. Take any system (18_O) and put $E_O(u) = \sum_{k \in O} |u_k|^2$.

Lemma 2 *(Energy Estimate). Let $\{u_O(t)\}, t \in [t_1, t_2]$ be a strong solution of (18_O), O is finite or $\mathbb{Z}^d$. Then*

$$\frac{d}{dt} E(u_O(t)) \leq -4\pi^2 \nu \, E(u_O(t)) + 2F\sqrt{E(u_O(t))},$$

where $F = \sum_k |f_k|^2$.

The proof is straight-forward (see for example [Si4]). We have

$$\begin{aligned} \tfrac{d}{dt} E(u(t)) &= \tfrac{d}{dt} \tfrac{1}{2} \sum_{k \in O} (u_k, u_k) = \frac{1}{2} \sum_{k \in O} \left(\frac{du_k}{dt}, u_k \right) + \frac{1}{2} \sum_{k \in O} \left(u_k, \frac{du_k}{dt} \right) = \\ &= -\pi i \sum_{k, k_1 \in O} (u_{k_1}, k)\,(u_{k-k_1}, u_k) - \pi i \sum_{k, k_1 \in O} (u_{k_1}, u_{k-k_1}) \cdot (u_{k_1}, k) \\ &\quad - 4\pi^2 \nu \sum_{k \in O} |k|^2 (u_k, u_k) + \sum_{k \in O} (f_k, u_k) + \sum_{k \in O} (u_k, f_k). \end{aligned}$$

It is easy to show that the first two sums are identically zero. A simple estimate of the other terms gives the result.

Lemma 2 shows that if the initial energy is finite then it remains bounded on the whole interval where a strong solution exists.

Theorem 9 (Local Existence Theorem) *Suppose that $\{u_k(0)\}$ is such that $\sum_{k \in \mathbb{Z}^d} |u_k(0)| = h < \infty$. Then the strong solution of (12) on the interval $[0, t_0]$, $t_0 = \frac{c\nu}{h}$, exists where c is a dimension-dependent absolute constant.*

There exist other more strong results which are not covered by this theorem (see [K], [M1]). One example is given by the following theorem.

Theorem 10 (Kaloshin, Sannikov) *Let $d = 3$ and $|u_k(0)| \leq \frac{c}{|k|^\gamma}$, $\gamma \geq 2$. If c is small enough then there exists a strong solution of (12) for all $t > 0$.*

There are several methods which give the local existence results. However, the values of t_0 are of the same order og magnitude.

The proof of global existence theorem is based on the notion of enstrophy. In what follows we assume $d = 2$ or 3.

Definition 6 *Enstrophy of $\{u_k, k \in O\}$ is*

$$V_O(\{u_k, k \in O\}) = \sum_{k \in O} |k|^2 \, |u_k|^2.$$

If $O = \mathbb{Z}^d$ we shall write $V\{u_k\})$. The basic result says that if for a strong solution of (18) or (18_O) the enstrophy $V_O(\{u_k(t),\, k \in O\})$ remains bounded then the modes u_k decay exponentially. Here is a more precise formulation.

Theorem 11 *Assume that $\{u_k(0), k \in O \text{ or } \mathbb{Z}^d\}$ is such that for some $\mathcal{D} < \infty$, $\gamma > \frac{d}{2} + 1$*

$$|u_k(0)| \,\leq\, \frac{\mathcal{D}}{|k|^{\gamma}}, k \in 0 \text{ or } \mathbb{Z}^d \tag{19}$$

and for a strong solution $\{u_k(t)\}$

$$V(\{u_k(t), k \in O \text{ or } \mathbb{Z}^d\}) \,\leq\, V_0, 0 \leq t \leq T.$$

Then there exist positive α, $\mathcal{D}_0$ such that for all k.

$$|u_k(t)| \,\leq\, \frac{\mathcal{D}_0\, e^{-\alpha t \cdot |k|}}{|k|^{\gamma}},\ 0 \,\leq\, t \,\leq T. \tag{20}$$

This theorem was proven by Foias and Temam (see [FT]). The proof below follows the papers [MS] and [Si4]. We start with the following preliminary statement.

Lemma 3 *If $\{u_k(0)\}$ satisfies (19) and the enstrophy is bounded then one can find another constant $\mathcal{D}_1$ such that $|u_k(t)| \,\leq\, \frac{\mathcal{D}_1}{|k|^{\gamma}}$ for all $0 \leq t \leq T$. This constant does not depend on O.*

Proof. In this lemma and in the subsequent lemmas we shall consider for simplicity the real version of (18)

$$\frac{du_k}{dt} \,=\, 2\pi \sum (u_{k_1}, k) \sqcap_k\, u_{k-k_1} \,-\, \nu\, k^2\, u_k \,+\, f_k. \tag{21}$$

Also we assume that u_k are one-dimensional. This is also a technical simplification. Take large enough K whose value will be determined later. Since $V(\{u_k(t), k \in \mathbb{Z}^d\}) \,\leq\, V_0$ we can find $\mathcal{D}_1(K)$ such that

$$|u_k(t)| \,\leq\, \frac{\sqrt{V_0}}{|k|} \,\leq\, \frac{\mathcal{D}_1(K)}{|k|^{\gamma}},\ |k| \,\leq\, K.$$

Let us prove that $|u_k(t)| \,<\, \frac{\mathcal{D}_1(K)}{|k|^{\gamma}}$ for all $k, |k| \,>\, K$ provided that K is large enough. Suppose that this is wrong and for some values of $\bar{k}$, $|\bar{k}| > K$ and $\bar{t} \leq T$ we have

$$|u_{\bar{k}}(\bar{t})| \,=\, \frac{\mathcal{D}_1(K)}{|\bar{k}|^{\gamma}}.$$

Take the smallest $\bar{t}$ and consider the case

$$u_{\bar{k}}(\bar{t}) \,=\, \frac{\mathcal{D}_1(K)}{|\bar{k}|^{\gamma}}.$$

The other case is considered in a similar way. Then we should have

$$\frac{du_{\bar{k}}(\bar{t})}{dt} \geq 0 \tag{22}$$

Our arguments below show that this is impossible because the viscous term dominates. Namely,

$$\nu|\bar{k}|^2 u_{\bar{k}} > \nu \mathcal{D}_1(K) \cdot |\bar{k}|^{2-\gamma} \tag{23}$$

We may always assume K is so large that $f_k = 0$ for $|k| \geq K$. In order to estimate the sum

$$\left| \sum_{k_1} (u_{k_1}, \bar{k}) \cdot \sqcap_{\bar{k}} u_{\bar{k}-k_1} \right| \leq \sum_{k_1} |u_{k_1}| \left|\bar{k} - k_1\right| \cdot \left|u_{\bar{k}\cdot k_1}\right|$$

we shall consider three cases.

Case I. $|k_1| \leq \frac{1}{2}|\bar{k}|$.

