Mathematics 323: Algebra and Applications

Lecture 11

Groups Acting on Sets -	The Sylow Theorems
First Sylow Theorem -	Group acts on Left Cosets by Left Multiplication
Second Sylow Theorem -	Same Group Action
Third Sylow Theorem -	Group acts on Sylow p-subgroups by Conjugation
Sylow 2 -	Subgroups of the Symmetric Group S_4

The Sylow Theorems

These describe the subgroups of prime power order in an arbitrary finite group.

$|G| = n = p^e m$ where p^e is the largest power of p dividing n.

First Sylow Theorem There is a subgroup with order p^e.

Group Action: G acts on subset of size p^e by left multiplication - we need to prove one of these subsets is a subgroup.

How many subsets of size p^e:

$$N = n \binom{n}{p^e} = \frac{n(n-1) \quad \ldots \quad (n - p^i l) \quad \ldots \quad (n - p^e + 1)}{p^e(p^e - 1) \quad \ldots \quad (p^e - p^i l) \quad \ldots \quad 1}$$

both terms divisible by p^i but not by p^{i+1}

N **is not divisible by** P

The First Sylow Theorem

What can we learn about the orbits of G:
Stab$(U) = \{g \in G \mid gU = U\}$ is always a subgroup
$|\text{Stab}(U)||\text{Orbit}_U| = |G| = p^e m$

If an orbit $\mathbf{O_U}$ has size prime to $\mathbf{p}$ then $\mathbf{p^e}$ divides $|$Stab$(\mathbf{U})|$

Look at Stab(U) acting on U by left multiplication:
U is a union of orbits and each orbit is a right coset of stab(U)

For every subset $\mathbf{U}$, we have $|$Stab$(\mathbf{U})|$ divides $|\mathbf{U}|$.

If an orbit O_U has size prime to p then $U =$ Stab(U) is a subgroup of order p^e

Formal Proof

$N = \sum_{\text{Orbits } O_U} |O_U|$ is not divisible by p
Hence there exists orbit O_U for which $|O_U|$ is not divisible by p
$U = \text{Stab}(U)$ is then a subgroup of order p^e $\square$

Corollary: If a prime p divides the order of a finite group G, then G contains an element of order p.

Proof: Let H be a subgroup of order p^e and let $x \in H, x \neq 1$.
x has order p^n so $x^{p^{n-1}}$ has order p. $\square$

Definition: $|G| = p^e m$. Subgroups of order p^e are called **Sylow p subgroups** of G.

Classifying Group of Order 6

Theorem: There are exactly two isomorphism classes of groups of order 6 - the cyclic group C_6 and the dihedral group D_3.

Proof: Let $x \in G$ be an element of order 3, and let $y \in G$ be an element of order 2.

$$G = \{1, x, x^2, y, xy, x^2y\}$$

$H = \{1, x, x^2\}$ is Sylow 3-subgroup and $\{1, x, x^2\}y$ is a different right coset. Therefore yx is one of $1, x, x^2, y, xy, x^2y$

In fact yx is one of y, xy, x^2y and $yx \neq y$ for otherwise $x = 1$

If $yx = xy$ then G is commutative and $G \approx C_6$

If $yx = x^2y$ then $G \approx D_3$.

In each case the relation uniquely determines the multiplication table for a group.

Second Sylow Theorem

Second Sylow Theorem Let K be a subgroup of G with order divisible by p, and let H be a Sylow p-subgroup of G. Then there is a conjugate subgroup $H' = gHg^{-1}$ such that $K \cap H'$ is a Sylow p-subgroup of K.

Group Action: G acts on left cosets of H by left multn

$$\begin{array}{lccc} \text{Stabilizer Group} & H & \rightarrow & aHa^{-1} \\ & | & & | \\ \text{Left Coset} & H & \rightarrow & aH \end{array}$$

The number of left cosets is $[G : H]$ which is not divisible by p.

Look at the orbits of the subgroup K - acting on left cosets of H by left multiplication.

One of these orbits has size not divisible by p.

Formal Proof

There is a left coset aH in orbit of K of size prime to p.

