THE MODULI SPACE OF STABLE QUOTIENTS

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Dedicated to William Fulton on the occasion of his 70th birthday

Abstract. A moduli space of stable quotients of the rank n trivial sheaf on stable curves is introduced. Over nonsingular curves, the moduli space is Grothendieck's Quot scheme. Over nodal curves, a relative construction is made to keep the torsion of the quotient away from the singularities. New compactifications of classical spaces arise naturally: a nonsingular and irreducible compactification of the moduli of maps from genus 1 curves to projective space is obtained. Localization on the moduli of stable quotients leads to new relations in the tautological ring generalizing Brill-Noether constructions.

The moduli space of stable quotients is proven to carry a canonical 2-term obstruction theory and thus a virtual class. The resulting system of descendent invariants is proven to equal the Gromov-Witten theory of the Grassmannian in all genera. Stable quotients can also be used to study Calabi-Yau geometries. The conifold is calculated to agree with stable maps. Several questions about the behavior of stable quotients for arbitrary targets are raised.

Contents

Date: September 2009.

1. Introduction

1.1. **Virtual classes.** Only a few compact moduli spaces in algebraic geometry carry virtual classes. The conditions placed on the associated deformation theories are rather strong. The principal cases (so far) are:

(i) stable maps to nonsingular varieties [2, 14, 19],
(ii) stable sheaves on nonsingular 3-folds [28, 31],
(iii) stable sheaves on nonsingular surfaces [19],
(iv) Grothendieck's Quot scheme on nonsingular curves [3, 23].

Of the above four families, the first three are understood to be related. The correspondences of [24, 25, 28] relate (i) and (ii). The connections [18, 33] between Gromov-Witten invariants and Donaldson/Seiberg-Witten invariants relate (i) and (iii). For equivalence with (ii) and (iii), the associated Gromov-Witten theories must be considered with domains varying in the moduli of stable curves $\overline{M}_g$.

The construction of the virtual class of the Quot scheme (iv) requires the curve C to be fixed in moduli. In fact, the Quot scheme of a nodal curve does *not* carry a virtual class via the standard deformation theory. In order to fully connect (i) and (iv), new moduli spaces are required.

1.2. **Stable quotients.** We introduce here a moduli space of *stable quotients*

$$\mathbb{C}^n \otimes \mathcal{O}_C \to Q \to 0$$

on m-pointed curves C with (at worst) nodal singularities. Two basic properties are satisfied:

- the torsion $\tau(Q)$ always has support away from the nodes and markings of C,
- the moduli of stable quotients is proper over $\overline{M}_{g,m}$.

The first property yields a virtual class, and the second property leads to a system of invariants over $\overline{M}_{g,m}$. Our main result equates the descendent theory of the moduli of stable quotients to the Gromov-Witten theory of the Grassmannian in all genera.

Stable quotients are defined in Section 2. The basic structures of the moduli space (including the virtual class) are discussed in Section 3. The important case of mapping to a point is studied in Section 4. Comparison results with the Gromov-Witten theory of Grassmannians in the strongest equivariant form are stated in Section 5. The construction of the moduli of stable quotients and proofs of the comparison results are presented in Section 6 - 7.

The intersection theory of the moduli of stable quotients leads to new tautological relations on the moduli of curves. Basic relations generalizing classical Brill-Noether constructions are presented in Section 8. Further calculations in the tautological rings of the moduli spaces of curves of compact type appear in [27].

Stable quotients can also be used to study Calabi-Yau geometries. The most accessible are the local toric cases. The conifold, given by the total space of

$$\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1,$$

is calculated in Section 9 and found to agree exactly with Gromov-Witten theory.

Given a projective embedding of an arbitrary scheme

$$X \subset \mathbb{P}^n,$$

a moduli space of stable quotients associated to X is defined in Section 10. We speculate, at least when X is a nonsingular complete intersection, that the moduli spaces carry virtual classes in all genera. Virtual classes may exist in even greater generality.

Stable quotient invariants in genus 1 for Calabi-Yau hypersurfaces are discussed in Section 10.2. Let

$$M_1(\mathbb{P}^n, d) \subset \overline{M}_1(\mathbb{P}^n, d)$$

be the open locus of the moduli of stable maps with nonsingular irreducible domain curves. Stable quotients provide a nonsingular[1], irreducible, modular compactification

$$M_1(\mathbb{P}^n, d) \subset \overline{Q}_1(\mathbb{P}^n, d).$$

For the Calabi-Yau hypersurface of degree $n+1$,

$$X_{n+1} \subset \mathbb{P}^n,$$

genus 1 invariants can be defined naturally as an Euler characteristic of a rank $(n+1)d$ vector bundle on $\overline{Q}_1(\mathbb{P}^n, d)$. The relationship to the

[1]Nonsingularity here is as a Deligne-Mumford stack.

Gromov-Witten invariants of X_{n+1} is not yet clear, but there will likely be a transformation.

The paper ends with several questions about the behavior of stable quotients. Certainly, our main results carry over to the hyperquot schemes associated to $\mathbf{SL}_n$-flag varieties. Other variants are discussed in Section 10.3. The toric case has been recently treated in [4].

Stable quotients should be considered to lie between stable maps and stable sheaves. Perhaps recent wall-crossing methods [13, 17] will be relevant to the study.

1.3. **Acknowledgments.** The results presented here were obtained at MSRI in 2009 during a program on modern moduli in algebraic geometry. We thank the organizers for creating a stimulating environment. Conversations with D. Abramovich, I. Ciocan-Fontanine, Y. Cooper, C. Faber, D. Maulik, R. Thomas, and A. Zinger have led to many improvements.

A.M. was partially supported by DMS-0812030. D.O. was partially supported by DMS-0852468. R.P. was partially supported by DMS-0500187 and the Clay Institute. The paper was written while R.P. was visiting the Instituto Superior Técnico in Lisbon.

2. Stability

2.1. **Curves.** A *curve* is a reduced and connected scheme over $\mathbb{C}$ of pure dimension 1. Let C be a curve of arithmetic genus

$$g = h^1(C, \mathcal{O}_C)$$

with at worst nodal singularities. Let

$$C^{ns} \subset C$$

denote the nonsingular locus. The data $(C, p_1, \ldots, p_m)$ with distinct markings $p_i \in C^{ns}$ determine a genus g, m-pointed, *quasi-stable curve.* A quasi-stable curve is *stable* if $\omega_C(p_1 + \ldots + p_m)$ is ample.

2.2. **Quotients.** Let q be a quotient of the trivial bundle on a pointed quasi-stable curve C,

$$\mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0.$$

If the torsion subsheaf $\tau(Q) \subset Q$ has support contained in

$$C^{ns} \setminus \{p_1, \ldots, p_m\},$$

then q is a *quasi-stable quotient.* Quasi-stability of q implies the associated kernel,

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0,$$

is a locally free sheaf on C. Let r denote the rank of S.

Let $(C, p_1, \ldots, p_m)$ be a quasi-stable curve equipped with a quasi-stable quotient q. The data $(C, p_1, \ldots, p_m, q)$ determine a *stable quotient* if the $\mathbb{Q}$-line bundle

$$\omega_C(p_1 + \ldots + p_m) \otimes (\wedge^r S^*)^{\otimes \epsilon} \tag{1}$$

is ample on C for every strictly positive $\epsilon \in \mathbb{Q}$. Quotient stability implies $2g - 2 + m \geq 0$.