Here $|\bar{k} - k_1| \geq \frac{1}{2}|\bar{k}|$ and therefore $|u_{\bar{k}-k_1}|\,|\bar{k} - k| \leq \frac{2^\gamma \cdot \mathcal{D}_1(K)}{|\bar{k}|^{\gamma-1}}$. From the other side by Cauchy-Schwarz inequality

$$\sum_{|k_1| \leq \frac{1}{2}\bar{k}} |u_{k_1}| \leq \sum_{|k_1| \leq \frac{1}{2}|\bar{k}|} |k_1| \cdot |u_{k_1}| \cdot \frac{1}{|k_1|} \leq \sqrt{\sum |k_1|^2 |u_{k_1}|^2} \cdot \sqrt{\sum \frac{1}{|k_1}|^2} \leq \text{const}\sqrt{V} \cdot |\bar{k}|^{\frac{1}{2}}.$$

The last inequality is valid for $d \leq 3$. We see that the viscous term in (21) dominates if K is large enough.

Case II. $\frac{1}{2}|\bar{k}| < |k_1| \leq 2|\bar{k}|$.

In this case $|u_{k_1}| \leq \frac{2^\gamma \mathcal{D}_1(K)}{|k_1|^\gamma}$ and again

$$\sum_{\frac{1}{2}|\bar{k}| \leq |k_1| \leq 2|\bar{k}|} |u_{\bar{k}_1}| \cdot |k - k_1| \cdot |u_{k-k_1}| \leq \frac{2^\gamma \mathcal{D}_1(K)}{|\bar{k}|^{\gamma-1}} \cdot \sum |u_{\bar{k}-k_1}| \cdot |\bar{k} - k_1|.$$

The same arguments as in I give

$$\sum |u_{\bar{k}-k_1}|\,|\bar{k} - k_1| \leq \text{const} \sqrt{\sum_{|k| \leq 3|\bar{k}|} |u_k|^2 |k|^2} \cdot |\bar{k}|^{3/2} \leq \text{const}\sqrt{V}\,|\bar{k}|^{3/2}.$$

We see again that for $d \leq 3$ the viscous term dominates for large enough K.

Case III. $|k_1| > 2|\bar{k}|$.

Here $|\bar{k} - k_1| > |\bar{k}|$. Therefore

$$\sum |u_{k_1}|\,|\bar{k}-k_1|\,|u_{\bar{k}-k_1}| \le \frac{1}{|\bar{k}|}\sum |u_{k_1}|\cdot|k_1|\cdot|\bar{k}-k_1|\,|u_{\bar{k}-k_1}| \le$$

$$\cdot\frac{1}{|\bar{k}|}\cdot\sqrt{\sum |u_{k_1}|^2\cdot|k_1|^2}\cdot\sqrt{\sum |\bar{k}-k_1|^2\,|u_{\bar{k}-k_1}|^2} \le$$

$$\le \frac{\sqrt{V}}{|\bar{k}|}\cdot\mathcal{D}_1(K)\sqrt{\sum_{|k_1|\ge 2|\bar{k}|}\frac{1}{|k_1|^{2\gamma-2}}} \le$$

$$\le \frac{\mathrm{const}\sqrt{V}\cdot\mathcal{D}_1(K)}{|\bar{k}|}\cdot\frac{|\bar{k}|^{1+d/2}}{|\bar{k}|^{\gamma}}.$$

Again the viscous term dominates and this completes the proof of Lemma 3.

Now we shall finish the proof of Theorem 6. Consider the case $O=\mathbb{Z}^d$. In view of Lemma 3 for all $k\in\mathbb{Z}^d$

$$|u_k(t)| \le \frac{\mathcal{D}_1}{|k|^{\gamma}},\ 0\le t\le T$$

Therefore for any K we can find $\alpha(K)=\alpha$ such that for all k, $|k|<K$

$$|u_k(t)| \le \frac{2\mathcal{D}_1}{|k|^{\gamma}}\,e^{-\alpha t|k|},\ 0\le t\le T. \tag{24}$$

Again we shall consider the real version of (18).

$$\frac{du_k}{dt} = 2\pi\sum_{k_1}(u_{k_1},k)\sqcap_k u_{k-k_1} - \nu|k|^2 u_k + f_k. \tag{12$'$}$$

Put $v_k(t)=e^{\alpha|k|t}u_k(t)$. Then for v_k we have the following system of equations

$$\frac{dv_k(t)}{dt} = \sum(v_{k_1},k)\sqcap_k v_{k-k_1}\cdot\frac{e^{-\alpha|k_1|t-\alpha|k-k_1|t}}{e^{-\alpha|k|t}}$$

$$-(\nu|k|^2-\alpha|k|)\,v_k + e^{\alpha|k|t}f_k \tag{25}$$

Now our arguments will be practically the same as in the proof of Lemma 3. For $|k|\le K$ we have the needed estimates which can be rewritten as

$$|v_k(t)| \le \frac{2\mathcal{D}_1}{|k|^{\gamma}}.$$

Assume that for some $\bar{k}$, $|\bar{k}|>K$ one can find t_0, $0<t_0\le T$ where $|v_{\bar{k}}(t_0)|=\frac{2\mathcal{D}_1}{|\bar{k}|^{\gamma}}$. Again for definiteness we consider the case $v_{\bar{k}}(t_0)=\frac{2\mathcal{D}_1}{|\bar{k}|^{\gamma}}$. We may also assume that t_0 is the smallest number with the needed properties. Then we should have $\frac{dv_{\bar{k}}(t_0)}{dt}\ge 0$. The

contradiction with this inequality comes in the same way as in Lemma 3. We remark that $e^{-\alpha|k_1|t-\alpha|k-k_1|t} \le e^{-\alpha|k|t}$ since $|k_1|+|k-k_1| \ge |k|$. Instead of finiteness of the enstrophy we shall use everywhere the estimates $|v_k(t_0)| \le \frac{2\mathcal{D}_1}{|k|^\gamma}$. The details are left to the reader.

In the two-dimensional case one can prove the existence of strong solutions for all $t > 0$ under very mild assumptions on initial conditions. In $d = 2$ the components of the velocity $u^{(1)}(x_1, x_2, t)$, $u^{(2)}(x_1, x_2, t)$ satisfy the incompressibility condition $\frac{\partial u^{(1)}}{\partial x_1} + \frac{\partial u^{(2)}}{\partial x_2} = 0$. Therefore we can introduce the vorticity $\omega = \frac{\partial u^{(1)}}{\partial x_2} - \frac{\partial u^{(2)}}{\partial x_1}$ so that the enstrophy becomes proportional to the L^2-norm of vorticity. It satisfies the so-called enstrophy inequality which is analogous to the energy inequality.

Lemma 4 *For any solution of (12) or* (12_O)

$$\frac{dV\{u_k(t)\})}{dt} \le -4\pi^2\,\nu V(\{u_k(t)) + 2F\sqrt{V(\{u_k(t)\})}.$$

The proof can be founded in many text-books, see also [Si4].

This lemma shows that if the enstrophy at $t = 0$ is finite then it remains bounded on any time-interval. Then Theorem 6 immediately implies the existence of global strong solutions for $d = 2$.

As we already mentioned the problem of existence of global strong solutions of NSS for $d = 3$ remains a challenging open problem.

10 Two-dimensional Navier-Stokes System with Periodic Boundary Conditions and Random Forcing

Navier-Stokes system with random forcing is considered as the most popular model of turbulence. However, some people express their concerns about its relevance to the actual hydrodynamics. We shall begin with the case $d = 2$ since it was studied more because the existence problems even in random case are simpler.

