

Dynamics of Filaments during the Isotropic – Smectic A Phase Transition

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Abstract: We study the dynamics of filaments formed when certain materials undergo the isotropic-smectic A phase transition. The basis of this work is a solidification model derived from first principles, taking into account elastic, hydrodynamic and thermodynamic effects. The permeation process, which is solely responsible for the growth of the filament, is analyzed in detail. Various characteristics of the filament, such as growth rates, velocity profile inside the filament and transport along the filament, are calculated by studying the asymptotic limit of a slender body problem. We also derive, in this limit, the equation governing the dynamics of the centerline of the filament.

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§1. Introduction

When certain materials are cooled below their isotropic-smectic A (abbreviated $I - S_A$) transition temperature, the low temperature smectic phase grows into a collection of spectacular spaghetti-like filaments (Figure 1). These filaments have a more or less fixed radius. The length of these filaments is carefully measured and is seen to grow exponentially [20], suggesting that the filament grows locally at a constant rate. The filaments buckle continuously as they grow and ultimately collapse to form compact domains. Occasionally thinner filaments buckling at a faster rate are shed from thicker ones.

(Note to printer: please place Figures 1 and 2 here)

This was first observed by F. Jones and R. B. Meyer in the unpublished thesis of Jones. Palffy-Muhoray et.al. carried out a more detailed experimental study of this process in a different material, and observed other interesting phenomena such as the pearling instability [20]. Experimental and theoretical results similar to [20] can also be found in [21, 22, 23, 19]. A similar form of growth is also observed in phospholipids [24].

Such a growth process is very different from the ones usually seen in similar situations, such as the growth of nematic phase during the isotropic-nematic transition and the growth of ice crystals during the liquid-solid transition. An immediate question is why filament morphology is selected? What determines the the diameter of the filament? What goes on at the $I - S_A$ interfacial region and inside the filament? How does the filament move and grow? What is the origin of the buckling instability? These questions and other related issues are the focus of this and a subsequent paper [7]. The present paper addresses the last three questions, leaving the first two to [7].

Before giving a more detailed description of the actual growth process, let us first make some introductory remarks about smectic A liquid crystals. The reader is referred to [10] for more details. The better-known nematic phase is a liquid crystalline phase in which the molecules tend to line up with each other. As a result, the material exhibits uniaxial symmetry at a macroscopic level. The local axis of symmetry is referred to as the director. In the nematic phase, the molecules acquire orientational order, but no positional order. In

the smectic A phase, molecules not only line up with each other, they also tend to position themselves into equally spaced layers. Within each layer, their positions are random. The director field coincides with the layer normal. Therefore we can think of smectic A liquid crystals as being a one-dimensional solid in the direction normal to the layers, and a two dimensional liquid within the layers. In this sense, smectic A liquid crystals provide the simplest example of a crystalline liquid.

What makes the dynamics of smectic A liquid crystal unique is the possibility of permeation – the velocity of the molecules and the velocity of the layers at the same spatial location may not be the same. As a result, molecules may permeate through the layers [10]. What makes this permeation process unique is that the permeating agent acts at the same time as the underlying porous medium (see Figure 3). We will see later that permeation is solely responsible for driving the growth of the filaments.

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The basic premise for the formation of filaments is that of an anisotropic surface tension: $\delta\gamma = \gamma_{\parallel} - \gamma_{\perp} < 0$, where γ_{\parallel} and γ_{\perp} are the surface tension for the $I - S_A$ interfaces parallel and perpendicular to the smectic layers. This has been confirmed for DCOB, the material used in most of the experimental results reported in [20]. The result of the anisotropic surface tension is that the layers arrange themselves to be parallel to the interface. In other words, the interface itself is a smectic layer. As we demonstrate in [7], the smectic nuclei first grow as spherical droplets. The outermost layer is increasingly dilated as the droplet grows. Eventually the droplet becomes unstable due to the build up of elastic dilational stress. Under normal growth conditions, this instability sets in earlier than the Mullins-Sekerka instability. The droplet then elongates and turns into a sphero-cylinder. At the same time, new layers are still nucleated via one of the following two ways: new layers are nucleated either directly at the $I - S_A$ interface, or, via permeation of molecules through smectic layers, they are nucleated at the inner core. (Notice that regardless of whether the smectic phase is in the shape of droplets, sphero-cylinders, or filaments, the inner core is a defect in the isotropic phase). In either case, nucleation of new layers becomes increasingly

difficult as the smectic phase grows, either because the barrier against nucleation increases, or because the obstacle through which the molecules have to permeate becomes greater. Eventually nucleation stops and the smectic phase grows as elongating filaments. As we show in [7], the diameter of the filament can be calculated from material constants and the undercooling. This explains the uniformity of filament size observed in experiments.

The advantage of filament growth is that no nucleation of new layers is required. Molecules permeate through the $I - S_A$ interface and the existing smectic layers, and the volume of the S_A phase is increased by elongation of the filaments (see Figure 2). From a dynamic point of view, it is interesting to see how the permeation process allows the concentric layers inside the filament to elongate at the same rate. This is one of the problems that we study here.

This paper has two main purposes. The first is to establish a framework within which the dynamics of the $I - S_A$ transition can be studied in detail. This interface model consists of the standard Navier-Stokes equations in the isotropic phase, the hydrodynamic equations for smectic liquid crystals [6] in the smectic phase, and the interface boundary conditions derived in Section 2. Our second purpose is to study the dynamics of the filaments using this interface model. This takes the entire Section 3. Specifically, we will:

(1). Calculate the growth rate of the filament. Experimental data suggest that the total length of the filament grows exponentially in time. We will verify this analytically and calculate the growth rate (3.3.2). This result gives the correct qualitative dependence of the growth rate on quench depth, although there does not exist experimental data to verify (3.3.2) quantitatively.

(2). Calculate the permeation current. Since no new layers are formed, at the interface, the permeation current is equal to the current of freezing molecules. Inside the filament, permeation is driven by the bending of the filament and permeation (see the discussions after equation (2.1.9)). At the same time, we also compute the velocity at which molecules are transported along the filament, and show that it is uniform to leading order. This provides the consistency check of the assumption that all layers elongate at the same rate.

(3). Study the dynamics of the centerline of the filament and the buckling instability. Not surprisingly, we obtain the canonical equation describing the viscous dynamics of an incompressible rod, except that in the present situation, the length of the rod increases with time, giving rise to the buckling instability. (3.9.1-3.9.2) has appeared in a variety of situations [2, 13, 12, 25], and has been used in [25] as a model for filament dynamics.

(4). Calculate the leading order temperature and stress distribution as well as the velocity field inside and outside the filament.

The present model can also be used to study the shedding and pearling instabilities observed in experiments [20, 7]. A qualitative study is presented in [7].

