MATH 306 — HOMEWORK 6

Due in class on Tuesday, March 25th

1. Let G be a digraph and for each edge e let $\phi(e) \geq 0$ be an integer, so that for every vertex v,
$$\sum_{e \in \delta^-(v)} \phi(e) = \sum_{e \in \delta^+(v)} \phi(e)$$
Show there is a list $C_1, ..., C_n$ of directed cycles (possibly with repetition) so that for every edge e of G,
$$|\{i : 1 \leq i \leq n,\ e \in E(C_i)\}| = \phi(e).$$

2. Let s, t be distinct vertices of a digraph G, and let $c : E(G) \to \mathbb{Z}_+$. Let ϕ be a c-admissible $s - t$ flow of total value k and suppose there is a c-admissible flow with total value $> k$. Does it follow that there is a c-admissible flow ψ of total value $> k$ so that $\phi(e) \leq \psi(e)$ for every edge e?

3. Let s, t be vertices of a digraph G, and let $\phi : E(G) \to \mathbb{R}_+$ be an $s - t$ flow. Show that there is an $s - t$ flow $\psi : E(G) \to \mathbb{Z}_+$ so that
   (i) its total value is at least that of ϕ, and
   (ii) $|\psi(e) - \phi(e)| < 1$ for every edge e of G.

4. Let T be a tree. Show (**without** using Tutte's 1-factor theorem) that T has a perfect matching if and only if for every vertex v, exactly one of the components of $T \backslash v$ has an odd number of vertices. [Hint for "if": use induction on $|V(T)|$, find a leaf with a neighbour of degree 2, and delete them both.]