The system of equations has the same form as above:

$$\begin{aligned}
\frac{\partial u^{(1)}}{\partial t} + u^{(1)}\frac{\partial u^{(1)}}{\partial x_1} + u^{(2)}\frac{\partial u^{(1)}}{\partial x_2} &= \nu\Delta u^{(1)} + g^{(1)} \\
\frac{\partial u^{(2)}}{\partial t} + u^{(1)}\frac{\partial u^{(2)}}{\partial x_1} + u^{(2)}\frac{\partial u^{(2)}}{\partial x_2} &= \nu\Delta u^{(2)} + g^{(2)} \\
\frac{\partial u^{(1)}}{\partial x_1} + \frac{\partial u^{(2)}}{\partial x_2} &= 0
\end{aligned} \tag{26}$$

with $g = (g^{(1)}, g^{(2)})$ being a random function of space and time. It is convenient to consider the vorticity $\omega = \frac{\partial u^{(1)}}{\partial x_2} - \frac{\partial u^{(2)}}{\partial x_1}$ and to write for it the equation

$$\frac{\partial \omega}{\partial t} + u^{(1)}\frac{\partial \omega}{\partial x_1} + u^{(2)}\frac{\partial \omega}{\partial x_2} = \nu\Delta\omega + f$$

with $f = \frac{\partial g^{(1)}}{\partial x_2} - \frac{\partial g^{(2)}}{\partial x_1}$. If we expand ω into Fourier series $\omega = \sum_{k\in\mathbb{Z}^2} w_k \, e^{2\pi i(k,x)}$ we have for w_k the system of equations

$$\frac{dw_k}{dt} = -2\pi i \sum_{k_1\in\mathbb{Z}^2} w_{k_1} \, w_{k-k_1} \, \frac{(k, k_1^{\perp})}{(k_1, k_1)} - \nu |k|^2 \, w_k + f_{L^k}(t), w_{-k} = \bar{w}_k. \tag{27}$$

We assume that f_K are independent white noises except the relations $f_{-k} = \bar{f}_k$ which imply the solutions of (27) to be real. (27) can be written as an infinite-dimensional system of stochastic differential equations. Various existence theorems valid in this situation can be found in [DPZ], [Ku], [FV].

We shall restrict ourselves to general properties of (27) and make a simplifying assumption: $f_k = 0$ if $|k| > K_0$. We shall need the notion of determining modes introduced by C. Foias (see [T]). Take K and arbitrary functions $\bar{w}_k(t)$, $-\infty < t \le 0$, $|k| \le K$. Consider the system (27) for $|k| \ge K$ and $t \ge -T$ with initial conditions $w_k(-T) = 0$, $k > K$. Denote this solution by $w_k(t; -T)$ and assume the existence of the limit $\lim_{T\to\infty} w_k(t; -T) = w_k(t, -\infty)$. This may hold if K is large enough and the viscous term dominates. Certainly the functions $w_k(t)$, $|k| \le K$ should not grow too fast as $t \to -\infty$.

If $w_k(t)$,$|k| \le K$ are determining modes then each high mode $w_k(t, -\infty)$,$|k| > K$ can be written as $w_k(t, -\infty) = \Phi_k(w_{k_1}(\zeta), \, s \le t, |k_1| \le K)$ where Φ_k is a functional depending on the whole pre-history of high modes with $|k| \le K$. Thus the whole system (27) is effectively reduced to a finite-dimensional system of stochastic differential equations

$$\begin{aligned} dw_k &= + \left[2\pi i \sum_{k_1\in\mathbb{Z}^2} w_{k_1} \, w_{k-k_1} \, \frac{(k, k_1^{\perp})}{(k_1, k_1)} - \nu |k|^2 w_k \right] dt \\ &+ \; f_k \, db_k(t) \end{aligned} \tag{28}$$

where high modes w_{k_1}, $|k_1| > K$ are functionals of low modes and b_k are independent Brownian motions except $b_{-k} = b_k$.

The first results concerning the existence of an invariant measure for the Markov process in the functional space corresponding to (28) were obtained by Fursikov and Vishik (see [FV]), Flandoli and Maslowski (see [FM]). The argument is simple. Consider the enstrophy $V(t) = \sum_k w_k(t)^2$. Its differential can be written with the help of Ito formula:

$$dV(t) = \sum dw_k(t) \, w_{-k}(t) + \sum w_k(t) \, dw_{-k} + \sum |f_k|^2.$$

If we substitute the *rhs* of (28) then we come easily to the inequality (see [FM], [Matt])

$$EV(t) \le C_1 + C_2(EV(t_0) - C_3)$$

for some constants $C_1 < \infty$, $C_2 < 1$, $C_3 < \infty$. This immediately implies the boundedness of $EV(t)$. Then general results of the theory of Markov process imply the existence of an invariant measure.

The uniqueness part is much harder. Flandoli and Maslowski (see [FM]) proved the uniqueness of an invariant measure under the assumption that the coefficients $f_k \neq 0$ for all k and have a regular decay at infinity. J. Mattingly (see [Matt]) proved the uniqueness for a less restrictive case of f_k and large ν. The uniqueness was proven in the case when $f_k \neq 0$ for all determining modes $|k| \leq K$ and $f_k = 0$ for $|k| > K$, was proven in a recent paper by S. Kuksin and A. Shirikyan.

Part III

Stochastic Passive Scalar Transport Equation

Consider the transport equation for the scalar field $\theta^\kappa(\vec{x}, t)$ in $\boldsymbol{R}^d$:

$$\frac{\partial \theta^\kappa}{\partial t} + (\vec{u}(\vec{x}, t) \cdot \nabla)\theta^\kappa = \kappa \Delta \theta^\kappa. \tag{29}$$

where $\vec{u}$ is a white-in-time random process (infinite dimensional Brownian motion) defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. It is well-known that this problem is closely related to the stochastic ODE

$$d\varphi^\omega_{s,t}(\vec{x}) = \vec{u}(\varphi^\omega_{s,t}(\vec{x}), t)dt, \qquad \varphi^\omega_{s,s}(\vec{x}) = \vec{x}, \tag{30}$$

In connection with the transport equation, it is most natural to consider this stochastic ODE in the Stratonovich sense. It is shown [Ku] that if the local characteristic of $\vec{u}$ is spatially twice continuously differentiable, then the system in (30) has a unique solution. Such conditions are not satisfied by typical turbulent velocity fields on the scale of interest. In fact Kolmogorov's theory of turbulence suggests that in the three-dimensional case $\vec{u}$ is only Hölder continuous with an exponent roughly equal to $\frac{1}{3}$. When the regularity condition on $\vec{u}$ fails, there are at least two natural ways to regularize (29). The first is to add diffusion:

$$d\varphi^{\omega;\kappa}_{s,t}(\vec{x}) = \vec{u}(\varphi^{\omega;\kappa}_{s,t}(\vec{x}), t)dt + \sqrt{2\kappa}d\vec{\beta}(t), \tag{31}$$

and consider the limit as $\kappa \to 0$, as in (29). We will call this case the κ-limit. The second is to smooth out the velocity field. Let ψ_ε be defined as

$$\psi_\varepsilon(\vec{x}) = \frac{1}{\varepsilon^d}\psi\left(\frac{\vec{x}}{\varepsilon}\right),$$

where ψ is a standard mollifier: $\psi \geq 0$, $\int_{\boldsymbol{R}^d} \psi d\vec{x} = 1$, ψ decays fast at infinity. Let $\vec{u}^\varepsilon = \vec{u} \star \psi_\varepsilon$ and consider

$$d\varphi^{\omega,\varepsilon}_{s,t}(\vec{x}) = \vec{u}^\varepsilon(\varphi^{\omega,\varepsilon}_{s,t}(\vec{x}), t)dt, \tag{32}$$

in the limit as $\varepsilon \to 0$. We will call this the ε-limit. Physically κ plays the role of molecular diffusivity, ε can be thought of as a crude model of the viscous cut-off scale. Therefore the κ-limit corresponds to the situation when the Prandtl number tends to zero, $Pr \to 0$, whereas the ε-limit corresponds to the situation when the Prandtl number diverges, $Pr \to \infty$.