Viewed in concrete terms, no amount of positivity of S^* can stabilize a genus 0 component

$$\mathbb{P}^1 \stackrel{\sim}{=} P \subset C$$

unless P contains at least 2 nodes or markings. If P contains exactly 2 nodes or markings, then S^* *must* have positive degree.

Of course, when considering stable quotients in families, flatness over the base is imposed on both the curve C and the quotient sheaf Q.

2.3. **Isomorphisms.** Let $(C, p_1, \ldots, p_m)$ be a quasi-stable curve. Two quasi-stable quotients

$$\mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0, \quad \mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q'} Q' \to 0 \tag{2}$$

on C are *strongly isomorphic* if the associated kernels

$$S, S' \subset \mathbb{C}^n \otimes \mathcal{O}_C$$

are equal.

An *isomorphism* of quasi-stable quotients

$$\phi : (C, p_1, \ldots, p_m, q) \to (C', p'_1, \ldots, p'_m, q')$$

is an isomorphism of curves

$$\phi : C \xrightarrow{\sim} C'$$

satisfying

(i) $\phi(p_i) = p'_i$ for $1 \leq i \leq m$,
(ii) the quotients q and $\phi^*(q')$ are strongly isomorphic.

Quasi-stable quotients (2) on the same curve C may be isomorphic without being strongly isomorphic.

Theorem 1. *The moduli space of stable quotients* $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ *parameterizing the data*

$$(C,\ p_1,\dots,p_m,\ 0\to S\to\mathbb{C}^n\otimes\mathcal{O}_C\stackrel{q}{\to}Q\to 0),$$

with $\mathrm{rank}(S)=r$ *and* $\deg(S)=-d$, *is a separated and proper Deligne-Mumford stack of finite type over* $\mathbb{C}$.

Theorem 1 is obtained by mixing the construction of the moduli of stable curves with the Quot scheme. Keeping the torsion of the quotient away from the nodes and markings is a twist motivated by relative geometry. The proof of Theorem 1 is given in Section 6.

2.4. **Automorphisms.** The automorphism group A_C of a quasi-stable curve $(C,p_1,\dots,p_m)$ may be positive dimensional. If the dimension is 0, A_C is finite. Stability of $(C,p_1,\dots,p_m)$ is well-known to be equivalent to the finiteness of A_C. If $(C,p_1,\dots,p_m,q)$ is a stable quotient, the ampleness condition (1) implies the automorphism group A_C of the underlying quasi-stable curve has dimension at most 1.

An *automorphism* of a quasi-stable quotient $(C,p_1,\dots,p_m,q)$ is a self-isomorphism. The automorphism group A_q of the quasi-stable quotient q embeds in the automorphism group of the underlying curve

$$\mathsf{A}_q\subset\mathsf{A}_C.$$

We leave the proof of the following elementary result to the reader.

Lemma 1. *Let* $(C,p_1,\dots,p_m,q)$ *be a quasi-stable quotient. If*

$$\dim(\mathsf{A}_C)\le 1,$$

then q *is stable if and only if* A_q *is finite.*

2.5. **First examples.** The simplest examples occur when $d=0$. Then, stability of the quotient implies the underlying pointed curve is stable. We see

$$\overline{Q}_{g,m}(\mathbb{G}(r,n),0)=\overline{M}_{g,m}\times\mathbb{G}(r,n)$$

where $\mathbb{G}(r,n)$ denotes the Grassmannian of r-planes in $\mathbb{C}^n$.

A more interesting example is $\overline{Q}_{1,0}(\mathbb{G}(1,n),1)$. A direct analysis yields

$$\overline{Q}_{1,0}(\mathbb{G}(1,n),1) = \overline{M}_{1,1} \times \mathbb{P}^{n-1}.$$

Given a 1-pointed stable genus 1 curve (E,p) and an element $\xi \in \mathbb{P}^{n-1}$, the associated stable quotient is

$$0 \to \mathcal{O}_E(-p) \xrightarrow{\iota_\xi} \mathbb{C}^n \otimes \mathcal{O}_E \to Q \to 0$$

where ι_ξ is the composition of the canonical inclusion

$$0 \to \mathcal{O}_E(-p) \to \mathcal{O}_E$$

with the line in $\mathbb{C}^n$ determined by ξ.

The open locus $Q_{g,0}(\mathbb{G}(r,n),d) \subset \overline{Q}_{g,0}(\mathbb{G}(r,n),d)$, corresponding to nonsingular domains C, is simply the universal Quot scheme over the moduli space of nonsingular curves.

3. Structures

3.1. **Maps.** Over the moduli space of stable quotients, there is a universal curve

$$\pi : U \to \overline{Q}_{g,m}(\mathbb{G}(r,n),d) \tag{3}$$

with m sections and a universal quotient

$$0 \to S_U \to \mathbb{C}^n \otimes \mathcal{O}_U \xrightarrow{q_U} Q_U \to 0.$$

The subsheaf S_U is locally free on U because of the restrictions imposed on the torsion by the stability condition.

The moduli space $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ is equipped with two basic types of maps. If $2g-2+m>0$, then the stabilization of $(C,p_1,\ldots,p_m)$ determines a map

$$\nu : \overline{Q}_{g,m}(\mathbb{G}(r,n),d) \to \overline{M}_{g,m}$$

by forgetting the quotient. For each marking p_i, the quotient is locally free over p_i, hence it determines an evaluation map

$$\mathrm{ev}_i : \overline{Q}_{g,m}(\mathbb{G}(r,n),d) \to \mathbb{G}(r,n).$$

The universal curve (3) is *not* isomorphic to $\overline{Q}_{g,m+1}(\mathbb{G}(r,n),d)$. In fact, there does *not* exist a forgetful map of the form

$$\overline{Q}_{g,m+1}(\mathbb{G}(r,n),d) \to \overline{Q}_{g,m}(\mathbb{G}(r,n),d)$$

since there is no canonical way to contract the quotient sequence.

The general linear group $\mathbf{GL}_n(\mathbb{C})$ acts on $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ via the standard action on $\mathbb{C}^n \otimes \mathcal{O}_C$. The structures π, q_U, ν and the evaluations maps are all $\mathbf{GL}_n(\mathbb{C})$-equivariant.

3.2. **Obstruction theory.** Even if $2g-2+m$ is not strictly positive, the moduli of stable quotients maps to the Artin stack of pointed domain curves

$$\nu^A : \overline{Q}_{g,m}(\mathbb{G}(r,n),d) \to \mathcal{M}_{g,m}.$$

The moduli of stable quotients with fixed underlying curve

$$(C, p_1, \ldots, p_m) \in \mathcal{M}_{g,m}$$

is simply an open set of the Quot scheme. The following result is obtained from the standard deformation theory of the Quot scheme.

Theorem 2. *The deformation theory of the Quot scheme determines a 2-term obstruction theory on $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ relative to ν^A given by $RHom(S,Q)$.*

An absolute 2-term obstruction theory on $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ is obtained from Theorem 2 and the smoothness of $\mathcal{M}_{g,m}$, see [2, 10]. The analogue of Theorem 2 for the Quot scheme of a *fixed* nonsingular curve was observed in [3, 23].