Although throughout this paper we use the thermal model of solidification, all results extend directly to the chemical model as in [17].

§2. The Solidification Problem

In this section, we derive the necessary equations for studying the solidification problem. We assume that the S_A phase and the I phase (occupying regions Ω_s and Ω_l respectively in space) are separated by a sharp interface Σ . For simplicity, we will neglect the volume change during the phase transition, and treat density as constant.

2.1. Smectic A hydrodynamics

Dynamics in the I phase Ω_l is governed by the standard Navier-Stokes equation for ordinary liquids:

$$(2.1.1) \quad \begin{cases} \rho_0 \frac{D\mathbf{u}}{Dt} &= -\nabla p + \mu_l \Delta \mathbf{u} \\ \nabla \cdot \mathbf{u} &= 0 \end{cases}$$

where \mathbf{u} is the velocity field, ρ_0 is the density of the fluid, p is the pressure, $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$ is the material derivative operator, μ_l is the viscosity in the isotropic liquid phase. We can also write the right hand side of the momentum equation as $\nabla \cdot \sigma$ where the stress tensor σ is given by $\sigma_{i,j} = +\frac{\mu_l}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. Since we are interested in the phase changes induced by a thermal quench, we will need the energy equation in the liquid phase:

$$(2.1.2) \quad E_t + \nabla \cdot J_E = 0$$

where E is the total energy density:

$$(2.1.3) \quad E = \frac{1}{2}\rho_0|\mathbf{u}|^2 + e_l(T),$$

$e_l(T)$ is the internal energy density of the liquid, T is the temperature; J_E is the total energy flux:

$$(2.1.4) \quad J_E = E\mathbf{u} + p\mathbf{u} - \sigma\mathbf{u} - \kappa_l \nabla T .$$

The last term is the heat flux, with κ_l being the thermal conductivity in the liquid phase.

To describe hydrodynamics in the smectic phase we introduce the layer variable ϕ such that the position of the smectic layers are given by the iso-surfaces of ϕ , and the director field is given by the layer normal $\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}$. Then hydrodynamics in the smectic phase Ω_s is described by ([6, 10, 18])

$$(2.1.5) \quad \left\{ \begin{array}{l} \rho_0 \frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla \cdot \sigma_e + \nabla \cdot \sigma_d \\ \nabla \cdot \mathbf{u} = 0 \\ E_t + \nabla \cdot J_E = 0 \\ \dot{\phi} = \frac{D\phi}{Dt} = -\beta \mathbf{n} \cdot \nabla T + \lambda \nabla \cdot \tau \end{array} \right.$$

where the elastic stress tensor σ_e is given by

$$(2.1.6) \quad \begin{aligned} \sigma_e &= \tau \otimes \mathbf{n} - K(\nabla \cdot \mathbf{n})(\nabla \mathbf{n})^T, \\ \tau &= -B(|\nabla\phi| - 1)\mathbf{n} + \frac{K}{|\nabla\phi|} \{ \nabla(\nabla \cdot \mathbf{n}) - (\nabla(\nabla \cdot \mathbf{n}) \cdot \mathbf{n})\mathbf{n} \} \end{aligned}$$

Here K and B are respectively the bending and compression elastic modulus of the layers.

The viscous stress tensor σ_d is given by:

$$(2.1.7) \quad \sigma_d = \mu_1 (\mathbf{n}^T A \mathbf{n}) \mathbf{n} \otimes \mathbf{n} + \mu_4 A + \mu_5 (A \mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes A \mathbf{n})$$

where the μ 's are the viscosity coefficients and A is the symmetric rate of strain tensor. As before, E is the total energy density: $E = \frac{1}{2}\rho_0|\mathbf{u}|^2 + e_s(T)$. The energy flux J_E is given by

$$(2.1.8) \quad J_E = E\mathbf{u} + p\mathbf{u} - (\sigma_e + \sigma_d)\mathbf{u} - \dot{\phi}\tau + \mathbf{q}$$

with the heat flux \mathbf{q} given by

$$(2.1.9) \quad \mathbf{q} = -\{\kappa_{\perp}I + (\kappa_{\parallel} - \kappa_{\perp})\mathbf{n} \otimes \mathbf{n}\} \nabla T - \beta \mathbf{n} \nabla \cdot \tau.$$

The terms with β describe a thermal-mechanical cross effect analogous to the Soret effect. λ in (2.1.5) is the mobility.

The ϕ equation describes a unique feature of the smectic phase – the permeation effect: If we assume that the layers are convected by a velocity field \mathbf{v} , then we can write $\dot{\phi} = (\mathbf{u} - \mathbf{v}) \cdot \mathbf{n} |\nabla \phi|$. If $\dot{\phi} = 0$, the layers move with the fluid. If $\dot{\phi} \neq 0$, there is a relative motion between the layers and the fluid, i.e. the molecules permeate through the layers. This is a unique example of a porous medium flow where the permeating agent acts at the same time as the porous medium. From (2.1.5) and (2.1.6) we see that permeation can be induced either by bending, compression or dilation of the layers, or by a thermal gradient. As we will see later, permeation is solely responsible for the growth of the filament observed in the experiment.

The possibility of having a nonzero velocity of layers relative to the fluid is the translational analog of a nonzero angular velocity of the director relative to the fluid found in nematics. In analogy with nematics, we call the force responsible for this relative velocity the molecular field.

In the following we will assume that the layers are incompressible, i.e. the distance between successive layers is fixed: $|\nabla \phi| = 1$. In this case τ has to be replaced by:

$$(2.1.10) \quad \tau = -\xi \mathbf{n} + K \{ \nabla(\nabla \cdot \mathbf{n}) - (\nabla(\nabla \cdot \mathbf{n}) \cdot \mathbf{n}) \mathbf{n} \}$$

where ξ is a Lagrange multiplier for the layer incompressibility condition. Although in general a small compression or dilation is always present, in the first approximation we neglect it in the filament dynamics.

2.2. Interfacial boundary conditions

There are three types of interfacial boundary conditions: The conservation laws (of mass, momentum and energy), the local equilibrium conditions (Gibbs-Thomson relation) and the anchoring condition for the director field \mathbf{n} .

Denote by $[f]$ the jump of f across the interface Σ : $[f] = f|_{\Omega_l} - f|_{\Omega_s}$. Let \mathbf{N} be the normal of Σ pointing to the liquid phase, V_n be the normal velocity of Σ . For $\Gamma(t) \subset \Sigma$,

define $R(\varepsilon) = \Gamma(t) \times [-\varepsilon, \varepsilon]$. $R(\varepsilon)$ is a “pillbox” around $\Gamma(t)$. We will make repeated use of the following identities [14]:

$$(2.2.1) \quad \lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \int_{R(\varepsilon)} \phi dV = - \int_{\Gamma(t)} [\phi] V_n dA$$

$$(2.2.2) \quad \lim_{\varepsilon \rightarrow 0} \int_{\partial R(\varepsilon)} \mathbf{f} \cdot \mathbf{m} dA = \int_{\Gamma(t)} [\mathbf{f}] \cdot \mathbf{N} dA$$

where \mathbf{m} denotes the unit outward normal of $R(\varepsilon)$.