Before proceeding further, we relate the regularized flows in (31), (32) to the solution of transport equations. Consider the κ-regularization first. It is convenient to introduce the backward transition probability

$$g^{\kappa}_{\omega}(\vec{x},t|d\vec{y},s) = \mathbf{E}_{\beta}\delta(\vec{y}-\varphi^{\omega,\kappa}_{t,s}(\vec{x}))d\vec{y}, \quad s<t, \tag{33}$$

where the expectation is taken with respect to $\vec{\beta}(t)$, and $\varphi^{\omega,\kappa}_{t,s}(\vec{x})$ is the flow inverse to $\varphi^{\omega,\kappa}_{s,t}(\vec{x})$ defined in (31) (i.e. $\varphi^{\omega,\kappa}_{s,t}(\vec{x})$ is the forward flow and $\varphi^{\omega,\kappa}_{t,s}(\vec{x})$ is the backward flow). The action of g^{κ}_{ω} generates a semi-group of transformation

$$S^{\omega,\kappa}_{t,s}\psi(\vec{x}) = \int_{\mathbf{R}^d}\psi(\vec{y})g^{\kappa}_{\omega}(\vec{x},t|d\vec{y},s), \tag{34}$$

for all test functions ψ. $\theta^{\kappa}_{\omega}(\vec{x},t) = S^{\omega,\kappa}_{t,s}\psi(\vec{x})$ solve the transport equation in (29) for the initial condition $\theta^{\kappa}_{\omega}(\vec{x},s) = \psi(\vec{x})$. Similarly, for the flow in (32), define

$$S^{\omega,\varepsilon}_{t,s}\psi(\vec{x}) = \psi(\varphi^{\omega,\varepsilon}_{t,s}(\vec{x})), \qquad s<t. \tag{35}$$

$\theta^{\varepsilon}_{\omega}(\vec{x},t) = S^{\omega,\varepsilon}_{t,s}\psi(\vec{x})$ solves the transport equation

$$\frac{\partial\theta^{\varepsilon}}{\partial t} + (\vec{u}^{\,\varepsilon}(\vec{x},t)\cdot\nabla)\theta^{\varepsilon} = 0. \tag{36}$$

with initial condition $\theta(\vec{x},s) = \psi(\vec{x})$. Similar definitions can be given for forward flows but we will restrict attention to the backward ones since we are primarily interested in scalar transport. The results given below generalize trivially to forward flows.

11 Kraichnan's Model

We will consider a generalization of Kraichnan's model [Kr] for which $\vec{u}$ is assumed to be a statistically homogeneous, isotropic and stationary Gaussian field with mean zero and covariance

$$\mathbf{E}\, u_{\alpha}(\vec{x},t)u_{\beta}(\vec{y},s) = (C_0\delta_{\alpha\beta} - c_{\alpha\beta}(\vec{x}-\vec{y}))\delta(t-s). \tag{37}$$

We assume that $\vec{u}$ has a correlation length ℓ_0, i.e. the covariance in (37) decays fast for $|\vec{x}-\vec{y}| > \ell_0$. Consistently, $c_{\alpha\beta}(\vec{x}) \to C_0\delta_{\alpha\beta}$ as $|\vec{x}|/\ell_0 \to \infty$. On the other hand, we will be mainly interested in small scale phenomena for which $|\vec{x}| \ll \ell_0$. In this range, we take $c_{\alpha\beta}(\vec{x}) = d_{\alpha\beta}(\vec{x}) + O(|\vec{x}|^2/\ell_0^2)$ with

$$d_{\alpha\beta}(\vec{x}) = Ad^{P}_{\alpha\beta}(\vec{x}) + Bd^{S}_{\alpha\beta}(\vec{x}), \tag{38}$$

and

$$
\begin{aligned}
d^P_{\alpha\beta}(\vec{x}) &= D\left(\delta_{\alpha\beta} + \xi \tfrac{x_\alpha x_\beta}{|\vec{x}|^2}\right)|\vec{x}|^\xi, \\
d^S_{\alpha\beta}(\vec{x}) &= D\left((d+\xi-1)\delta_{\alpha\beta} - \xi \tfrac{x_\alpha x_\beta}{|\vec{x}|^2}\right)|\vec{x}|^\xi.
\end{aligned} \tag{39}
$$

D is a parameter having the dimension $[\text{length}]^{2-\xi}[\text{time}]^{-1}$. The dimensionless parameters A and B measure the divergence and rotation of the field $\vec{u}$. $A = 0$ corresponds to incompressible fields with $\nabla \cdot \vec{u} = 0$. $B = 0$ corresponds to irrotational fields with $\nabla \times \vec{u} = 0$. The parameter ξ measures the spatial regularity of $\vec{u}$. For $\xi \in (0, 2)$, the local characteristic of $\vec{u}$ fails to be twice differentiable and this fact has important consequences on both the transport equation in (29) and the systems of ODEs in (30) or (31).

Existing physics literature concentrates on the κ-limit for Kraichnan's model [GK, CFKL, SS]. Let $\mathcal{S}^2 = A + (d-1)B$, $\mathcal{C}^2 = A$, $\mathcal{P} = \mathcal{C}^2/\mathcal{S}^2$. $\mathcal{P} \in [0,1]$ is a measure of the degree of compressibility of $\vec{u}$. The pioneering work of Gawędzki and Vergassola [GV] identifies two different regimes for the κ-limit:

1. The strongly compressible regime when

$$
\mathcal{P} \geq \frac{d}{\xi^2}. \tag{40}
$$

In this regime g^κ_ω converges to a flow of maps, i.e. there exists a two-parameter family of maps $\{\varphi^\omega_{t,s}(\vec{x})\}$ such that

$$
g^\kappa_\omega(\vec{x}, t|d\vec{y}, s) \to \delta(\vec{y} - \varphi^\omega_{t,s}(\vec{x}))d\vec{y}. \tag{41}
$$

Moreover particles have finite probability to coalesce under the flow of $\{\varphi^\omega_{t,s}(\vec{x})\}$.