The $\mathbf{GL}_n(\mathbb{C})$-action lifts to the obstruction theory, and the resulting virtual class is defined in $\mathbf{GL}_n(\mathbb{C})$-equivariant cycle theory,

$$[\overline{Q}_{g,m}(\mathbb{G}(r,n),d)]^{vir} \in A_*^{\mathbf{GL}_n(\mathbb{C})}(\overline{Q}_{g,m}(\mathbb{G}(r,n),d), \mathbb{Q}).$$

A system of $\mathbf{GL}_n(\mathbb{C})$-equivariant descendent invariants is defined by the brackets

$$\langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_m}(\gamma_m) \rangle_{g,d} = \int_{[\overline{Q}_{g,m}(\mathbb{G}(r,n),d)]^{vir}} \prod_{i=1}^m \psi_i^{a_i} \cup \mathrm{ev}_i^*(\gamma_i)$$

where $\gamma_i \in A^*_{\mathbf{GL}_n(\mathbb{C})}(\mathbb{G}(r,n),\mathbb{Q})$. The classes ψ_i are obtained from the cotangent lines on the domain (or, equivalently, pulled-back from the Artin stack by ν^A).

3.3. **Nonsingularity.** Let E be a nonsingular curve of genus 1, and let

$$f : E \to \mathbb{G}(1,n)$$

be a morphism of degree $d > 0$. The pull-back of the tautological sequence on $\mathbb{G}(1, n)$ determines a stable quotient on E. The moduli space of maps is an open[2] subset

$$M_{1,0}(\mathbb{G}(1, n), d) \subset \overline{Q}_{1,0}(\mathbb{G}(1, n), d) \tag{4}$$

for $d > 0$.

Let (C, q) be a stable quotient parameterized by $\overline{Q}_{1,0}(\mathbb{G}(1, n), d)$. By stability, C is either a nonsingular genus 1 curve or a cycle of rational curves. The associated sheaf S is a line bundle of degree $-d < 0$. The vanishing

$$\mathrm{Ext}^1(S, Q) = 0$$

holds since there are no nonspecial line bundles of positive degree on such curves.

Proposition 1. *$\overline{Q}_{1,0}(\mathbb{G}(1, n), d)$ is a nonsingular irreducible Deligne-Mumford stack of dimension nd for $d > 0$.*

Proof. Nonsingularity has already been established. The dimension is obtained from a Riemann-Roch calculation[3] of $\chi(S, Q)$. Irreducibility is clear since $Q_{1,0}(\mathbb{G}(1, n), d)$ is an open set of a projective bundle over the moduli of elliptic curves. □

For simplicity, we will denote the moduli space by $\overline{Q}_{1,0}(\mathbb{P}^{n-1}, d)$. Stable quotients provide an efficient compactification (4) of $M_{1,0}(\mathbb{P}^{n-1}, d)$. Instead of desingularizing the moduli of maps by blowing-up the closure of

$$M_{1,0}(\mathbb{P}^{n-1}, d) \subset \overline{M}_{1,0}(\mathbb{P}^{n-1}, d)$$

in the moduli of stable maps [12, 34], the stable quotient space achieves a simple modular desingularization by blowing-down.

For large degree d, all line bundles on nonsingular curves are nonspecial. As a result, the following nonsingularity result holds.

Proposition 2. *For $g \geq 2$ and $d \geq 2g - 1$, the forgetful morphism*

$$\nu : Q_{g,0}(\mathbb{P}^{n-1}, d) \to M_g$$

is smooth of expected relative dimension.

[2]If $d > 1$, the subset is nonempty. If $d = 1$, the subset is empty.

[3]The calculation is done in general in Lemma 4 below.

The result does *not* hold over the boundary or even over the interior if markings are present.

4. Stable quotients for $\mathbb{G}(n,n)$

4.1. $n = 1$. Consider $\overline{Q}_{g,m}(\mathbb{G}(1,1),d)$ for $d > 0$. The moduli space parameterizes stable quotients

$$0 \to S \to \mathcal{O}_C \to Q \to 0.$$

Hence, S is an ideal sheaf of C.

Let $\overline{M}_{g,m|d}$ be the moduli space of genus g curves with markings

$$\{p_1 \ldots, p_m\} \ \cup \ \{\widehat{p}_1, \ldots, \widehat{p}_d\} \in C^{ns} \subset C$$

satisfying the conditions

(i) the points p_i are distinct,
(ii) the points $\widehat{p}_j$ are distinct from the points p_i,

with stability given by the ampleness of

$$\omega_C(\sum_{i=1}^{m} p_i + \epsilon \sum_{j=1}^{d} \widehat{p}_j)$$

for every strictly positive $\epsilon \in \mathbb{Q}$. The conditions allow the points $\widehat{p}_j$ and $\widehat{p}_{j'}$ to coincide.

The moduli space $\overline{M}_{g,m|d}$ is a nonsingular, irreducible, Deligne-Mumford stack.[4] Given an element

$$[C, p_1, \ldots, p_m, \widehat{p}_1, \ldots, \widehat{p}_d] \in \overline{M}_{g,m|d} \ ,$$

there is a canonically associated stable quotient

$$0 \to \mathcal{O}_C(-\sum_{j=1}^{d} \widehat{p}_j) \to \mathcal{O}_C \to Q \to 0. \qquad (5)$$

We obtain a morphism

$$\phi : \overline{M}_{g,m|d} \to \overline{Q}_{g,m}(\mathbb{G}(1,1),d).$$

The following result is proven by matching the stability conditions.

[4]In fact, $\overline{M}_{g,m|d}$ is a special case of the moduli of pointed curves with weights studied by [11, 22].

Proposition 3. *The map ϕ induces an isomorphism*

$$\overline{M}_{g,m|d}/\mathbb{S}_d \stackrel{\sim}{\to} \overline{Q}_{g,m}(\mathbb{G}(1,1),d)$$

where the symmetric group $\mathbb{S}_d$ acts by permuting the markings $\widehat{p}_j$.

The first example to consider is $\overline{Q}_{0,2}(\mathbb{G}(1,1),d)$ for $d>0$. The space has a rather simple geometry. For example, the Poincaré polynomial

$$p_d = \sum_{k=0}^{2d-2} B_k t^k$$

where B_k is the k^{th} Betti number of $\overline{Q}_{0,2}(\mathbb{G}(1,1),d)$, is easily obtained.

Lemma 2. $p_d = (1+t^2)^{d-1}$ *for* $d>0$.

Proof. Let (C,p_1,p_2,q) be an element of $\overline{Q}_{0,2}(\mathbb{G}(1,1),d)$. By the stability condition, (C,p_1,p_2) must be a simple chain of rational curves with the markings p_1 and p_2 on opposite extremal components. We may stratify $\overline{Q}_{0,2}(\mathbb{G}(1,1),d)$ by the number n of components of C and the distribution of the degree d on these components. The associated quasi-projective strata

$$S_{(d_1,\ldots,d_n)} \subset \overline{Q}_{0,2}(\mathbb{G}(1,1),d)$$

are indexed by vectors

$$(d_1,\ldots,d_n), \quad d_i>0, \quad \sum_{i=1}^n d_i = d.$$

Moreover, each stratum is a product,

$$S_{(d_1,\ldots,d_n)} \cong \prod_{i=1}^n (\mathrm{Sym}^{d_i}(\mathbb{C}^*)/\mathbb{C}^*).$$

To calculate p_d, we must compute the virtual Poincaré polynomial of the quotient space $\mathrm{Sym}^k(\mathbb{C}^*)/\mathbb{C}^*$ for all $k>0$. We start with the virtual Poincaré polynomial of $\mathrm{Sym}^k(\mathbb{C})$,

$$p(\mathrm{Sym}^k(\mathbb{C})) = p(\mathbb{C}^k) = t^{2k}.$$

Filtering by the order at $0\in\mathbb{C}$, we find

$$p(\mathrm{Sym}^k(\mathbb{C})) = \sum_{i=0}^k p(\mathrm{Sym}^i(\mathbb{C}^*)).$$

