

Symmetry and Regularity of Solutions for Nonlinear Systems of Wolff Type.

Wenxiong Chen Congming Li

May 1, 2010

Abstract

In this talk, we will consider radial symmetry and regularity for positive solutions of the fully nonlinear integral systems involving Wolff potentials:

$$\begin{cases} u(x) = W_{\beta,\gamma}(v^q)(x), & x \in R^n; \\ v(x) = W_{\beta,\gamma}(u^p)(x), & x \in R^n; \end{cases} \quad (1)$$

where

$$W_{\beta,\gamma}(f)(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} f(y) dy}{t^{n-\beta\gamma}} \right]^{\frac{1}{\gamma-1}} \frac{dt}{t}.$$

In a special case when $\beta = \frac{\alpha}{2}$ and $\gamma = 2$, system (1) reduces to

$$\begin{cases} u(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} v(y)^q dy, & x \in R^n, \\ v(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} u(y)^p dy, & x \in R^n. \end{cases} \quad (2)$$

The solutions (u, v) of (2) are critical points of the functional associated with the well-known Hardy-Littlewood-Sobolev inequality. The classification of solutions would provide the best constant in the HLS inequality.

We can also prove that the integral system (2) is equivalent to the system of partial differential equations

$$\begin{cases} (-\Delta)^{\alpha/2} u = v^q, u > 0, & \text{in } R^n, \\ (-\Delta)^{\alpha/2} v = u^p, v > 0, & \text{in } R^n. \end{cases} \quad (3)$$

And in particular when $\alpha = 2$, it reduces to the well-known Lane-Emden system. And even more particularly, when $p = q = \frac{n+2}{n-2}$, it becomes the Yamabe equation.

The symmetry is obtained by the integral form of the method of moving planes. This method is quite different from the ones for PDEs. Instead of using maximum principles, some global norms are estimated.

We will also mention two convenient ways to lift regularity for solutions: one by contracting operators and the other by the combined use of contracting and shrinking operators. The latter is a new idea which has just been applied in our recent paper to establish Lipschitz continuity of positive solutions for system (1), and we believe that this idea will become a useful tool in nonlinear analysis.