2. When

$$
\mathcal{P} \leq \frac{d}{\xi^2}, \tag{42}
$$

g^κ_ω converges to a "generalized stochastic flow" [EV3]

$$
g^\kappa_\omega(\vec{x}, t|d\vec{y}, s) \to g_\omega(\vec{x}, t|d\vec{y}, s), \tag{43}
$$

and the limit g_ω is a nontrivial probability distribution in $\vec{y}$. This means that the image of a particle under the flow defined by the velocity field $\vec{u}$ is non-unique and has a nontrivial distribution. In other words, particle trajectories branch.

The following result of E and Vanden Eijnden provides rigorous justification of these predictions and also extends the result to the ε-limit. In particular, it points out that there are three regimes when both the κ- and the ε-limits are considered.

Theorem 12 *In the strongly compressible regime when*

$$\mathcal{P} \geq \frac{d}{\xi^2}, \tag{44}$$

there exists a two-parameter family of random maps $\{\varphi^{\omega}_{t,s}(\vec{x})\}$, such that for all smooth test functions ψ and for all $(s,t,\vec{x})$, $s<t$,

$$\mathbf{E}\left(S^{\omega,\kappa}_{t,s}\psi(\vec{x}) - \psi(\varphi^{\omega}_{t,s}(\vec{x}))\right)^2 \to 0, \tag{45}$$

as $\kappa \to 0$, and

$$\mathbf{E}\left(\psi(\varphi^{\omega,\varepsilon}_{t,s}(\vec{x})) - \psi(\varphi^{\omega}_{t,s}(\vec{x}))\right)^2 \to 0, \tag{46}$$

as $\varepsilon \to 0$. Moreover, the limiting flow $\{\varphi^{\omega}_{t,s}(\vec{x})\}$ coalesces in the sense that for almost all $(s,\vec{x},\vec{y})$, $\vec{x}\neq\vec{y}$, one can define a time τ, $0<\tau<\infty$, such that

$$\varphi^{\omega}_{s,t}(\vec{x}) = \varphi^{\omega}_{s,t}(\vec{y}), \tag{47}$$

for $t \geq \tau$.

In the weakly compressible regime when

$$\mathcal{P} \leq \frac{d+\xi-2}{2\xi}, \tag{48}$$

there exists a random family of generalized flow $g_\omega(\vec{x},t|d\vec{y},s)$, such that for all test function ψ,

$$S^{\omega}_{t,s}\psi(\vec{x}) = \int_{\mathbf{R}^d} \psi(\vec{y}) g_\omega(\vec{x},t|d\vec{y},s), \tag{49}$$

satisfies

$$\mathbf{E}\left(S^{\omega,\kappa}_{t,s}\psi(\vec{x}) - S^{\omega}_{t,s}\psi(\vec{x})\right)^2 \to 0, \tag{50}$$

as $\kappa \to 0$ for all $(s,t,\vec{x})$, $s<t$, and

$$\mathbf{E}\Big(\int_{\mathbf{R}^d} \eta(\vec{x})\left(\psi(\varphi^{\omega,\varepsilon}_{t,s}(\vec{x})) - S^{\omega}_{t,s}\psi(\vec{x})\right) d\vec{x}\Big)^2 \to 0, \tag{51}$$

as $\varepsilon \to 0$ for all (s,t), $s<t$, and for all test function η. Moreover, $g_\omega(\vec{x},t|d\vec{y},s)$ is non-degenerate in the sense that

$$S^{\omega}_{t,s}\psi^2(\vec{x}) - \left(S^{\omega}_{t,s}\psi(\vec{x})\right)^2 > 0 \quad a.s. \tag{52}$$

In the intermediate regime when

$$\frac{d+\xi-2}{2\xi} \leq \mathcal{P} \leq \frac{d}{\xi^2}, \tag{53}$$

there exists a random family of maps $\{\varphi_{t,s}^{\omega}(\vec{x})\}$, and a random family of generalized flows $g_\omega(\vec{x},t|d\vec{y},s)$, such that for all test function ψ and for all $(s,t,\vec{x})$, $s<t$,

$$\mathbf{E}\left(S_{t,s}^{\omega,\kappa}\psi(\vec{x}) - S_{s,t}^{\omega}\psi(\vec{x})\right)^2 \to 0 \tag{54}$$

as $\kappa \to 0$, and

$$\mathbf{E}\left(\psi(\varphi_{t,s}^{\omega,\varepsilon}(\vec{x})) - \psi(\varphi_{t,s}^{\omega}(\vec{x}))\right)^2 \to 0 \tag{55}$$

as $\varepsilon \to 0$. Furthermore, the limit $\{\varphi_{t,s}^{\omega}(\vec{x})\}$ coalesces in the sense of (47), and the limit g_ω is non-degenerate in the sense of (52).

12 One Force—One Solution Principle

We now turn to the issue of invariant measure for the transport equation when it is suitably forced. We will first restrict our attention to the non-degenerate case. This includes the weakly compressible regime and the intermediate regime in the κ-limit. The non-degeneracy of $g_\omega(\vec{x},t|d\vec{y},s)$ as a probability distribution in $\vec{y}$ implies dissipation of energy or, phrased differently, decay in memory in the semi-group $S_{t,s}^{\omega}$ generated by $\{g_\omega\}$. This is the main reason that the forced transport equation has a unique invariant measure, as we explain now.

We will consider (compare with (29))

$$\frac{\partial \theta}{\partial t} + (\vec{u}(\vec{x},t)\cdot\nabla)\theta = b(\vec{x},t). \tag{56}$$

where b is a white-noise forcing such that

$$\mathbf{E}\, b(\vec{x},t)b(\vec{y},s) = B(|\vec{x}-\vec{y}|)\delta(t-s). \tag{57}$$

$B(r)$ is assumed to be smooth and rapidly decaying to zero for $r>L$; L will be referred to as the forcing scale. The solution of (56) for the initial condition $\theta_\omega(\vec{x},s)=\theta_0(\vec{x})$ is understood as

$$\theta_\omega(\vec{x},t) = S_{t,s}\theta_0(\vec{x}) + \int_s^t S_{t,\tau} b(\vec{x},\tau)d\tau. \tag{58}$$

Define the product probability space $(\Omega_u\times\Omega_b, \mathcal{F}_u\times\mathcal{F}_b, \mathcal{P}_u\times\mathcal{P}_b)$, and the shift operator

$$T_\tau\omega(t) = \omega(t+\tau), \tag{59}$$

with $\omega=(\omega_u,\omega_b)$.

Theorem 13 (One force–one solution I) *For almost all ω, there exists a unique solution of (56) defined on $\boldsymbol{R}^d\times(-\infty,\infty)$. This solution can be expressed as*

$$\theta_\omega^{\star}(\vec{x},t) = \int_{-\infty}^{t} S_{t,s} b(\vec{x},s)ds. \tag{60}$$

Furthermore the map $\omega \to \theta^\star_\omega$ satisfies the invariance property

$$\theta^\star_{T_\tau \omega}(\vec{x}, t) = \theta^\star_\omega(\vec{x}, t + \tau). \tag{61}$$

Theorem 13 is the "one force—one solution principle" articulated in [EKMS1]. Because of the invariance property (61), the map in (60) leads to a natural invariant measure. As a consequence we have

Theorem 14 *There exists a unique invariant measure on $L^2_{loc}(\boldsymbol{R}^d \times \Omega)$ for the dynamics defined by (56).*

The connection between the map (60) and the invariant measure, together with uniqueness, is explained in [EKMS1].