We conclude

$$p(\mathrm{Sym}^k(\mathbb{C}^*)) = t^{2k} - t^{2k-2}$$

for $k > 0$. The quotient by $\mathbb{C}^*$ can be handled simply by dividing by $t^2 - 1$, see [8]. Hence,

$$p(\mathrm{Sym}^k(\mathbb{C}^*)/\mathbb{C}^*) = t^{2k-2}.$$

The Lemma then follows by elementary counting. □

4.2. **Classes.** There are several basic classes on $\overline{M}_{g,m|d}$. As in the study of the standard moduli space of stable curves, there are strata classes

$$\mathcal{S} \in A^*(\overline{M}_{g,m|d}, \mathbb{Q})$$

given by fixing the topological type of a degeneration. New diagonal classes are defined for every subset $J \subset \{1, \ldots, d\}$ of size at least 2,

$$D_J \in A^{|J|-1}(\overline{M}_{g,m|d}, \mathbb{Q}),$$

corresponding to the locus where the points $\{\widehat{p}_j\}_{j\in J}$ are coincident. In fact, the subvariety

$$D_J \subset \overline{M}_{g,m|d}$$

is isomorphic to $\overline{M}_{g,m|(d-|J|+1)}$. The cotangent bundles

$$\mathbb{L}_i \to \overline{M}_{g,m|d}, \quad \widehat{\mathbb{L}}_j \to \overline{M}_{g,m|d}$$

corresponding to the two types of markings have respective Chern classes

$$\psi_i = c_1(\mathbb{L}_i), \ \widehat{\psi}_j = c_1(\widehat{\mathbb{L}}_j) \in A^1(\overline{M}_{g,m|d}, \mathbb{Q}).$$

The Hodge bundle with fiber $H^0(C, \omega_C)$ over the curve $[C] \in \overline{M}_{g,m|d}$,

$$\mathbb{E} \to \overline{M}_{g,m|d},$$

has Chern classes

$$\lambda_i = c_i(\mathbb{E}) \in A^i(\overline{M}_{g,m|d}, \mathbb{Q}).$$

4.3. **Cotangent calculus.** Assume $2g-2+m \geq 0$. Canonical contraction defines a fundamental birational morphism

$$\tau : \overline{M}_{g,m+d} \to \overline{M}_{g,m|d}.$$

By the stability conditions, the cotangent lines at the points p_i are unchanged by τ,

$$\tau^*(\psi_i) = \psi_i, \quad 1 \leq i \leq m.$$

However, contraction affects the cotangent line classes at the other points,

$$\psi_{m+j} = \tau^*(\widehat{\psi}_j) + \Delta_{m+j}. \tag{6}$$

Here, Δ_{m+j} is the sum

$$\Delta_{m+j} = \sum_{j' \neq j} \Delta_{j,j'}$$

where $\Delta_{j,j'}$ is the boundary divisor of $\overline{M}_{g,m+d}$ parameterizing curves

$$C = C' \cup C'', \quad g(C') = 0, \quad g(C'') = g$$

with a single separating node and the markings labeled $m+j$ and $m+j'$ distributed to C'.

Let $\prod_{j=1}^d \widehat{\psi}_j^{y_j}$ be a monomial class on $\overline{M}_{g,m|d}$. Since τ is birational,

$$\tau_* \tau^* (\prod_{j=1}^d \widehat{\psi}_j^{y_j}) = \prod_{j=1}^d \widehat{\psi}_j^{y_j}. \tag{7}$$

After using relations (6) and (7), we see for example

$$\tau_*(\psi_{m+j}) = \widehat{\psi}_j + \sum_{j' \neq j} D_{j,j'} \ .$$

The method proves the following result.

Lemma 3. *There exists a universal formula*

$$\tau_* \left(\prod_{i=1}^m \psi_i^{x_i} \prod_{j=1}^d \psi_{m+j}^{y_j} \right) = \prod_{i=1}^m \psi_i^{x_i} \left(\prod_{j=1}^d \widehat{\psi}_j^{y_j} + \ldots \right)$$

where the dots are polynomials in the $\widehat{\psi}_j$ and D_J classes which are independent of g and m.

4.4. **Canonical forms.** Let $J, J' \subset \{1, \ldots, d\}$. The cotangent line classes

$$\widehat{\psi}_j | D_J = \widehat{\psi}_J \tag{8}$$

are all equal for $j \in D_J$. If J and J' have nontrivial intersection, we obtain

$$D_J \cdot D_{J'} = (-\widehat{\psi}_{J \cup J'})^{|J \cap J'|-1} D_{J \cup J'} \ . \tag{9}$$

by examining normal bundles.

If $M(\widehat{\psi}_j, D_J)$ is any monomial in the cotangent line and diagonal classes, we can write M in a canonical form in two steps:

(i) multiply the diagonal classes using (9) until the result is a product of cotangent line classes with $D_{J_1} D_{J_2} \cdots D_{J_l}$ where all the subsets J_i are disjoint,

(ii) collect the equal cotangent line classes using (8).

Let M^C denote the resulting canonical form.

By extending the operation linearly, we can write any polynomial $P(\widehat{\psi}_j, D_J)$ in canonical form P^C. In particular, the universal formulas of Lemma 3 can be taken to be in canonical form.

4.5. **Example.** The cotangent class intersections on $\overline{M}_{0,2|d}$,

$$\int_{\overline{M}_{0,2|d}} \psi_1^{x_1} \psi_2^{x_2} \widehat{\psi}_1^{y_1} \cdots \widehat{\psi}_d^{y_d}, \tag{10}$$

for $d > 0$ are straightforward to calculate. Since the dimension of $\overline{M}_{0,2|d}$ is $d-1$, at least one of the y_j must vanish. After permuting the indices, we may take $y_d = 0$. By studying the geometry of the map

$$\overline{M}_{0,2|d} \to \overline{M}_{0,2|d-1}$$

forgetting $\widehat{p}_d$ in case $d > 1$, we deduce

$$\int_{\overline{M}_{0,2|d}} \psi_1^{x_1} \psi_2^{x_2} \widehat{\psi}_1^{y_1} \cdots \widehat{\psi}_{d-1}^{y_{d-1}} =$$
$$\int_{\overline{M}_{0,2|d-1}} \psi_1^{x_1-1} \psi_2^{x_2} \widehat{\psi}_1^{y_1} \cdots \widehat{\psi}_{d-1}^{y_{d-1}} + \int_{\overline{M}_{0,2|d-1}} \psi_1^{x_1} \psi_2^{x_2-1} \widehat{\psi}_1^{y_1} \cdots \widehat{\psi}_{d-1}^{y_{d-1}}.$$

Solving the recurrence, we conclude (10) vanishes unless all $y_j = 0$ and

$$\int_{\overline{M}_{0,2|d}} \psi_1^{x_1} \psi_2^{x_2} = \binom{d-1}{x_1, x_2}.$$

4.6. **Tautological complexes.** Consider the universal curve

$$\pi : U \to \overline{M}_{g,m|d}$$

with universal quotient sequence

$$0 \to S_U \to \mathcal{O}_U \to Q_U \to 0$$

obtained from (5). The complex $R\pi_*(S_U^*) \in D^b_{coh}(\overline{M}_{g,m|d})$ will arise naturally in localization calculations on the moduli of stable quotients. Base change of the complex to

$$[C, p_1, \ldots, p_m, \widehat{p}_1, \ldots, \widehat{p}_d] \in \overline{M}_{g,m|d}$$

computes the cohomology groups

$$H^0(C, \mathcal{O}_C(\sum_{j=1}^{d} \widehat{p}_j)), \quad H^1(C, \mathcal{O}_C(\sum_{j=1}^{d} \widehat{p}_j))$$

with varying ranks.