The stabilizer of aH in G is aHa^{-1} and the stabilizer of aH in K is $K \cap (aHa^{-1})$

$[K : (K \cap aHa^{-1})]$ is the size of the K-orbit of aH and is not divisible by p Hence $K \cap aHa^{-1}$ is a Sylow p-subgroup of K. $\square$

Corollary:

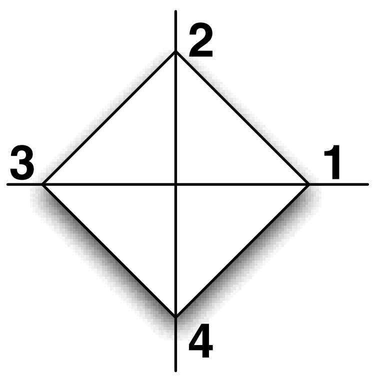
1. If K is any subgroup of G which is a p-group, then K is contained in a Sylow p-subgroup of G.
2. The Sylow p-subgroups of G are all conjugate.

Proof: The Sylow subgroup of a p-group K is the group K itself
Any conjugate of a Sylow p-subgroup is also a Sylow p-subgroup.
K is contained in some conjugate aHa^{-1}
If K is itself a Sylow p-subgroup then $K = aHa^{-1}$

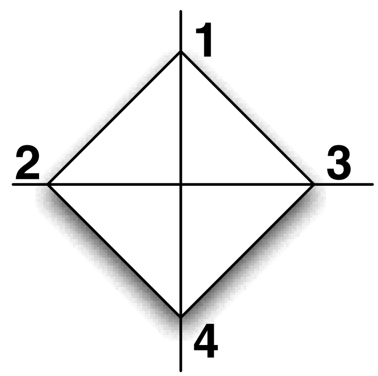
Sylow 2-Subgroups of the Symmetric Group S_4

There are 3 Sylow 2-subgroups - each is dihedral and all are conjugate.

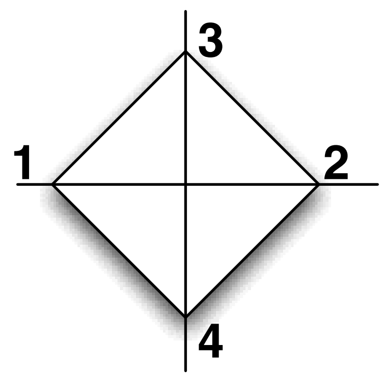
Every subgroup of order 2 or 4 appears in at least one of these subgroups - which subgroup appears in all Sylow 2-subgroups?



$H \approx D_4$
$x = (1234)$
$y = (24)$



$H' = (132)H(123)$
$x' = (1243)$
$y' = (14)$



$H'' = (123)H(132)$
$x'' = (1423)$
$y'' = (34)$

The Third Sylow Theorem

Third Sylow Theorem $|G| = n = p^e m$ and s is the number of Sylow p-subgroups. Then $s \mid m$ and $s \equiv 1 (\mod p)$.

Group Action G acts on Sylow p-subgroups by conjugation and there is one orbit.

$$\begin{array}{lccc} \text{Stabilizer Group} & N & \rightarrow & aNa^{-1} \\ & | & & | \\ \text{Sylow p-subgroup} & H & \rightarrow & aHa^{-1} \end{array} \qquad \begin{array}{c} N = \{g \in G \mid gHg^{-1} = H\} \\ \text{(normalizer)} \end{array}$$

H is the unique Sylow p-subgroup of N
aHa^{-1} is the unique Sylow p-subgroup of aNa^{-1}

How to prove $\mathbf{s \equiv 1(mod\ p)}$: let H act by conjugation on the set of Sylow p-subgroups.

Formal Proof

An orbit consists of a single Sylow p-subgroup H' if and only if H is contained in the normalizer N' of H'