We sketch the proof of Theorem 13. Basically, it amounts to verifying that the dissipation and loss of memory in the system is strong enough in the sense that

$$\mathbf{E}\Big(\int_{T_1}^{T_2} \int_{\boldsymbol{R}^d} b(\vec{y}, s) g(\vec{x}, 0 | d\vec{y}, s) ds\Big)^2 \to 0, \tag{62}$$

as T_1, $T_2 \to -\infty$ for fixed $\vec{x}$ and t. The average in (62) is given explicitly by

$$\int_{T_1}^{T_2} \int_0^\infty B(r) P(0|r, s) dr ds, \tag{63}$$

Here $P(\rho | r, s)$ is defined for all test function η as

$$\begin{aligned} &\int_0^\infty \eta(r) P(\rho | r, s - t) dr \\ &= \int\limits_{\boldsymbol{R}^d \times \boldsymbol{R}^d} \eta(|x - x'|) \mathbf{E}\left(g_\omega(y, t|x, s) g_\omega(z, t|x', s)\right) dx dx', \end{aligned} \tag{64}$$

in the non-degenerate case, and as

$$\int_0^\infty \eta(r) P(\rho | r, s - t) dr = \mathbf{E}\, \eta(|\varphi_{t,s}(y) - \varphi_{t,s}(z)|), \tag{65}$$

in the coalescence case. Here $\rho = |y - z|$ and $s < t$. $P(\rho|r, s)$ can be thought of as the probability density that two particles have distance r at time $s < t$ if their final distance is ρ at time t. For Kraichnan model, P satisfies the backward equation

$$-\frac{\partial P}{\partial s} = -\frac{\partial}{\partial r}\left(b(r) P\right) + \frac{\partial^2}{\partial r^2}\left(a(r) P\right), \tag{66}$$

for the final condition $\lim_{s \to 0-} P(\rho | r, s) = \delta(r - \rho)$, and with $a(r)$, $b(r)$ such that

$$\begin{aligned} a(r) &= D(\mathcal{S}^2 + \xi \mathcal{C}^2) r^\xi + O(r^2/\ell_0^2), \\ b(r) &= D((d - 1 + \xi)\mathcal{S}^2 - \xi \mathcal{C}^2) r^{\xi - 1} + O(r/\ell_0^2). \end{aligned} \tag{67}$$

For $r \gg \ell_0$, $a(r)$ tends to C_0, $b(r)$ to $C_0(d-1)/r$, and the equation in (61) reduces to a diffusion equation with constant coefficient. The equation in (61) is singular at $r = 0$. The convergence of the integral in (62) depends on the rate of decay in $|s|$ of $P(0|r, s)$. Because of the integral in r in (63) has a cut-off at the forcing scale L due to $B(r)$, we can restrict attention to the behavior of $P(0|r, s)$ for $r < L$. For r not too large, it follows from the equation in (61) that P can be approximated by

$$P = C(s)r^{d-1} + o(r^{d-1}), \tag{68}$$

where $C(s)$ is yet to be determined. The range of value for r in which the approximation in (68) is valid increases with $|s|$. For $|s|$ large enough, most of the mass of $P(0|r, s)$ is in the range $r \gg \ell_0$, where P satisfies a diffusion equation with constant diffusion coefficient C_0 whose exact solution is known. A standard matching argument between this solution and the approximation in (68) can be used to estimate $C(s)$. This gives

$$P(0|r, s) = \frac{Cr^{d-1}}{|s|^{d/2}} + o(r^{d-1}), \tag{69}$$

Using (69) gives the following leading order estimate for the average in (62)

$$C \int_0^\infty B(r)r^{d-1}dr \int_{T_1}^{T_2} |s|^{-d/2}ds. \tag{70}$$

The integral in s in this expression tends to zero as T_1, $T_2 \to -\infty$ if $d > 2$. It follows that the invariant measure in (60) exists provided $d > 2$. The restriction on the dimensionality can be relaxed by restricting attention to forcing obeying $\int_0^\infty B(r)r^{d-1}dr = 0$, as is easily shown be computing the next order term in the expansion in (69).

Consider now the coalescence case, i.e the strongly compressible regime and the intermediate regime in the ε-limit. Since no anomalous dissipation is present in this case, no invariant measure for the temperature field as the one in (60) exists. It makes sense, however, to ask about the existence of an invariant measure for the temperature difference, i.e. to consider

$$\delta\theta^\star_\omega(\vec{x}, \vec{y}, t) = \int_{-\infty}^t S_{t,s}(b(\vec{x}, s) - b(\vec{y}, s))ds. \tag{71}$$

When $\theta^\star_\omega$ exists, one has $\delta\theta^\star_\omega(\vec{x}, \vec{y}, t) = \theta^\star_\omega(\vec{x}, t) - \theta^\star_\omega(\vec{y}, t)$, but it is conceivable that $\delta\theta^\star_\omega$ exists in the coalescence case even though $\theta^\star_\omega$ is not defined. The reason is that coalescence of the generalized flow implies that the temperature field flattens with time, which is a dissipation mechanism as far as the temperature difference is concerned. Of course, this effect has to overcome the fluctuations produced by the forcing, and the existence of an invariant measure such as (71) will depend on how fast particles coalesce under the flow.

At this point a difficulty arises. If we were to consider two particles separated by much more than the correlation length ℓ_0, the dynamics of their distance under the flow is governed by the equation (61) with $\eta \approx C_0$, i.e. by a diffusion equation with constant diffusion coefficient on the scale of interest. It follows that no tendency of coalescence is observed before the distance becomes smaller than ℓ_0, which, as is shown below, does not happen fast enough in order to overcome the the fluctuations produced by the forcing. In other words,

Lemma 5 *In the coalescence case, for finite ℓ_0, there is no invariant measure with finite energy for the temperature difference.*

The obvious question to ask next is what happens if we let $\ell_0 \to \infty$? This question, however, has to be considered carefully because the velocity field with the covariance in (37) diverges as $\ell_0 \to \infty$. The right way to proceed is to consider an alternative velocity $\vec{v}$, taken to be Gaussian, white-in-time, but *non-homogeneous*, with covariance

$$\begin{aligned}&\mathbf{E}\, v_\alpha(\vec{x},t) v_\beta(\vec{y},s)\\ &= (c_{\alpha\beta}(\vec{x}) + c_{\alpha\beta}(\vec{y}) - c_{\alpha\beta}(\vec{x}-\vec{y}))\delta(t-s).\end{aligned} \tag{72}$$