2.2.1. Conservation of mass

Assuming that mass transport along the interface can be neglected, we have

$$(2.2.3) \quad \frac{d}{dt} \int_{R(\varepsilon)} \rho dV = - \int_{\partial R(\varepsilon)} (\rho \mathbf{u} \cdot \mathbf{m}) dA$$

As $\varepsilon \rightarrow 0$, we get

$$(2.2.4) \quad - \int_{\Gamma(t)} [\rho] V_n dA = - \int_{\Gamma(t)} [\rho \mathbf{u}] \cdot \mathbf{N} dA.$$

Since $\Gamma(t)$ is an arbitrary region in Σ , we obtain

$$(2.2.5) \quad [\rho] V_n = [\rho \mathbf{u}] \cdot \mathbf{N}$$

2.2.2. Conservation of momentum

The only surface force we will take into account is the tension: For $\Gamma(t) \subset \Sigma$, surface tension acts on $\partial\Gamma(t)$ (where ∂ is taken with respect to Σ) in the direction of the normal of $\partial\Gamma(t)$ on the surface. We denote this normal by \mathbf{T} . Force balance gives:

$$(2.2.6) \quad \frac{d}{dt} \int_{R(\varepsilon)} \rho \mathbf{u} dV = - \int_{\partial R(\varepsilon)} (\rho \mathbf{u} \otimes \mathbf{u}) \cdot \mathbf{m} dA + \int_{\partial R(\varepsilon)} \boldsymbol{\sigma} \cdot \mathbf{m} dA + \int_{\partial\Gamma(t)} \gamma \mathbf{T} ds$$

where γ is the surface tension coefficient ($= \gamma_{||}$ in anticipation of the result of Section 2.2.5), assumed to be constant. As $\varepsilon \rightarrow 0$, we get using the surface divergence theorem,

$$(2.2.7) \quad - \int_{\Gamma(t)} [\rho \mathbf{u}] V_n dA = - \int_{\Gamma(t)} [\rho \mathbf{u} \otimes \mathbf{u}] \cdot V_n dA + \int_{\Gamma(t)} [\boldsymbol{\sigma}] \cdot \mathbf{N} dA - \gamma \int_{\Gamma(t)} H \mathbf{N} dA$$

where H is the mean curvature of Σ . This implies

$$(2.2.8) \quad [\rho \mathbf{u}] V_n = [\rho \mathbf{u} \otimes \mathbf{u}] \mathbf{N} - [\sigma] \mathbf{N} + \gamma H \mathbf{N}.$$

In the special case when ρ and \mathbf{u} are continuous across Σ , and $\sigma = -pI$, this reduces to

$$(2.2.9) \quad -[p] = \gamma H$$

which is the Laplace condition.

2.2.3. Conservation of energy

For $\Gamma(t) \subset \Sigma$, the surface energy on $\Gamma(t)$ is equal to $\int_{\Gamma(t)} \gamma dA$. The rate of work done by the surface tension is $\int_{\partial\Gamma} \gamma \mathbf{T} \cdot \mathbf{v} ds$ where \mathbf{v} is the local velocity of $\partial\Gamma(t)$ on Σ . Therefore we have

$$(2.2.10) \quad \frac{d}{dt} \left(\int_{R_\epsilon} E dV + \int_{\Gamma(t)} \gamma dA \right) = - \int_{\partial R_\epsilon} (J_E \cdot \mathbf{m}) dA + \int_{\partial\Gamma} \gamma \mathbf{T} \cdot \mathbf{v} ds$$

Using (2.2.1-2.2.2), we get

$$(2.2.11) \quad \int_{\Gamma} \{-[E] V_n + \gamma H V_n\} dA + \gamma \int_{\partial\Gamma} (\mathbf{T} \cdot \mathbf{v}) ds = - \int_{\Gamma} [J_E] \cdot \mathbf{N} dA + \gamma \int_{\partial\Gamma} (\mathbf{T} \cdot \mathbf{v}) ds$$

This is

$$(2.2.12) \quad ([E] - \gamma H) V_n = [J_E] \cdot \mathbf{N}$$

2.2.4. Local equilibrium conditions – the Gibbs-Thomson relation

This is the consequence of the assumption that the interface is approximately in local equilibrium. Let F be the (Helmholtz) free energy density. The Gibbs-Thomson [15] relation states:

$$(2.2.13) \quad [F] = \gamma H.$$

In the absence of elastic distortions we have

$$(2.2.14) \quad [F] = F_l - F_s = -L \frac{T_l - T^*}{T^*}$$

where T_I is the temperature at the interface, T^* is the transition temperature, and L is the latent heat: $L = T^*[S]$, S is the entropy. In the presence of elastic distortions this changes to [20]

$$(2.2.15) \quad [F] = -L \frac{T_I - T^*}{T^*} - \frac{1}{2} K H^2$$

where the last term accounts for the elastic energy. Therefore we get

$$(2.2.16) \quad -L \frac{T_I - T^*}{T^*} = \gamma H + \frac{1}{2} K H^2$$

In terms of the internal energy, we have

$$(2.2.17) \quad [e] = [F] + T_I[S] = -L \frac{T_I - T^*}{T^*} - \frac{1}{2} K H^2 + T_I \frac{L}{T^*} = L - \frac{K}{2} H^2$$

This will be used later.

2.2.5. Anchoring condition

As we remarked in the introduction, a necessary condition for filament growth is $\gamma_{\parallel} < \gamma_{\perp}$.

We will make this assumption and impose normal anchoring condition at the interface:

$$(2.2.18) \quad \mathbf{n} \parallel \mathbf{N}$$

This is seen in experiments [20] and a justification using diffusive interface theory is given in the Appendix. This elimination of growth in energetically unfavorable directions is a result of the nonlinear interaction between the director field and the growth process.

In the opposite situation when $\gamma_{\perp} < \gamma_{\parallel}$, the director will have parallel anchoring at the interface. As a result, the $S_A - I$ transition will proceed via the formation and growth of focal conic domains [9].

