MATH 306 — HOMEWORK 5

Due in class on Tuesday, March 11th

Let G be a graph and $X \subseteq V(G)$. We denote by $\delta(X)$ the set of all edges of G with one end in X and the other in $V(G) - X$. If $D \subseteq E(G)$, D is called a *cut* if $D = \delta(X)$ for some $X \subseteq V(G)$ with $\emptyset \neq X \neq V(G)$.

1. Show that if C is a cycle of G and D is a cut then $|E(C) \cap D|$ is even. Deduce that if $Z \subseteq E(G)$, then there is a cut D with $Z \subseteq D$ if and only if there is no cycle C of G with odd length and with $E(C) \subseteq Z$.

2. $A \subseteq E(G)$ is *even-degree* if every vertex of G is incident with an even number of non-loop edges in A. Show that if A and B are both even-degree then so is $(A-B) \cup (B-A)$. Deduce that if T is a spanning tree of G, there is an even-degree set $A \subseteq E(G)$ with $A \cup E(T) = E(G)$.

3. Show that if G is connected and loopless and $Z \subseteq E(G)$, there is an even-degree set $A \subseteq E(G)$ with $Z \subseteq A$ if and only if there is no cut $D \subseteq Z$ with $|D|$ odd. (Hint for "if": use induction on $|Z|$. If there is no cut included in Z, show there is a spanning tree T with $Z \subseteq E(G) - E(T)$, and use question 2. If there is a cut included in Z, remove one of its elements from Z and use induction).

4.
   (i) Say distinct $u, v \in V(G)$ are *k-linked* if there are k paths $P_1, ..., P_k$ of G from u to v so that $E(P_i \cap P_j) = \emptyset$ $(1 \leq i < j \leq k)$. Suppose u, v, w are distinct and u, v are k-linked, and so are v, w. Does it follow that u, w are k-linked?

   (ii) Say subsets $X, Y \subseteq V(G)$ are *k-joined* if $|X| = |Y| = k$ and there are k paths $P_1, ..., P_k$ of G from X to Y so that $V(P_i \cap P_j) = \emptyset$ $(1 \leq i < j \leq k)$. Suppose $X, Y, Z \subseteq V(G)$ and X, Y are k-joined, and so are Y, Z. Does it follow that X, Z are k-joined?

5. Show that if G is a 2-connected graph with no loops, then for any two edges e, f there is a cycle C of G with $e, f \in E(C)$.