For finite ℓ_0, one has $\vec{v}(\vec{x},t) = \vec{u}(\vec{x},t) - \vec{u}(\vec{a},t)$, where $\vec{a}$ is arbitrary but fixed. However, $\vec{v}$ makes sense in the limit as $\ell_0 \to \infty$. Denote by $\vartheta_\omega(\vec{x},t)$ the temperature field advected by $\vec{v}$, i.e. the solution of the transport equation (56) with $\vec{u}$ replaced by $\vec{v}$. Restricting to zero initial condition, it follows from the homogeneity of the forcing that the single-time moments of θ_ω and ϑ_ω coincide for finite ℓ_0, but in contrast to θ_ω, ϑ_ω makes sense as $\ell_0 \to \infty$. Thus, ϑ_ω is a natural process to study the limit as $\ell_0 \to \infty$, and from now on we restrict attention to this case. Let $\delta\vartheta_\omega(\vec{x},\vec{y},t) = \vartheta_\omega(\vec{x},t) - \vartheta_\omega(\vec{y},t)$. The temperature difference $\delta\vartheta_\omega$ satisfies of the transport equation

$$\frac{\partial \delta\vartheta}{\partial t} + (\vec{v}(\vec{x},t)\cdot\nabla_x + \vec{v}(\vec{y},t)\cdot\nabla_y)\delta\vartheta = b(\vec{x},t) - b(\vec{y},t). \tag{73}$$

We have

Theorem 15 (One force–one solution II) *For almost all ω, in the strongly and the weakly compressible regimes, as well as in the intermediate regime if the flow is non-degenerate, there exists a unique solution of (73) defined on $\boldsymbol{R}^d \times (-\infty,\infty)$. This solution can be expressed as*

$$\delta\vartheta^\star_\omega(\vec{x},\vec{y},t) = \int_{-\infty}^t S_{t,s}(b(\vec{x},s) - b(\vec{y},s))ds. \tag{74}$$

Furthermore the map $\omega \to \delta\vartheta^\star_\omega$ satisfies the invariance property

$$\delta\vartheta^\star_{T_\tau\omega}(\vec{x},\vec{y},t) = \delta\vartheta^\star_\omega(\vec{x},\vec{y},t+\tau). \tag{75}$$

In contrast, in the intermediate regime if the flow coalesces (ε-limit) there is no solution such as (74) with finite covariance.

An immediate consequence of this theorem is

Theorem 16 *In the strongly and the weakly compressible regimes, as well as in the intermediate regime if the flow is non-degenerate, there exists a unique invariant measure on* $L^2_{loc}(\boldsymbol{R}^d \times \Omega)$ *for the dynamics defined by (73). In the intermediate regime if the generalized flow coalesce, there is no invariant measure for the equation in (73) with finite energy.*

In regimes for which the generalized flow is non-degenerate, Theorem 15 follows from Theorem 13. In the coalescence cases one proceeds similarly as in the proof of Theorem 13 and study the convergence as T_1, $T_2 \to -\infty$ of

$$\mathbf{E}\Big(\int_{T_1}^{T_2} (b(\varphi^\omega_{t,s}(\vec{x}, s) - b(\varphi^\omega_{t,s}(\vec{y}, s))ds\Big)^2. \tag{76}$$

The average in (62) is given explicitly by

$$2\int_{T_1}^{T_2}\int_0^\infty (B(0) - B(\rho))P(r|\rho, s)drds, \tag{77}$$

where $r = |\vec{x} - \vec{y}|$, P satisfies 61 with $\ell_0 = \infty$. Since in the coalescence case $r = 0$ is an exit boundary, it follows that P looses mass at $r = 0+$. The convergence of the time integral in 77 depends on the rate at which mass is lost (i.e. the rate at which particles coalesce). The analysis of the equation in 61 shows that the process is fast enough in order that the integral over s in (77) tends to zero as T_1, $T_2 \to -\infty$ in the strongly compressible regime. In contrast, the integral diverges in the weakly compressible regime in the coalescence case.

Acknowledgment: The first author is partially supported by a Presidential Faculty Fellowship from NSF. The second author is partially supported by NSF grant DMS-97067994 and RFBR grant 99-01-00314.

References

[A] Aubry, S., The twist map, the extended Frenkel-Kontorova model and the Devil's staircase, *Physica $\mathcal{D}$*, **7**, 1-3 (1983), 240-258.

[AE] Avellaneda, M. and E, W., Statistical properties of shocks in Burgers turbulence, *Comm. Math. Phys.* 122 (1995), No. 1, 13-38.

[ARE] Avellaneda, M., Ryan, R. and E, W., PDFs for velocity and velocity gradients in Burgers turbulence, *Phys. Fluids* 7 (1994), No. 12, 3067-3071.

[Av] Avellaneda, M., Statistical properties of shocks in Burgers turbulence, II. Tail probabilities for velocities, shock-strength and rarefunction intervals, *Comm. Math. Phys.* 169 (1995), No. 1, 45-59.

[Ba] Baxendale, P.H., Statistical Equilibrium and two-point motion for a stochastic flow of diffeomorphims spatial stochastic processes, (Boston, MA), *Progress in probability*, vol. 19, Birkhäuser Boston, Boston, MA, 1991, 189-218.

[Be] Bertoin, J., The Inviscid Burgers Equation with Brownian Initial Velocity, *Comm. in Math. Physics*, v.193, n2, 1998, 397-406.

[Bu] Burgers, J.M., *The Nonlinear Diffusion Equation*, Reidel, 1974.

[BFK] Bec, J., Frisch, U., Khanin, K., Kicked Burgers Turbulence, submitted to *J. Fluid Mech.*, 1999, chao-dyn/9910001.

[C] Cole, J., On a quasi-linear parabolic equation occuring in hydrodynamics, *A. Appl. Math.*, **9**, 1951, 225-236.

[CF] Constantin, P., Foias, C., *Navier-Stokes equations*, University of Chicago Press, Chicago, 1988.

[CFKL] Chertkov, M., Falkovich, G., Kolokolov, I., Lebedev, V., Normal and anomalous scaling of the fourth-order correlation function of a randomly advected passive scalar, *Phys. Rev. E*, **52**, No. 5, 4924-4941 (1995).

[CY] Chekhlov, A., and Yakhot, V., Algebraic Tails of Probability Density Functions in the Random-Force-Driven Burgers Turbulence, *Phys. Rev. Lett.* **77**, 3118–3121 (1996).

[DPZ] Da Prato, G. & Zabczyk, J., *Ergodicity for infinite-dimensional systems*, Cambridge, 1996.

[EKMS1] E, W., Khanin, K., Mazel, A., Sinai, Y., Invariant measure for Burgers Equation with Random Forcing, *Annals of Mathematics*, accepted for publication, 101p.

[EKMS2] E, W., Khanin, K., Mazel, A., Sinai, Y., Probability distribution function for the random forced Burgers equation, *Phys. Rev. Lett.*, **78**, (1997), 1904-1907.

[ERS] E, W., Rykov, Y. & Sinai, Y., Generalized Variational Principles, Global Weak Solutions and Behaviour with Random Initial Data for Systems of Conservation Laws Arising in Adhesion Particle Dynamics, *Comm. in Math. Phys.*, v.177, n2, 1996, 349-380.

[EV1] E, W., Vanden Eijnden, E. (1999) Asymptotic theory for the probability density functions in Burgers turbulence, *Phys. Rev. Lett.* **83** 2572–2575

[EV2] E, W., and Vanden Eijnden, E., Statistical theory of the stochastic Burgers equation in the inviscid limit, to appear in *Comm. Pure Appl. Math.*

[EV3] E, W. and Vanden Eijnden, E., Generalized flows, intrinsic stochasticity and turbulent transport, submitted to *Proc. Nat. Acad. Sci.*, 2000.

[Fa] Fan, H., Elementary waves of Burgers equation under white noise perturbation, preprint.