In the case when $[\rho] = 0$, we have from (2.2.8)

$$[\sigma] \mathbf{N} = \gamma H \mathbf{N}$$

Combining (2.2.8) with (2.2.12) and using the fact that \mathbf{n} is parallel to \mathbf{N} , we get

$$(2.2.19) \quad ([E] - \gamma H)(V_n - \mathbf{u} \cdot \mathbf{N}) = [\mathbf{q}] \cdot \mathbf{N}$$

Since

$$(2.2.20) \quad [E] - \gamma H = [e] - \gamma H = L - \frac{K}{2} H^2 - \gamma H = L \frac{T_I}{T^*}$$

we obtain

$$(2.2.21) \quad L \frac{T_I}{T^*} (V_n - \mathbf{u} \cdot \mathbf{N}) = [\mathbf{q}] \cdot \mathbf{N}$$

This is a very general result that holds in the presence of permeation, elastic and hydrodynamic effects.

§3. Filament Growth and Dynamics

In this section we obtain fairly detailed information about the filament, including its mechanism of growth, growth rate, temperature distribution inside and outside the filament, transport along the filament, etc. We will also derive the effective equation governing the dynamics of the centerline of the filament, and study the buckling instability.

We will make the following assumptions.

(1). The time scale for the equilibration of thermal gradients is much smaller than the time scale for the motion of the centerline curve. Therefore we will neglect time derivatives in the temperature equation.

(2). We assume that the centerline curve is planar, i.e. effects due to torsion will be neglected.

(3). Anisotropic transport effects will be neglected.

(4). The transport coefficients (diffusivity and viscosity) in the I and S_A phases are comparable.

(5). Effects of surface tension will be neglected in the Gibbs-Thomson relation. It is true that surface energy increases as the filament grows, and this factor may contribute to the ultimate collapse of the filament, but it can be easily seen that this term has the same effect as the elastic term in the Gibbs-Thomson relation.

All these assumptions are made only to simplify the derivations. The essential features of the $I - S_A$ phase transition are not affected by them.

The main parameters are:

(1). The two length scales: filament diameter r_0 and the typical modulation wavelength of the filament R . The ratio $\varepsilon = \frac{r_0}{R}$ will be used as a small parameter to carry out the asymptotics.

(2). The material properties, represented by K , μ and κ .

(3). There are three different velocities: velocity of the centerline V_c , the permeation velocity \tilde{u}_1 and the velocity at which molecules are transported along the filament \tilde{u}_3 .

Nondimensionalization:

We will redefine T as $T = \frac{T}{T^*}$, and x as $x = \frac{x}{R}$. We let $K_1 = \frac{1}{2} \frac{K}{L\varepsilon^2}$.

Coordinates and parametrization:

We will use $\Gamma(t)$ to denote the position of the centerline curve at time t . We will parametrize $\Gamma(t)$ using equal arc-length coordinate s , with $0 \leq s \leq 1$. This means that if \tilde{s} is the arc-length of the curve $\Gamma(t)$ measured from some suitably chosen reference point, and $l(t)$ is the total length of $\Gamma(t)$, then

$$\frac{\partial \tilde{s}}{\partial s} = l(t) = \eta$$

Let $\Gamma(t)$ be defined parametrically by $\mathbf{X}(s, t)$ and let $\hat{\mathbf{s}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}$ be the unit tangent, principal normal and binormal vectors associated with Γ , then we have the Serret-Frenet formula:

$$\frac{\partial \mathbf{X}}{\partial s} = \eta \hat{\mathbf{s}}, \quad \frac{\partial \hat{\mathbf{s}}}{\partial s} = k\eta \hat{\mathbf{n}}, \quad \frac{\partial \hat{\mathbf{n}}}{\partial s} = -k\eta \hat{\mathbf{s}}$$

where k is the curvature of the planar curve Γ . We can write any point \mathbf{x} in R^3 as $\mathbf{x} = \mathbf{X} + r\hat{\mathbf{r}}$ where $\hat{\mathbf{r}}$ is a unit vector in the $\hat{\mathbf{n}} - \hat{\mathbf{b}}$ plane. In this way we define a curvilinear coordinate system (r, θ, s) where θ is the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{n}}$. We have

$$d\mathbf{x} = \hat{\mathbf{r}}dr + r\hat{\theta}d\theta + h_3\hat{\mathbf{s}}ds$$

where $\hat{\theta}$ is the unit vector in the θ coordinate,

$$h_3 = \eta(1 - kr \cos \theta)$$

3.1. Temperature distribution

For the time being, we will neglect dependence on t in the notations.

From Assumptions (1) and (3) we have

$$(3.1.1) \quad \Delta T = 0$$

with the boundary condition

$$(3.1.2) \quad T|_{r=\varepsilon} = 1 - K_1 \varepsilon^2 H^2$$

The curvature at the boundary of the filament can be evaluated easily:

$$(3.1.3) \quad H = \nabla \cdot \hat{\mathbf{r}} = \frac{1}{\varepsilon} - k \cos \theta$$

Therefore we have

$$(3.1.4) \quad \begin{aligned} T|_{r=\varepsilon} &= 1 - K_1 \varepsilon^2 \left(\frac{1}{\varepsilon^2} - 2\frac{k}{\varepsilon} \cos \theta + k^2 \cos^2 \theta \right) + O(\varepsilon^2) \\ &= 1 - K_1 + 2K_1 \varepsilon k \cos \theta + O(\varepsilon^2) = \bar{T}_I + 2K_1 \varepsilon k \cos \theta + O(\varepsilon^2) \end{aligned}$$

where $\bar{T}_I = 1 - K_1$. We will solve (3.1.1) and (3.1.4) asymptotically inside and outside the filament.

Inside the filament, we let $\rho = \frac{r}{\varepsilon}$ and write (3.1.1) as:

$$(3.1.5) \quad \begin{aligned} \frac{1}{\varepsilon} \frac{\partial}{\partial \rho} \left(\varepsilon \rho \eta (1 - \varepsilon k \rho \cos \theta) \frac{1}{\varepsilon} \frac{\partial T}{\partial \rho} \right) + \frac{\partial}{\partial \theta} \left(\frac{\eta (1 - \varepsilon k \rho \cos \theta)}{\varepsilon \rho} \frac{\partial T}{\partial \theta} \right) \\ + \frac{\partial}{\partial s} \left(\frac{\varepsilon \rho}{\eta (1 - \varepsilon k \rho \cos \theta)} \frac{\partial T}{\partial s} \right) = 0 \end{aligned}$$

Assuming that $T = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \dots$, we obtain from (3.1.5) that:

$$(3.1.6) \quad T_0 = \bar{T}_I$$

$$(3.1.7) \quad T_1 = 2K_1 k \rho \cos \theta$$

Therefore we have

$$(3.1.8) \quad T = \bar{T}_I + 2K_1 k r \cos \theta + O(r^2)$$

inside the filament.