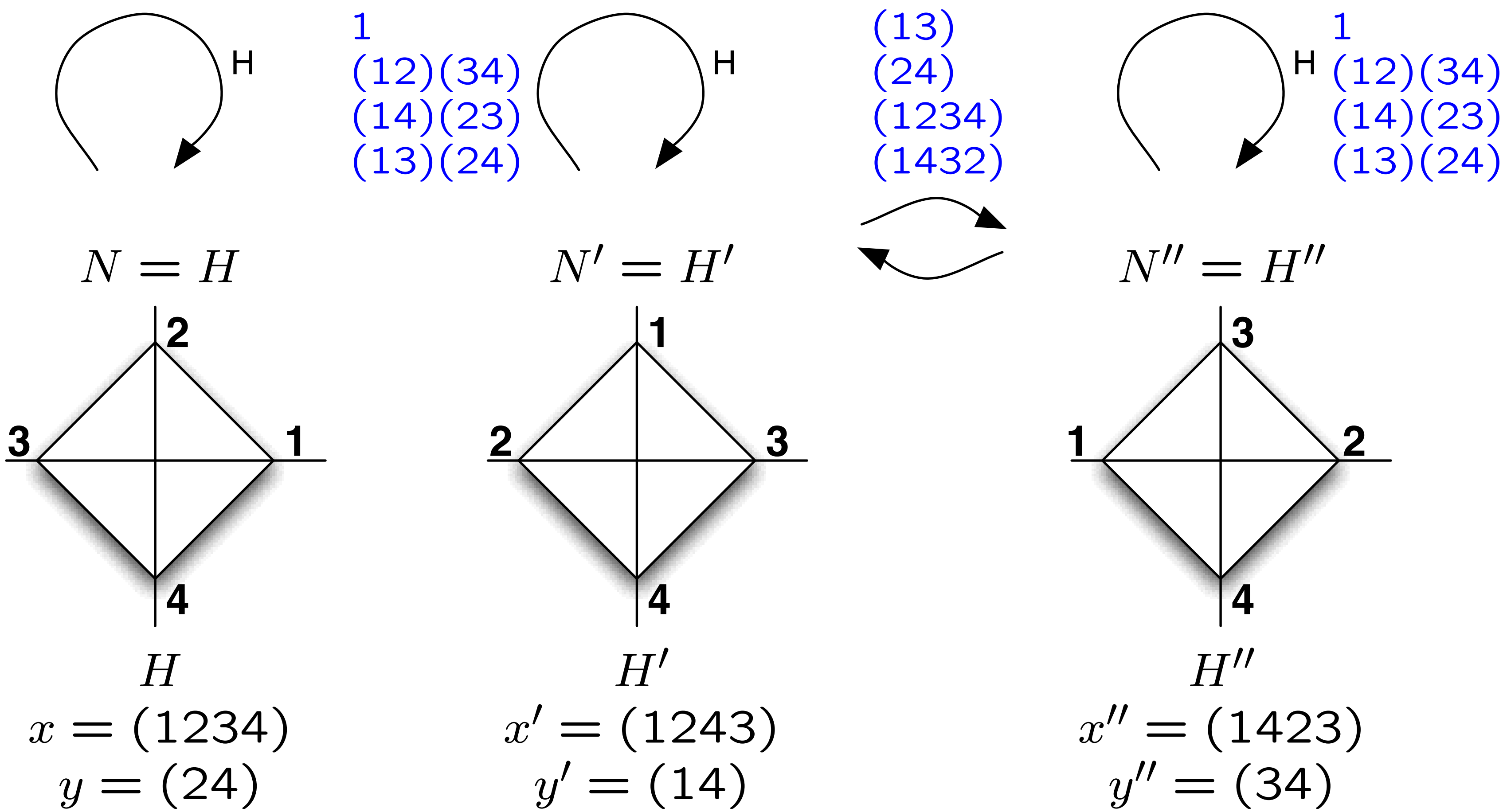
In this case H and H' are Sylow p-subgroups of N'

Hence $H = H'$.
This means that there is only one orbit consisting of a single Sylow p-subgroup (namely H) and that all other orbits have size divisible by p.

Hence $s \equiv 1 \pmod{p}$ $\square$

Sylow 2 - Subgroups of the Symmetric Group S_4

Group Action conjugation by elements of H.



Every Group of Order 15 is Cyclic

Let G be a group of order 15

The number of Sylow 3-subgroups divides 5 and is congruent to 1 mod 3.

Hence there is a unique Sylow 3-subgroup H that is normal in G
Similarly - there is a unique Sylow 5-subgroup K and it is normal in G

H commutes with K:
$H \cap K = \{1\}$ and $hkh^{-1}k{-}1$ is in both H and K

$\mathbf{G = HK}$: HK is a group that properly contains both H and K
$\Phi : (h,k) \to hk$ is a homomorphism from $H \times K$ to G

ker $\Phi = \{1\}$ so $G \approx H \times K \approx C_3 \times C_5 \approx C_{15}$