A canonical resolution by vector bundles of $R\pi_*(S_U^*)$ is easily obtained from the sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(\sum_{j=1}^{d} \widehat{p}_j) \to \mathcal{O}_C(\sum_{j=1}^{d} \widehat{p}_j)|_{\sum_{j=1}^{d} \widehat{p}_j} \to 0. \tag{11}$$

The rank d bundle

$$\mathbb{B}_d \to \overline{M}_{g,m|d}$$

with fiber

$$H^0(C, \mathcal{O}_C(\sum_{j=1}^{d} \widehat{p}_j)|_{\sum_{j=1}^{d} \widehat{p}_j})$$

is obtained from the geometry of the points $\widehat{p}_j$. The Chern classes of $\mathbb{B}_d$ are universal polynomials in the $\widehat{\psi}_j$ and D_J classes. Up to a rank 1 trivial factor, $R\pi_*(S_U^*)$ is equivalent to the complex

$$\mathbb{B}_d \to \mathbb{E}^*$$

obtained from the derived push-forward of (11).

4.7. **General n.** While the moduli space

$$\overline{Q}_{g,m}(\mathbb{G}(1,1),d) \to \overline{M}_{g,m}$$

may be viewed simply as a compactification of the symmetric product of the universal curve over $\overline{M}_{g,m}$, the moduli space $\overline{Q}_{g,m}(\mathbb{G}(n,n),d)$ is more difficult to describe since the stable subbundles have higher rank. Nevertheless, since $\mathrm{Ext}^1(S,Q)$ always vanishes, we obtain the following result.

Proposition 4. *$\overline{Q}_{g,m}(\mathbb{G}(n,n),d)$ is nonsingular of expected dimension $3g-3+m+nd$.*

5. Gromov-Witten comparison

5.1. **Dimensions.** The moduli space of stable maps $\overline{M}_{g,m}(\mathbb{G}(r,n),d)$ also carries a perfect obstruction theory and a virtual class. In order to compare with the moduli space of stable quotients, we will always assume $2g-2+m \geq 0$ and $0 < r < n$.

Lemma 4. *The virtual dimensions of the spaces $\overline{M}_{g,m}(\mathbb{G}(r,n),d)$ and $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ are equal.*

Proof. The virtual dimension of the moduli space of stable maps is

$$\int_\beta c_1(T)+(\dim_{\mathbb{C}} \mathbb{G}(r,n)-3)(1-g)+m = nd+(r(n-r)-3)(1-g)+m.$$

where β is the degree d curve class and T is the tangent bundle of $\mathbb{G}(r,n)$. Similarly, the virtual dimension of the moduli of stable quotients is

$$\chi(S,Q)+3g-3+m = nd + r(n-r)(1-g)+3g-3+m,$$

by Riemann-Roch, which agrees. □

5.2. **Stable maps to stable quotients.** There exists a natural morphism

$$c : \overline{M}_{g,m}(\mathbb{G}(1,n),d) \to \overline{Q}_{g,m}(\mathbb{G}(1,n),d).$$

Given a stable map

$$f : (C, p_1, \ldots, p_m) \to \mathbb{G}(1,n)$$

of degree d, the image $c([f]) \in \overline{Q}_{g,m}(\mathbb{G}(1,n),d)$ is obtained by the following construction.

The first step is to consider the minimal contraction

$$\kappa : C \to \widehat{C}$$

of rational components yielding a quasi-stable curve $(\widehat{C}, p_1, \ldots, p_m)$ with an automorphism group $\mathsf{A}_{\widehat{C}}$ of dimension at most 1. The minimal contraction κ is unique — the exceptional curves of κ are the maximal connected trees $T \subset C$ of rational curves which

(i) contain no markings,
(ii) meet $\overline{C \setminus T}$ in a single point.

Let $T_1, \ldots, T_t$ be the set of maximal trees satisfying (i) and (ii). Then,

$$\widehat{C} = \overline{C \setminus \cup_i T_i}$$

is canonically a subcurve of C. Let $x_1, \ldots, x_t \in \widehat{C}^{ns}$ be the points of incidence with the trees $T_1, \ldots, T_t$ respectively.

Let d_i be the degree of the restriction of f to T_i. Let

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_{\widehat{C}} \to Q \to 0$$

be the pull-back by the restriction of f to $\widehat{C}$ of the tautological sequence on $\mathbb{G}(1, n)$. The canonical inclusion

$$0 \to S(-\sum_{i=1}^{t} d_i x_i) \to S$$

yields a new quotient

$$0 \to S(-\sum_{i=1}^{t} d_i x_i) \to \mathbb{C}^n \otimes \mathcal{O}_{\widehat{C}} \stackrel{\widehat{q}}{\to} \widehat{Q} \to 0.$$

Stability of the map f implies $(\widehat{C}, p_1, \ldots, p_m, \widehat{q})$ is a stable quotient. We define

$$c([f]) = (\widehat{C}, p_1, \ldots, p_m, \widehat{q}) \in \overline{Q}_{g,m}(\mathbb{G}(1, n), d).$$

The morphism c has been studied earlier for genus 0 curves in the linear sigma model constructions of [9]. See [20] for a scheme theoretic discussion by J. Li. The morphism c is considered for the Quot scheme of a fixed nonsingular curve of arbitrary genus in [30].

5.3. **Equivalence.** The strongest possible comparison result holds for $\mathbb{G}(1,n)$.

Theorem 3. $c_*[\overline{M}_{g,m}(\mathbb{G}(1,n),d)]^{vir} = [\overline{Q}_{g,m}(\mathbb{G}(1,n),d)]^{vir}$.

If $r>1$, a morphism c for $\mathbb{G}(r,n)$ does not in general exist. However, the following construction provides a substitute. Recall the Plücker embedding

$$\iota : \mathbb{G}(r,n) \to \mathbb{G}(1, \binom{n}{r}).$$

The Plücker embedding induces canonical maps

$$\iota_M : \overline{M}_{g,m}(\mathbb{G}(r,n),d) \to \overline{M}_{g,m}(\mathbb{G}(1, \binom{n}{r}), d),$$

$$\iota_Q : \overline{Q}_{g,m}(\mathbb{G}(r,n),d) \to \overline{Q}_{g,m}(\mathbb{G}(1, \binom{n}{r}), d).$$

The morphism ι_M is obtained by composing stable maps with ι. The morphism ι_Q is obtained by associating the subsheaf

$$0 \to \wedge^r S \to \wedge^r \mathbb{C}^n \otimes \mathcal{O}_C$$

to the subsheaf $0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_C$.

Theorem 4. *For* $0<r<n$ *and all classes* $\gamma_i \in A^*_{\mathbf{GL}_n(\mathbb{C})}(\mathbb{G}(r,n),\mathbb{Q})$,

$$c_*\iota_{M*}\Big(\prod_{i=1}^m \mathrm{ev}_i^*(\gamma_i) \ \cap \ [\overline{M}_{g,m}(\mathbb{G}(r,n),d)]^{vir}\Big) =$$
$$\iota_{Q*}\Big(\prod_{i=1}^m \mathrm{ev}_i^*(\gamma_i) \ \cap \ [\overline{Q}_{g,m}(\mathbb{G}(r,n),d)]^{vir}\Big).$$

Since descendent classes in both cases are easily seen to be pulled-back via $c \circ \iota_M$ and ι_Q respectively, there is no need to include them in the statement of Theorem 4. In particular, Theorem 4 implies the fully equivariant stable map and stable quotient brackets (and CoFT) are equal.