Outside the filament, we divide the isotropic phase into two regions, an inner region near the filament and an outer region far from the filament. In the inner region, we proceed as in [16] and obtain:

$$(3.1.9) \quad T = \bar{T}_I + \tilde{T} \log(\rho) + O(\varepsilon) = \bar{T}_I + \tilde{T}(\log r + \log \frac{1}{\varepsilon}) + O(\varepsilon)$$

\tilde{T} is an unknown to be determined later.

In the outer region, we write T as a sum of heat sources:

$$(3.1.10) \quad T(\mathbf{x}) = T_\infty + \int_\Gamma \frac{1}{|\mathbf{x} - \mathbf{X}(s)|} \beta(s) ds$$

where T_∞ is the normalized temperature at infinity. The unknown density β can be determined by matched asymptotics as in [16]. In the inner end of the outer region, if we let $r = \text{dist}(\mathbf{x}, \Gamma) = |\mathbf{x} - \mathbf{X}(s)|$, then

$$(3.1.11) \quad \int_\Gamma \frac{1}{|\mathbf{x} - \mathbf{X}(s)|} \beta(s) ds = 2\beta(s) \log r + I_\Gamma(\beta)(s)$$

where $I_\Gamma : \beta \rightarrow I_\Gamma(\beta)$ is a bounded integral operator.

Matching of (3.1.9) and (3.1.11) gives:

$$(3.1.12) \quad \tilde{T} = 2\beta$$

$$(3.1.13) \quad I_\Gamma(\beta)(s) = \bar{T}_I - T_\infty + 2\beta(s) \log \frac{1}{\varepsilon} = \delta T + 2\beta(s) \log \frac{1}{\varepsilon}$$

where $\delta T = \bar{T}_I - T_\infty$ is a measure of the undercooling. Therefore

$$(3.1.14) \quad \beta(s) = -(2 \log \frac{1}{\varepsilon} - I_\Gamma)^{-1} \delta T = -\frac{\delta T}{2 \log \frac{1}{\varepsilon}}$$

to leading order.

3.2. Energy balance across the interface

We will denote by $\tilde{\mathbf{u}}$ the velocity field as seen from the moving curvilinear coordinate system, and write $\tilde{\mathbf{u}} = \tilde{u}_1 \hat{\mathbf{r}} + \tilde{u}_2 \hat{\boldsymbol{\theta}} + \tilde{u}_3 \hat{\mathbf{s}}$. Denote by V_c the velocity of the centerline Γ in

the direction of its principal normal $\hat{\mathbf{n}}$. The total fluid velocity is then $\mathbf{u} = V_c \hat{\mathbf{n}} + \tilde{\mathbf{u}}$. From (2.2.21), we have

$$(3.2.1) \quad \tilde{u}_1 T_I = \kappa_l \frac{\partial T_l}{\partial r} - \kappa_s \frac{\partial T_s}{\partial r}$$

Using (3.1.14) and (3.1.8), we get

$$(3.2.2) \quad \tilde{u}_1 (\bar{T}_I + 2K_1 k \varepsilon \cos \theta) = -\kappa_l \frac{\delta T}{\varepsilon \log \frac{1}{\varepsilon}} - 2\kappa_s K_1 k \cos \theta$$

To leading order, this gives:

$$(3.2.3) \quad \tilde{u}_1 = -\frac{\kappa_l}{\varepsilon \log \frac{1}{\varepsilon}} \frac{\delta T}{\bar{T}_I}$$

This is the velocity at which molecules permeate through the skin of the filament. In dimensional units, we have:

$$\tilde{u}_1 = -\frac{\kappa_l}{r_0 \log \frac{r_0}{R}} \frac{\delta T}{L}$$

3.3. Permeation and growth rates

(3.2.3) gives the permeation rate at the surface of the filament. To leading order it only depends on material constants. This uniformity of the permeation rate is a consequence of the uniformity of the temperature gradient at the interface, given that the filament diameter and the temperature distribution at the interface is nearly uniform. We will denote this constant permeation rate as C_1 .

Knowing the permeation rate, we can compute easily the growth rate α . Let $V(t)$ be the total volume of the filament at time t . We have:

$$(3.3.1) \quad \frac{dV(t)}{dt} = \pi \varepsilon^2 \frac{dL(t)}{dt} = -\tilde{u}_1 \times \text{surface area} = 2\pi \varepsilon L(t) C_1$$

This gives

$$(3.3.2) \quad \alpha = \frac{2C_1}{\varepsilon} = \frac{2\kappa_l \delta T}{\varepsilon^2 (\log \frac{1}{\varepsilon}) \bar{T}_I}$$

In dimensional units, this gives

$$(3.3.2') \quad \alpha = \frac{2\kappa_l \delta T}{r_0^2 (\log \frac{r_0}{R}) L}$$

This is consistent with the experimental observation that the growth rate increases with the increase of the quench depth.

3.4. Transport of molecules along the filament

It is natural to assume that $\tilde{u}_2 = 0$, i.e. the azimuthal velocity can be neglected. Since $\hat{\mathbf{n}} = \hat{\mathbf{r}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta$, we have

$$(3.4.1) \quad \mathbf{u} = (V_c \cos \theta + \tilde{u}_1) \hat{\mathbf{r}} - V_c \sin \theta \hat{\boldsymbol{\theta}} + \tilde{u}_3 \hat{\mathbf{s}}$$

The incompressibility condition gives:

$$(3.4.2) \quad \begin{aligned} & \frac{\partial}{\partial r}(r h_3 \tilde{u}_1) + \frac{\partial}{\partial r}(r h_3 V_c \cos \theta) - \frac{\partial}{\partial \theta}(V_c \sin \theta h_3) + \frac{\partial}{\partial s}(r \tilde{u}_3) \\ &= \frac{\partial}{\partial r}(r h_3 \tilde{u}_1) + r \frac{\partial}{\partial s} \tilde{u}_3 - V_c k \eta r = 0 \end{aligned}$$

This implies

$$(3.4.3) \quad \frac{\partial}{\partial r}(r h_3 \tilde{u}_1) = r(V_c k \eta - \frac{\partial}{\partial s} : \tilde{u}_3)$$

Integrating in r , and letting $\bar{u}_3 = \frac{1}{\varepsilon} \int_0^\varepsilon \tilde{u}_3 dr$, we get

$$(3.4.4) \quad V_c k \eta - \frac{\partial \bar{u}_3}{\partial s} = \frac{2 \tilde{u}_1 h_3}{r}$$

To leading order at the filament surface, we have

$$(3.4.5) \quad \frac{\partial \bar{u}_3}{\partial s} - V_c k \eta = \frac{2 \eta \kappa_l \delta T}{\varepsilon^2 \log \frac{1}{\varepsilon} \bar{T}_I} = \alpha \eta$$

Aside from the curvature effects, the rate of transport of molecules along the filament is uniform.