[FM] Flandoli, F. & Maslowski, B., Ergodicity of the $2\mathcal{D}$ Navier-Stokes equation under random perturbation, *Comm. in Math. Physics*, **171**, n2 (1995), 119-141.

[FrM] Frachebourg L. and Martin, Ph., Exact statistical properties of the Burgers equation, cond-mat/9905056.

[FBV] Frisch U., Bec J., and Villone B., Singularities and the distribution of density in the Burgers adhension model, preprint, submitted to *Physica D*, 1999.

[FT] Foias, C. & Temam, R., Gevrey Class regularity for the solutions of the Navier-Stokes equations, *J. Funct. Anal.*, **87**, 2, 1989, 359-369.

[FV] Fursikov, A. & Vishik, M., *Mathematical problems of statistical hydrodynamics*, Kluwer Acad. Publishers, 1988.

[G] Groenboom, P., The concave majorant of Brownian motion, *Annals of Probability* **11**, No. 4, 1016-1027, 1983.

[GMS] Gurbatov, S., Malakhov, A. and Saichev, A., *Nonlinear Random Waves and Turbulence in Nondispersive Media; Waves, Rays and Particles*, Manchester University Press, 1991.

[GK] Gawędzki, K., and Kupiainen, A., Anomalous scaling of a passive scalar, *Phys. Rev. Lett.* **75**, 3834–3837 (1995).

[GV] Gawędzki, K., Vergassola, M. (1998) cond-mat/9811399.

[H] Handa, K., A remark on shocks in inviscid Burgers turbulence, in *Nonlinear Waves and Weak Turbulence*, N. Fitzmaurice et al. Eds., Birkhäuser-Boston, 1993.

[Ho1] Hopf, E., The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure Appl. Math.*, **3**, (1950), 201-230.

[Ho2] Hopf, E., Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, *Math. Nachr.*, **4**, 1951, 213-231.

[JW] Janicki, A. & Woyczynski, W., Hausdorff dimension of regular points in stochastic Burgers flows with Levy α-stable initial data, *J. Stat. Physics*, **86**, (1997), 277-299.

[K] Kato, T., Nonstationary flows of viscous and ideal fluids in R^3, *J. Funct. Analysis*, **9**, 1972, 296-305.

[Ki] Kida, S., Asymptotic properties of Burgers turbulence, *J. Fluid Mech.*, **93**, 337-377, 1979.

[KS] Kuksin S., and Shirikyan A., Stochastic dissipative PDEs and Gibbs measures, submitted to *Comm. Math. Phys.*

[Ku] Kunita, H., *Stochastic differential equations*, Cambridge University Press, 1990.

[L] Lachal, A., *Quelgues applications de la théorie des excursions à l'integrabe du mouvement brownien*, Lecture Notes in Mathematics, Springer-Verlag, in press.

[La] Ladyzenskaya, O.A., *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

[Lax] Lax, P.D., Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.*, **10**, 1957, 537-556.

[Le] Leray, J., Essai sur le mouvement d'un liguide visquex emplissant l'espace, *Acta Math.*, **63**, 1934, 193-248.

[LLR] Leadbetter, M., Lindgren G. and Rootzen, H., *Extremes and Related Properties of Random Sequences and Processes*, Springer-Verlag, 1983.

[M] Majda, A., Lectures on hydrodynamics, Princeton University (unpublished).

[M2] Majda, A. J. (1993) *J. Stat. Phys.* **73**, 515–542.

[Ma] Mather, J., Existence of quasi-periodic orbits for twist homeomorphisms of the Annulus, *Topology*, **21**, (4), 1982, 457-567.

[Mo] Molchan, G., Burgers' equation with self-similar Gaussian initial data: Tail probabilities.

[MP] Martin, Ph. & Piasecki, J., Aggregation Dynamics in a Self-Graviting One-Dimensional Gas, *Journal of Statistical Physics*, v.84, n3/4, 1996, 837-857.

[Matt] Mattingly, J., Ergodicity of $2\mathcal{D}$ Navier-Stokes Equations with Random Forcing and Large Viscosity, *Comm. Math. Phys.,* v.206, n2, 1999, 273-288.

[MS] Mattingly, J. & Sinai, Y., An Elementary Proof of the Existence and Uniqueness Theorem for the Navier-Stokes Equations, *Comm. in Contemporary Mathematics*, vol.1, n4, 1999, 497-516.

[O] Oleinik, O., Discontinuous solutions of nonlinear differential equations, *Uspekhi Math. Nauk*, **12**, 1957, 3-73.

[Pi] Pitman, J. M., *Seminar on Stochastic Processes*, E. Cinlar et.al. editors, Birkhauser, 219-227, 1982.

[R] Ryan, R., Large deviation analysis of Burgers turbulence With white-noise initial data, *Comm. Pure Appl. Math.*

[Si1] Sinai, Y.G., Statistics of shocks in solutions of inviscid Burgers equation, *Comm. in Math. Phys.*, **148**, 601-621.

[Si2] Sinai, Y.G., Distribution of some functionals of the integral of the Brownian motion, *Theor. and Math. Physics*, v.90, (1992), n3, 323-353.

[Si3] Sinai, Y.G., Two results concerning asymptotic behaviour of solutions of the Burgers equation with force, *Journal of Statistical Phys.*, **64**, 1992, 1-12.

[Si4] Sinai, Y.G., Navier-Stokes Systems with Periodic Boundary Conditions, *Regular & Chaotic Dynamics*, v.4, n2, 1999, 3-15.

[Si5] Sinai, Y.G., *Topics in Ergodic Theory*, Princeton University Press, 1994.

[Sm] Smoller, J., *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, Berlin, New York, 1994.

[SAF] She, Z-S, Aurell, E., Frisch, U., The inviscid Burgers equation with initial data of Brownian type, *Comm. in Math. Phys.*, **148**, 1992, 623-641.

[SS] Shraiman, B., Siggia, E. (1994) *Phys. Rev. E* **49**, 2912–2927.

[T] Temam, R., Some Developments on Equations in the Second Half of the 20^{th} Century, 44p., to appear in *"Development des Mathematiques an cours de la second moitie du XX-eme siecle"*, J.P. Pier Managing Editor

[VDFN] Vergassola, M, Dubrulle, B., Frisch, U., Noullez, A., Burgers equation, devil's staircases and the mass distribution function for large-scale structures, *Astron & Astrophys.*, **289**, 1994, 325-356.

[Vo] Vol'pert, A., The space BV and quasilinear equations, *Math. USSR*, Sb. **2** 225–267 (1967).

[W] Woyczynski, W., *Burgers-KPZ Turbulence*, Lecture Notes in Mathematics, Springer-Verlag, v.1700, 1998.

[WX] Wehr, J. and Xin, J., White noise perturbation of the viscous shock fronts of the Burgers equation, *Comm. Math. Phys.* **181**, 183-203 (1996).

[Y] Yudovich, V., *Mathematical questions of the theory of stability of flows of a fluid*, Doctor Thesis, Institut of Problems of Mechanics, Moscow, USSR, 1972.

[Z] Zeldovitch, Ya.B., Gravitational Instability: An approximate theory for large density perturbations, *Astron & Astrophys.*, **5**, 1970, 84-89.