5.4. **Example.** To see Theorems 3 and 4 are not purely formal, we can study the case of genus 1 maps to $\mathbb{P}^{n-1}$ of degree 1 for $n \geq 2$. Let

$$I \subset \mathbb{P}^{n-1} \times \mathbb{G}(2,n)$$

be the incidence correspondence consisting of points and lines (p, L) with $p \in L$. The moduli space of stable maps is

$$\overline{M}_{1,0}(\mathbb{P}^{n-1}, 1) = \overline{M}_{1,1} \times I.$$

We will denote elements of the moduli space of stable maps by (E, p, L) where $(E, p) \in \overline{M}_{1,1}$ and $(p, L) \in I$. We have already seen

$$\overline{Q}_{1,0}(\mathbb{P}^{n-1}, 1) = \overline{M}_{1,1} \times \mathbb{P}^{n-1}.$$

The morphism

$$c : \overline{M}_{1,0}(\mathbb{P}^{n-1}, 1) \to \overline{Q}_{1,0}(\mathbb{P}^{n-1}, 1)$$

is given by the projection

$$I \to \mathbb{P}^{n-1}$$

onto the first factor. The virtual class of the moduli space of stable maps is easily computed from deformation theory,

$$[\overline{M}_{1,0}(\mathbb{P}^{n-1}, 1)]^{vir} = c_{n-2}(\text{Obs}) \cap [\overline{M}_{1,0}(\mathbb{P}^{n-1}, 1)],$$

where the rank $n-2$ obstruction bundle is

$$\text{Obs}_{(E,p,L)} = \frac{\mathbb{E}^* \otimes T_p(\mathbb{P}^{n-1})}{\Psi_p^* \otimes T_p(L)} = \mathbb{E}^* \otimes N_p(\mathbb{P}^{n-1}/L) \ .$$

Here, $\mathbb{E}$ is the Hodge bundle on $\overline{M}_{1,1}$, Ψ_p is the cotangent line, and $N_p(\mathbb{P}^{n-1}/L)$ is the normal space to $L \subset \mathbb{P}^{n-1}$ at p. We see

$$\begin{aligned} c_{n-2}(\text{Obs}) = c_{n-2}(N_p(\mathbb{P}^{n-1}/L)) - \lambda c_{n-3}(N_p(\mathbb{P}^{n-1}/L)) \\ + \lambda^2 c_{n-4}(N_p(\mathbb{P}^{n-1}/L)) + \ldots \end{aligned}$$

where $\lambda = c_1(\mathbb{E})$. Since $I \to \mathbb{P}^{n-1}$ is a $\mathbb{P}^{n-2}$-bundle,

$$\begin{aligned} c_*[\overline{M}_{1,0}(\mathbb{P}^{n-1}, 1)]^{vir} &= c_*(c_{n-2}(N_p(\mathbb{P}^{n-1}/L)) \cap [\overline{M}_{1,0}(\mathbb{P}^{n-1}, 1)]) \\ &= [\overline{Q}_{1,0}(\mathbb{P}^{n-1}, 1)] \\ &= [\overline{Q}_{1,0}(\mathbb{P}^{n-1}, 1)]^{vir}. \end{aligned}$$

For the second equality, we use the elementary projective geometry calculation

$$c_{n-2}(Q) = 1$$

where Q is universal rank $n-2$ quotient on the projective space of lines in $\mathbb{C}^{n-1}$. The last equality follows since the moduli space of stable quotients is nonsingular of expected dimension.

6. Construction

6.1. **Quotient presentation.** Let g, m, and d satisfy

$$2g - 2 + m + \epsilon d > 0$$

for all $\epsilon > 0$. We will exhibit the moduli space $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ as a quotient stack.

To begin, fix a stable quotient $(C, p_1, \ldots, p_m, q)$ where

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0.$$

By assumption, the line bundle

$$\mathcal{L}_\epsilon = \omega_C(p_1 + \ldots + p_m) \otimes (\Lambda^r S^*)^\epsilon$$

is ample for all $\epsilon > 0$. The genus 0 components of C must contain at least 2 nodes or markings with strict inequality for components of degree 0. As a consequence, ampleness of $\mathcal{L}_\epsilon$ for $\epsilon = \frac{1}{d+1}$ is enough to ensure the stability of a degree d quotient. We will fix $\epsilon = \frac{1}{d+1}$ throughout.

By standard arguments, there exists a sufficiently large and divisible integer f such that the line bundle $\mathcal{L}^f$ is very ample with no higher cohomology

$$H^1(C, \mathcal{L}^f) = 0.$$

We will show[5] that for all $k \geq 5$, the choice

$$f = k(d+1)$$

suffices. Then,

$$\mathcal{L}^f = \left(\omega_C\left(\sum_{i=1}^m p_i\right)\right)^{k(d+1)} \otimes (\Lambda^r S^*)^k.$$

To check very ampleness, we verify

$$H^1\left(C, \mathcal{L}^f \otimes I_{q_1} I_{q_2}\right) = 0 \tag{12}$$

for all pairs of (not necessarily distinct) points $q_1, q_2 \in C$. By duality, the vanishing (12) is equivalent to

$$\operatorname{Ext}^0(I_{q_1} I_{q_2}, \omega_C \otimes \mathcal{L}^{-f}) = 0.$$

If $q_1, q_2 \in C^{ns}$, we can check instead

$$H^0(C, \omega_C(q_1 + q_2) \otimes \mathcal{L}^{-f}) = 0,$$

[5]In fact, the result is true for $k \geq 3$, but the arguments for $k \geq 5$ are simpler.

which is clear since the line bundle has negative degree on each component. The following three cases also need to be taken into account:

(i) q_1 is node and $q_2 \in C^{ns}$,
(ii) q_1 and q_2 are distinct nodes,
(iii) $q_1 = q_2$ are coincident nodes.

Cases (i-iii) can be easily handled. For instance, to check (i), consider the normalization at q_1,

$$\pi : \widetilde{C} \to C,$$

and let $\pi^{-1}(q_1) = \{q_1', q_1''\}$. We have

$$\mathrm{Ext}^0(I_{q_1} I_{q_2}, \omega_C \otimes \mathcal{L}^{-f}) = H^0(\widetilde{C}, \omega_{\widetilde{C}} \otimes \pi^* \mathcal{L}^{-f}(q_1' + q_1'' + q_2))$$

which vanishes since the line bundle on the right has negative degree on each component. The other two cases are similar. The condition $k \geq 5$ is used in (iii).

By the vanishing of the higher cohomology, the dimension

$$h^0(C, \mathcal{L}^f) = 1 - g + k(d+1)(2g - 2 + m) + kd \tag{13}$$

is independent of the choice of stable quotient. Let V be a vector space of dimension (13). Given an identification

$$H^0(C, \mathcal{L}^f) \cong \mathsf{V}^*,$$

we obtain an embedding

$$i : C \hookrightarrow \mathbb{P}(\mathsf{V}),$$

well-defined up to the action of the group $\mathbf{PGL}(\mathsf{V})$. Let Hilb denote the Hilbert scheme of curves in $\mathbb{P}(\mathsf{V})$ of genus g and degree (13). Each stable quotient gives rise to a point in

$$\mathcal{H} = \mathsf{Hilb} \times \mathbb{P}(\mathsf{V})^m,$$

where the last factors record the locations of the markings $p_1, \ldots, p_m$.