3.5. Viscous forces exerted on the filament by the isotropic fluid

This has been calculated in the fluid dynamics literature, assuming that inertial effects can be neglected [5, 16, 1]. Denote the velocity of the filament centerline by $\mathbf{v}(V_c = \mathbf{v} \cdot \hat{\mathbf{n}})$, the viscous drag exerted on the filament by the isotropic fluid outside is:

$$(3.5.1) \quad \mathbf{F}_1 = -\frac{2\pi\mu_l\eta}{4\log\frac{1}{\varepsilon}} \int_{\Gamma} (2I - \hat{\mathbf{s}} \otimes \hat{\mathbf{s}})(\mathbf{v} + \text{lower order terms}) ds$$

The lower order terms are of a non-local nature. They represent the force due to fluids at a distance. Notice the presence of an anisotropic tensor in front of \mathbf{v} . A consequence of

this result is that a horizontally falling rod experiences twice as much drag as a vertically falling rod at the same speed.

3.6. Elastic forces inside the filament

Again assuming that inertial effects can be neglected, we have

$$(3.6.1) \quad \nabla \cdot \sigma = 0$$

where $\sigma = -pI + \sigma_e + \sigma_d$.

(Note to printer: please place Figure 4 near here.)

Take a cylindrical domain Ω along the filament centerline such that the side surface Σ which generates the cylinder is located on a layer. We will denote the circular surfaces at the two ends by A_1 and A_2 . Then

$$(3.6.2) \quad \iiint_{\Omega} (\nabla \cdot \sigma_e) dV = \iint_{\Sigma} (\sigma_e \cdot \mathbf{n}) dA + \iint_{A_2} (\sigma_e \cdot \hat{\mathbf{s}}) dA - \iint_{A_1} (\sigma_e \cdot \hat{\mathbf{s}}) dA$$

Recall that $\sigma_e = \tau \otimes \mathbf{n} - K(\nabla \cdot \mathbf{n})\nabla \mathbf{n}^T$ where $\tau = -\xi \mathbf{n} + K\{\nabla(\nabla \cdot \mathbf{n}) - (\nabla(\nabla \cdot \mathbf{n}) \cdot \mathbf{n})\mathbf{n}\}$. Below we will write $K\xi$ in place of ξ . We will write $\mathbf{n} = (n_1, n_2, n_3)$, $\hat{\mathbf{s}} = (s_1, s_2, s_3)$ in a fixed rectangular coordinate system. On Σ we have (using summation convention)

$$(3.6.3) \quad \{[\tau \otimes \mathbf{n} - (\nabla \cdot \mathbf{n})(\nabla \mathbf{n})^T] \cdot \mathbf{n}\}_i = -\xi n_i$$

$$(3.6.4) \quad -\xi(\mathbf{n} \otimes \mathbf{n})\mathbf{n} = -\xi \mathbf{n}$$

$$(3.6.5) \quad \{\nabla(\nabla \cdot \mathbf{n}) - (\nabla(\nabla \cdot \mathbf{n}) \cdot \mathbf{n})\} \cdot \mathbf{n} = 0$$

$$(3.6.6) \quad \{(\nabla \mathbf{n})^T \mathbf{n}\}_i = \frac{\partial n_j}{\partial x_i} n_j = 0$$

Hence we obtain

$$(3.6.7) \quad \iint_{\Sigma} (\sigma_e \cdot \mathbf{n}) dA = -K \iint_{\Sigma} \xi \mathbf{n} dA = K\eta r_0 \int \tilde{\xi}(s) \hat{\mathbf{n}}(s) ds$$

where r_0 is the radius of A_1 and A_2 , $\tilde{\xi} = \int_0^{2\pi} \xi \cos \theta d\theta$. By symmetry, the coefficient of the $\hat{\mathbf{b}}$ term must be zero.

On A_2 the normal to the surface is $\hat{\mathbf{s}}$. Since

$$(3.6.8) \quad \begin{aligned} \{\sigma_e \cdot \hat{\mathbf{s}}\}_i &= \left\{ K (\nabla(\nabla \cdot \mathbf{n}) \cdot \hat{\mathbf{s}}) \mathbf{n} - K (\nabla \cdot \mathbf{n}) (\nabla \mathbf{n})^T \hat{\mathbf{s}} \right\}_i \\ &= K (\nabla(\nabla \cdot \mathbf{n}) \cdot \hat{\mathbf{s}}) n_i + K (\nabla \cdot \mathbf{n}) n_j \frac{\partial s_j}{\partial x_i} \end{aligned}$$

and

$$\nabla(\nabla \cdot \mathbf{n}) \cdot \hat{\mathbf{s}} = -\frac{1}{h_3} k_s \cos \theta$$

we have

$$(3.6.9) \quad \begin{aligned} \iint_{A_2} K (\nabla(\nabla \cdot \mathbf{n}) \cdot \hat{\mathbf{s}}) \mathbf{n} dA &= -K \iint_{A_2} \frac{k_s \cos \theta}{\eta(1 - kr \cos \theta)} \mathbf{n} r dr d\theta \\ &= -K \frac{k_s}{\eta} \int_0^{r_0} r dr \int_0^{2\pi} d\theta \left\{ \hat{\mathbf{n}} (\cos^2 \theta + kr \cos^3 \theta) + \hat{\mathbf{b}} (\sin \theta \cos \theta + kr \cos^2 \theta \sin \theta) \right\} \\ &= -K \frac{k_s}{\eta} \int_0^{r_0} r dr \pi \hat{\mathbf{n}} = -\frac{\pi K k_s}{2\eta} r_0^2 \hat{\mathbf{n}} \end{aligned}$$

To compute $\frac{\partial s_i}{\partial x_i} n_j$, we choose the coordinate system such that A_2 is in the $y - z$ plane, and the centerline curve Γ is in the $x - y$ plane, i.e. $s_3 = 0, n_1 = 0$. We then have

$$(3.6.10) \quad (\nabla \mathbf{s}) \mathbf{n} = n_2 \nabla s_2$$

Using the Frenet formula, we have

$$(3.6.11) \quad \nabla s_2 = \hat{\mathbf{s}} \frac{1}{h_3} \frac{\partial s_2}{\partial s} = \hat{\mathbf{s}} \frac{1}{h_3} \eta k n_2 = \hat{\mathbf{s}} \frac{1}{h_3} \eta k$$

Therefore we have, to leading order:

$$(3.6.12) \quad \begin{aligned} \iint_{A_2} K (\nabla \cdot \mathbf{n}) n_2 \nabla s_2 dA &= K \hat{\mathbf{s}} \iint_{A_2} \left(\frac{1}{r} - k \cos \theta \right) \frac{\cos \theta}{\eta} (1 + kr \cos \theta) k \eta dA \\ &= K k \hat{\mathbf{s}} \int_0^{r_0} dr \int_0^{2\pi} \cos \theta (1 - k^2 r^2 \cos^2 \theta) d\theta = \mathcal{O}(r_0^4) \end{aligned}$$

In summary, the leading order contribution from the elastic stress is:

$$(3.6.13) \quad \mathbf{F}_2 = -\frac{\pi K}{2\eta} r_0^2 \{ (k_s \hat{\mathbf{n}})|_{A_2} - (k_s \hat{\mathbf{n}})|_{A_1} \} + K \eta r_0 \int \tilde{\xi}(s) \hat{\mathbf{n}}(s) ds$$

3.7. Viscous forces inside the filament

The calculation of the viscous forces inside the filament proceeds in the same way.

