

Title “Optimal error estimates in stochastic homogenization”

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In applications like water flow through a porous medium, one has to solve an elliptic equation  $-\nabla \cdot (a(x)\nabla u) = f$  with heterogeneous coefficients  $a(x)$  that vary on a characteristic length scale  $\ell$  much smaller than the domain size. The coefficients  $a(x)$  are typically characterized in statistical terms: Their statistics are assumed to be stationary (i. e. translation invariant) and to decorrelate over distances large compared to  $\ell$ . In this situation, it is known from the theory of stochastic homogenization that the solution operator, i. e. the inverse of the elliptic operator  $-\nabla \cdot a(x)\nabla$ , behaves – on scales large compared to  $\ell$  – like the inverse of  $-\nabla \cdot a_{hom}\nabla$  with *homogeneous, deterministic* coefficients  $a_{hom}$ . This is a major reduction in complexity.

Hence the relation between the statistics of the heterogeneous coefficients  $a(x)$  and the value of the homogenized coefficient  $a_{hom}$  is of practical importance. Stochastic homogenization also provides a formula for  $a_{hom}$  in terms of the ensemble of  $a(x)$ . It involves the solution of the “corrector problem”  $-\nabla \cdot (a(x)(\nabla\phi_\xi + \xi)) = 0$  in the whole space  $\mathbb{R}^d$  for a given direction  $\xi$ . The formula then is given by  $a_{hom}\xi = \langle a(\nabla\phi_\xi + \xi) \rangle$ , where  $\langle \cdot \rangle$  denotes the ensemble average. Despite its simplicity, this formula has to be approximated in practise:

- 1) The corrector problem can only be solved for a small number of realizations of the coefficients  $a(x)$ . Thus, appealing to ergodicity, the ensemble average has to be replaced by a spatial average over a region of large diameter  $L$ .
- 2) The corrector problem can only be solved in a finite domain of large diameter  $L$ , thereby introducing some artificial boundary conditions.

In this talk, we present optimal estimates on both errors for the simplest possible model problem: We consider a discrete elliptic equation on the  $d$ -dimensional lattice  $\mathbb{Z}^d$  with random coefficients  $a$ , which are identically distributed and independent from edge to edge. This makes a connection with the area of “random walks in random environments”. We establish the optimal scaling of both errors in the ratio  $L/\ell \gg 1$  (where the correlation length  $\ell$  is unity in our model problem). It turns out that the scaling is the same as in the case of coefficients  $a(x)$  that are very close to the identity

(where the corrector problem can be linearized). Hence with respect to the error scaling, the highly nonlinear relation between  $a(x)$  and  $a_{hom}$  behaves like the linearized one.

Our methods involve spectral gap estimates and estimates on the Green's function which only depend on the ellipticity ratio  $\lambda$  of  $a$ .

This is joint work with Antoine Gloria, INRIA Lille.