Elements of $\mathcal{H}$ are tuples $(C, p_1, \ldots, p_m)$. A quasi-projective subscheme $\mathcal{H}' \subset \mathcal{H}$ is defined by requiring

(i) the points $p_1, \ldots, p_m$ are contained in C,
(ii) the curve $(C, p_1, \ldots, p_m)$ is quasi-stable.

We denote the universal curve over $\mathcal{H}'$ by

$$\pi : \mathcal{C}' \to \mathcal{H}'.$$

Next, we construct the π-relative Quot scheme

$$\mathsf{Quot}(n-r,d) \to \mathcal{H}'$$

parametrizing rank $n-r$ degree d quotients

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_C \to Q \to 0$$

on the fibers of π. A locally closed subscheme

$$\mathcal{Q}' \subset \mathsf{Quot}(n-r,d)$$

is further singled out by requiring

(iii) the torsion $\tau(Q)$ lies in $C^{ns} \setminus \{p_1, \ldots, p_m\}$,
(iv) the restriction of $\mathcal{O}_{\mathbb{P}(\mathsf{V})}(1)$ to C agrees with the line bundle

$$\left(\omega\left(\sum p_i\right)\right)^{k(d+1)} \otimes (\Lambda^r S^*)^k.$$

The action of $\mathbf{PGL}(\mathsf{V})$ extends to $\mathcal{H}'$ and $\mathcal{Q}'$. A $\mathbf{PGL}(\mathsf{V})$-orbit in $\mathcal{Q}'$ corresponds to a stable quotient up to isomorphism. By stability, each orbit has finite stabilizers. The moduli space $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ is the stack quotient $[\mathcal{Q}'/\,\mathbf{PGL}(\mathsf{V})]$.

6.2. **Separatedness.** We prove the moduli stack $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ is separated by the valuative criterion.

Let $(\Delta, 0)$ be a nonsingular pointed curve with complement

$$\Delta^0 = \Delta \setminus \{0\}.$$

We consider two flat families of quasi-stable pointed curves

$$\mathcal{X}_i \to \Delta, \qquad p_1^i, \ldots, p_m^i : \Delta \to \mathcal{X}_i,$$

and two flat families of stable quotients

$$0 \to S_i \to \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}_i} \to Q_i \to 0,$$

for $1 \leq i \leq 2$. We assume the two families are isomorphic away from the central fiber. We will show the isomorphism extends over 0. In fact, by the separatedness of the Quot functor, we only need to show that the isomorphism extends to the families of curves $\mathcal{X}_i \to \Delta$ in a manner preserving the sections.

By the semistable reduction theorem, possibly after base change ramified over 0, there exists a third family

$$\mathcal{Y} \to \Delta, \qquad p_1, \ldots, p_m : \Delta \to \mathcal{Y}$$

of quasi-stable pointed curves and dominant morphisms

$$\pi_i : \mathcal{Y} \to \mathcal{X}_i$$

compatible with the sections. Since the central fibers

$$(\mathcal{X}_i)_0 \subset \mathcal{X}_i$$

have automorphism groups of dimension bounded by 1, the central fiber of $\mathcal{Y}_0 \subset \mathcal{Y}$ can also taken to have automorphism group of dimension bounded by 1.

By the automorphism condition, π_i must restrict to an isomorphism away from the nodes of $(\mathcal{X}_i)_0$. After pull-back, we obtain exact sequences of quotients

$$0 \to \pi_i^* S_i \to \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{Y}} \to \pi_i^* Q_i \to 0 \tag{14}$$

on $\mathcal{Y}$ of the same degree and rank. Exactness holds after pull-back since the quotient Q_i is locally free at the nodes of $(\mathcal{X}_i)_0$.

The two pull-back sequences (14) must agree on the central fiber by the separatedness of the Quot functor. We claim the central fiber $\mathcal{Y}_0$ cannot contain components which are contracted over the nodes of $(\mathcal{X}_1)_0$ but uncontracted over the nodes of $(\mathcal{X}_2)_0$. Indeed, if such a component E existed, the quotient $\pi_1^* Q_1$ would be trivial on E, whereas by stability, the quotient $\pi_2^* Q_2$ could not be trivial. We conclude the families $\mathcal{X}_1$ and $\mathcal{X}_2$ are isomorphic. □

6.3. **Properness.** We prove the moduli stack $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ is proper by the valuative criterion. Let

$$\pi^0 : \mathcal{X}^0 \to \Delta^0, \qquad p_1, \ldots, p_m : \Delta^0 \to \mathcal{X}^0$$

carry a flat family of stable quotients

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}^0} \to Q \to 0 \tag{15}$$

which we must extend over Δ, possibly after base-change. By standard reductions, after base change and normalization, we may assume the fibers of π^0 are nonsingular and irreducible curves, possibly after adding the preimages of the nodes to the marking set. The original family is reconstructed by gluing stable quotients on different components via the evaluation maps at the nodes.

Once the general fiber of π^0 is assumed to be nonsingular, we construct an extension

$$\pi : \mathcal{X} \to \Delta, \qquad p_1, \ldots, p_m : \Delta \to \mathcal{X}$$

with central fiber an m-pointed stable curve.[6] After resolving the possible singularities of the total space at the nodes of $\mathcal{X}_0$ by blow-ups, we may take $\mathcal{X}$ to be a nonsingular surface. Using the properness of the relative Quot functor, we complete the family of quotients across the central fiber:

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}} \to Q \to 0.$$

The extension may fail to be a quasi-stable quotient since torsion may occur at the nodes or the markings of the central fiber. Further blow-ups will be necessary to move the torsion away from the special points.

We will first treat the case when S has rank 1. Consider the image T of the map

$$(\mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}})^* \to S^*$$

which can be written as

$$T = S^* \otimes I_Z$$

for a subscheme $Z \subset \mathcal{X}$. The quotient Q will have torsion supported on Z. By the flatness of Q, the subscheme Z is not supported on any components of the central fiber.

We consider a point $\xi \in \mathcal{X}$ which is a node or marking of the central fiber. After restriction to an open set containing ξ, we may assume all components of Z pass through ξ. After a sequence of blow-ups

$$\mu : \widetilde{\mathcal{X}} \to \mathcal{X},$$

we may take

$$\widetilde{Z} = \mu^{-1}(Z) = \sum_i m_i E_i + \sum_j n_j D_j,$$

where the $E_i \subset \mathcal{X}$ are the exceptional curves of μ and the D_j intersect the E_i away from the nodes and markings. Since we are only interested in constraining the behavior of $\widetilde{Z}$ at the nodes or markings over ξ, the

[6]There are exactly two cases where the central fiber can not be taken to stable,

$$(g, m) = (0, 2) \ \text{ or } \ (1, 0)\ .$$

In both cases, the central fiber can be taken to be irreducible and nodal. The argument afterwards is the same. We leave the details to the reader.

morphism μ can be achieved by repeatedly blowing-up only nodes or markings of the fiber over ξ.

On the blow-up, the image of the map

$$(\mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}})^* \to \mu^* S^*$$

factors though $\mu^* S^*(-\widetilde{Z})$. Setting

$$\widetilde{S} = \mu^* S(\sum_i m_i E_i) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}},$$

we obtain a flat family

$$0 \to \widetilde{S} \to \mathbb{C}^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}} \to \widetilde{Q} \to 0 \tag{16}$$

on $\widetilde{\mathcal{X}}$ where the quotient $\widetilde{Q}$ does not have torsion at the nodes or the markings of the (reduced) central fiber.