Neglecting effects of anisotropy, we have:

$$(3.7.1) \quad \iint_{\Sigma} (\sigma_d \cdot \hat{\mathbf{n}}) dA = \mu_s \iint_{\Sigma} (e_{11} \hat{\mathbf{r}} + e_{12} \hat{\boldsymbol{\theta}} + e_{13} \hat{\mathbf{s}}) dA$$

where (e_{ij}) denotes the rate of strain tensor in the curvilinear coordinates. Using $\hat{\mathbf{r}} = \hat{\mathbf{n}} \cos \theta + \hat{\mathbf{b}} \sin \theta$, we get

$$(3.7.12) \quad \begin{aligned} \iint_{\Sigma} e_{11} \hat{\mathbf{r}} dA &= \eta \iint_{\Sigma} \hat{\mathbf{n}}(s) e_{11} \cos \theta r_0 d\theta ds + \hat{\mathbf{b}} \iint_{\Sigma} e_{11} \sin \theta r_0 d\theta ds \\ &= \eta r_0 \int \hat{\mathbf{n}}(s) \lambda_1(s) ds \end{aligned}$$

where $\lambda_1 = \int_0^{2\pi} e_{11} \cos \theta d\theta$. Similarly,

$$(3.7.3) \quad \iint_{\Sigma} e_{12} \hat{\boldsymbol{\theta}} dA = \eta r_0 \int \hat{\mathbf{n}}(s) \lambda_2(s) ds$$

$$(3.7.4) \quad \begin{aligned} \iint_{\Sigma} e_{13} r_0 \hat{\mathbf{s}} d\theta ds &= r_0 \int ds \int d\theta \left\{ \frac{1}{2h_3} \frac{\partial}{\partial s} : u_1 + \frac{h_3}{2} \frac{\partial}{\partial r} \left(\frac{u_3}{h_3} \right) \right\} \hat{\mathbf{s}} \\ &= r_0 \int \hat{\mathbf{s}} ds \int \frac{h_3}{2} \frac{\partial}{\partial r} \left(\frac{u_3}{h_3} \right) d\theta \\ &= \frac{1}{2} : r_0 \int \hat{\mathbf{s}} ds \int \frac{\partial u_3}{\partial r} : d\theta = \frac{1}{2} : r_0 \int \frac{\partial \bar{u}_3}{\partial r} \hat{\mathbf{s}} ds \end{aligned}$$

where $\bar{u}_3 = \int_0^{2\pi} u_3 d\theta$, and

$$(3.7.5) \quad \iint_{A_2} (\sigma_d \cdot \hat{\mathbf{s}}) dA = \mu_s \iint_{A_2} (e_{13} \hat{\mathbf{r}} + e_{23} \hat{\boldsymbol{\theta}} + e_{33} \hat{\mathbf{s}}) dA$$

$$(3.7.6) \quad \begin{aligned} \iint_{A_2} e_{13} \hat{\mathbf{r}} dA &= \hat{\mathbf{n}} \iint_{A_2} e_{13} \cos \theta dA \\ &= \hat{\mathbf{n}} \iint_{A_2} \left(\frac{1}{2h_3} \frac{\partial V_c}{\partial s} : \cos \theta + \frac{h_3}{2} \frac{\partial}{\partial r} \left(\frac{\tilde{u}_3}{h_3} \right) \right) \cos \theta r dr d\theta \\ &= \frac{\pi}{4\eta} \frac{\partial V_c}{\partial s} : r_0^2 \hat{\mathbf{n}} + \mathcal{O}(r_0^2) = \mathcal{O}(r_0^2) \end{aligned}$$

$$(3.7.7) \quad \iint_{A_2} e_{23} \hat{\boldsymbol{\theta}} dA = \mathcal{O}(r_0^2)$$

$$(3.7.8) \quad \iint_{A_3} e_{33} \hat{\mathbf{s}} dA = \mathcal{O}(r_0^2)$$

In summary, the leading order contribution from the viscous stress is:

$$(3.7.9) \quad \mathbf{F}_3 = \mu_s \eta r_0 \int_{\Gamma} (\lambda_1(s) + \lambda_2(s)) \hat{\mathbf{n}}(s) ds + \frac{\mu_s}{2} \eta r_0 \int_{\Gamma} \frac{\partial \bar{u}_3}{\partial r} \hat{\mathbf{s}} ds$$

The contribution of pressure can also be computed:

$$(3.7.10) \quad \iint_{\Sigma} -p \mathbf{n} dA = -\eta r_0 \int \bar{p}(s) \hat{\mathbf{n}}(s) ds$$

where $\bar{p}(s) = \int_0^{2\pi} p d\theta$.

$$(3.7.11) \quad \iint_{A_2} -p \hat{\mathbf{s}} dA = -\hat{\mathbf{s}} \iint_{A_2} p dA = -\hat{\mathbf{s}} p_0$$

Therefore the total contribution from the pressure is:

$$(3.7.12) \quad \mathbf{F}_4 = -\eta r_0 \int \bar{p}(s) \hat{\mathbf{n}}(s) ds - (\hat{\mathbf{s}} p_0|_{A_2} - \hat{\mathbf{s}} p_0|_{A_1})$$

Balancing the forces, we get

$$\mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = 0$$

This gives

$$(3.7.13) \quad K \eta r_0 \tilde{\xi} \hat{\mathbf{n}} - \frac{\pi K r_0^2}{2\eta} (k_s \hat{\mathbf{n}})_s + \mu_s \eta r_0 (\lambda_1 + \lambda_2) \hat{\mathbf{n}} + \frac{\mu_s}{2} \eta r_0 \frac{\partial \bar{u}_3}{\partial r} \hat{\mathbf{s}} - \eta r_0 \bar{p} \hat{\mathbf{n}} - (p_0 \hat{\mathbf{s}})_s = 0$$

3.8. Motion of the centerline

Integrating the momentum equation on a cylindrical domain covering a piece of the filament Γ , we get

$$(3.8.1) \quad \mathbf{F}_1 = \frac{\pi K \varepsilon^2}{2\eta} \{k_s \hat{\mathbf{n}}|_{A_2} - k_s \hat{\mathbf{n}}|_{A_1}\} + \{p_0 \hat{\mathbf{s}}|_{A_2} - p_0 \hat{\mathbf{s}}|_{A_1}\}$$

where \mathbf{F}_1 was given in Section 3.5. Hence we get

$$(3.8.2) \quad \frac{2\pi \mu_l}{4 \log \frac{1}{\varepsilon}} (2I - \hat{\mathbf{s}} \otimes \hat{\mathbf{s}}) (\mathbf{v} + \text{smaller terms}) + \frac{\pi K \varepsilon^2}{2\eta^2} (k_s \hat{\mathbf{n}})_s + \frac{1}{\eta} (p_0 \hat{\mathbf{s}})_s = 0$$

This is the equation governing the dynamics of the filament. It coincides with the model proposed in [25] on phenomenological grounds. (3.8.2) represents a balance of viscous forces

exerted on the filament by the isotropic fluid and the elastic forces inside the filament. In the terminology of elasticity theory of thin rods, the second term is a bending force, the third term is a tension force. p_0 can be thought of as the line tension, which acts as the Lagrange multiplier to maintain the constant rate of growth for the filament. As can be seen from the derivation, the physical origin of the tension is the average pressure over the cross-section of the filament.

Equations of the type (3.8.2) have been studied in many different contexts. In the absence of the viscous term, this equation describes the classical Euler's problem of elastica. A huge literature exists on this problem both in the mathematics and mechanics community. [13] contains a detailed study of the inertial dynamics of rods undergoing twist and bend. The viscous dynamics (3.8.2) was used in [2] as a model for the dynamics of actins gliding over a surface, and in [12] as a model for stiff polymers. Most relevant to the present paper are the careful numerical studies in [25] where this equation was proposed as a model for the dynamics of filaments.

3.9. Buckling instability

We next give a simple analysis of the buckling instability, triggered by the growth. For simplicity of presentation we will neglect the anisotropic mobility tensor in (3.8.2), as well as unimportant constants.

We write (3.8.2) in the following form for the position of the filament \mathbf{X} (recall that s is the scaled arclength: $0 \leq s \leq 1$):

$$(3.9.1) \quad \bar{\mu} \mathbf{X}_t = (-\bar{K} k_{ss} + p \eta k) \hat{\mathbf{n}} + (\bar{K} \eta k_s k + p_s) \hat{\mathbf{s}}$$

$$(3.9.2) \quad \bar{\mu} \eta_t = p_{ss} + \bar{K} \eta (k_s k)_s + \bar{K} \eta k k_{ss} - \eta^2 k^2 p$$

where $\bar{\mu} = \mu \eta$, $\bar{K} = \frac{\pi K e^2}{2 \eta}$. (3.9.2) is the equation for the Lagrange multiplier p which ensures the correct rate of growth. (3.9.1) and (3.9.2) admit a solution in the form of circular filaments:

$$\mathbf{X}(s, t) = \bar{r}(t) (\cos(2\pi s), \sin(2\pi s)),$$

where $\bar{r}(t) = \frac{1}{2\pi}\eta(t)$, $\bar{k}(t) = \frac{1}{\bar{r}(t)}$, $\bar{p} = -\frac{1}{(2\pi)^2}\bar{\mu}\eta(t)_t$.

Consider perturbations of the circular solution in the form: $r(s, t) = \bar{r}(t) + \delta r(t) \sin(m\theta)$, $\theta = 2\pi s$, and $p(s, t) = \bar{p}(t) + \delta p(t) \sin(m\theta)$. Using the curvature formula: $k = \frac{r^2 + 2r_\theta^2 - rr_{\theta\theta}}{(r^2 + r_\theta^2)^{3/2}}$, we can calculate the leading order perturbation of curvature:

$$\delta k = \frac{m^2 - 1}{\bar{r}^2} \delta r(t) \sin(m\theta)$$

The linearized equation takes the form:

$$\bar{\mu}(\delta r)_t \sin(m\theta) = \bar{K}(\delta k)_{ss} - \bar{p}\eta\delta k + 2\pi\delta p = 0$$

$$(\delta p)_{ss} + 2\bar{K}\eta\bar{k}(\delta k)_{ss} - \eta^2(\bar{k}^2\delta p + 2\bar{k}\bar{p}\delta k) = 0$$

We obtain then:

$$(\delta r)_t = \omega \delta r$$

where $\omega = \frac{\bar{k}^2}{\bar{\mu}}[-(2\pi^2)\bar{K}m^2(m^2 - 1) - \bar{p}\eta(m^2 - 1) + 2(\bar{K}m^2 + \eta\bar{p})]$. The maximum growth occurs at the mode number $m^* = (-\frac{\bar{p}\eta}{\bar{K}})^{1/2}$. In dimensional units, we have:

$$(m^*)^2 = \frac{\mu\eta^3\eta_t}{Kr_0^2} = \frac{1}{A} \left(\frac{\eta}{r_0}\right)^4$$

where

$$A = \frac{KL}{\mu\kappa\delta T}$$

is the a dimensionless quantity measuring the ratio of solidification current and the permeation current. This quantity plays a very important role in [7]. The maximum growth rate is approximately:

$$\omega^* = \frac{\mu\eta^2\eta_t^2}{Kr_0^2} = \frac{1}{A} \left(\frac{\eta}{r_0}\right)^4 \frac{\kappa\delta T}{r_0^2 L}$$

At very early times the circular filament is stable since the length is too short to accommodate any unstable mode. The critical time for instability can be estimated by requiring $m^* > 1$. This gives

$$t^* = \frac{3}{4\alpha} \log(A)$$

where α is the growth rate calculated in (3.3.2).

§4. Concluding Remarks

Let us summarize what we have done in this paper. By setting up a solidification model which describes the details of the hydrodynamic, elastic and thermodynamic processes during the $I - S_A$ transition, we are able to calculate the growth rate, permeation velocity, transport velocities along the filament, elastic and viscous stresses inside and outside the filament. We have also derived the dynamical equation for the filament motion, and analysed the growth induced buckling instability of the filament.

Many other aspects of the filament growth can be studied within the framework of the present paper. These include in particular the various dynamic instabilities of the filament. A quantitative discussion of these will be presented in [7].

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Figure Captions

Figure 1. Appearance of filaments when DCOB is cooled below the $I - S_A$ transition temperature. The material is in the smectic phase inside the filament and isotropic phase outside the filament.

Figure 2. Same as in Figure 1, but at a later time.

Figure 3. Permeation and growth process inside the filament. The material is organized into concentric cylinders inside the filament. Smectic molecules permeate through the isotropic-smectic interface. Different layers elongate at approximately the same rate due to the permeation process between layers.

Figure 4. Coordinate system and notations used in the calculations in Sections 3.6 and 3.7.