Unfortunately, the above blow-up process yields a family

$$\widetilde{\mathcal{X}} \to \Delta$$

with possible nonreduced components occuring in chains over nodes and markings of $\mathcal{X}_0$. The multiple components can be removed by base change and normalization,

$$\mathcal{X}' \to \widetilde{\mathcal{X}},$$

with the nodes and markings of $\mathcal{X}'_0$ mapping to the nodes and markings of $\widetilde{\mathcal{X}}_0^{red}$.

The pull-back of (16) to $\mathcal{X}'$ yields a quotient

$$\mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}'} \to Q' \to 0.$$

The quotient Q' is certainly locally free (and hence flat) over the nodes and markings of $\mathcal{X}'_0$. The quotient Q' may fail to be flat over finitely many nonsingular points of $\mathcal{X}'_0$. A flat limit

$$0 \to S'' \to \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}'} \to Q'' \to 0 \tag{17}$$

can then be found after altering Q' only at the latter points. Since (17) has no torsion over the nodes and markings of $\mathcal{X}'_0$, we have constructed a quasi-stable quotient. However (17) may fail to be stable because of possible unstable genus 0 components in the central fiber.

By the economical choice of blow-ups (occuring only at nodes and markings over ξ), all unstable genus 0 curves P carry exactly 2 special points and

$$S''|_P \stackrel{\sim}{=} \mathcal{O}_P.$$

All such unstable components are contracted by the line bundle

$$\mathcal{L} = \omega_C(p_1 + \ldots + p_m)^{d+1} \otimes \Lambda^r(S'')^*.$$

Indeed, $\mathcal{L}^k$ is π'-relatively[7] basepoint free for $k \geq 2$ and trivial over the unstable genus 0 curves. As a consequence, $\mathcal{L}^k$ determines a morphism

$$q : \mathcal{X}' \to \mathcal{Y} = \mathrm{Proj}\left(\oplus_m L^{km}\right).$$

The push-forward

$$0 \to q_* S'' \to \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{Y}} \to q_* Q'' \to 0$$

is stable. We have constructed the limit of the original family (15) of stable quotients over Δ^0.

The case when the subsheaf S has arbitrary rank is similar. The cokernel K of the map

$$(\mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}})^* \to S^*$$

has support of dimension at most 1. The initial Fitting ideal of K, denoted $\mathcal{F}_0(K)$, endows the support of K with a natural scheme structure. After a suitable composition of blow-ups

$$\mu : \widetilde{\mathcal{X}} \to \mathcal{X},$$

we may take

$$\mathcal{F}_0(p^* K) = p^* \mathcal{F}_0(K)$$

to be divisorial with only exceptional components passing through the nodes and markings of the central fiber. Let V be the exceptional part of $\mathcal{F}_0(p^*K)$ We set

$$K' = \mu^* K \otimes \mathcal{O}_V,$$

and define the sheaves $\widetilde{K}$ and $\widetilde{S}$ by the diagram

$$\begin{array}{ccccc} (\mathbb{C}^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}})^* & \longrightarrow & \widetilde{S}^* & \longrightarrow & \widetilde{K} \\ \| & & \downarrow & & \downarrow \\ (\mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}})^* & \longrightarrow & \mu^* S^* & \longrightarrow & \mu^* K \\ & & \downarrow & & \downarrow \\ & & K' & = & K' \end{array} .$$

[7]Here, $\pi' : \mathcal{X}' \to \Delta$.

The Fitting ideal $\mathcal{F}_0(\widetilde{K})$ does not vanish on exceptional divisors of μ. Therefore, the quotient

$$0 \to \widetilde{S} \to \mathbb{C}^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}} \to \widetilde{Q} \to 0$$

has no torsion at the nodes or the markings of the (reduced) central fiber. The remaining steps exactly follow the rank 1 case. □

7. Proofs of Theorems 3 and 4

7.1. **Localization.** The idea is to proceed by localization with respect to the maximal torus $\mathbf{T} \subset \mathbf{GL}_n(\mathbb{C})$ acting with diagonal weights $\mathsf{w}_1, \ldots, \mathsf{w}_n$. By the usual splitting principle, the torus calculation is enough for the full equivariant result. Localization formulas for the virtual classes of the moduli of stable maps and stable quotients[8] are both given by [10].

Theorem 3 is a special case of Theorem 4, so will consider only the latter. We will compare fixed point residues pushed-forward to

$$\overline{Q}_{g,m}(\mathbb{G}(1, \binom{n}{r}), d).$$

7.2. **T-fixed loci for stable maps.** The $\mathbf{T}$-fixed loci of the moduli space $\overline{M}_{g,m}(\mathbb{G}(r,n), d)$ are described in detail in [10]. We briefly recall here that the fixed loci are indexed by decorated graphs $(\Gamma, \nu, \gamma, \epsilon, \delta, \mu)$ where

(i) $\Gamma = (V, E)$ such that V is the vertex set and E is the edge set (with no self-edges),
(ii) $\nu : V \to \mathbb{G}(r,n)^{\mathbf{T}}$ is an assignment of a $\mathbf{T}$-fixed point $\nu(v)$ to each element $v \in V$,
(iii) $\gamma : V \to \mathbb{Z}_{\geq 0}$ is a genus assignment,
(iv) ϵ is an assignment to each $e \in E$ of a $\mathbf{T}$-invariant curve $\epsilon(e)$ of $\mathbb{G}(r,n)$ together with a covering number $\delta(e) \geq 1$,
(v) μ is a distribution of the m markings to the vertices V.

The graph Γ is required to be connected. The two vertices incident to the edge $e \in E$ must correspond via ν to the two $\mathbf{T}$-fixed points

[8] An analogous localization computation for the virtual class of the Quot scheme of a fixed nonsingular curve was carried out in [23]. In particular, the fixed loci and their contributions were explicitly determined. The localization for the stable quotient space is conceptually similar.

incident to $\epsilon(e)$. The sum of γ over V together with $h^1(\Gamma)$ must equal g. The sum of δ over E must equal d.

The $\mathbf{T}$-fixed locus corresponding to a given graph is, up to automorphisms, the product

$$\prod_v \overline{M}_{\gamma(v),\, \mathrm{val}(v)},$$

where $\mathrm{val}(v)$ counts all incident edges and markings. The stable maps in the $\mathbf{T}$-fixed locus are easily described. If the condition

$$2\gamma(v) - 2 + \mathrm{val}(v) > 0$$

holds[9], then the vertex v corresponds to a collapsed curve varying in $\overline{M}_{\gamma(v),\mathrm{val}(v)}$. Moreover, each edge e gives a degree $\delta(e)$ covering of the invariant curve $\epsilon(e)$, ramified only over the two torus fixed points. The stable map is obtained by gluing along the graph incidences.

7.3. T-fixed loci for stable quotients.

7.3.1. *The indexing set.* The $\mathbf{T}$-fixed loci of $\overline{Q}_{g,m}(\mathbb{G}(r,n), d)$ are similarly indexed by decorated graphs $(\Gamma, \nu, \gamma, s, \epsilon, \delta, \mu)$ where

- (i) $\Gamma = (V, E)$ such that V is the vertex set and E is the edge set (no self-edges are allowed),
- (ii) $\nu : V \to \mathbb{G}(r,n)^{\mathbf{T}}$ is an assignment of a $\mathbf{T}$-fixed point $\nu(v)$ to each element $v \in V$,
- (iii) $\gamma : V \to \mathbb{Z}_{\geq 0}$ is a genus assignment,
- (iv) $s(v) = (s_1(v), \ldots, s_r(v))$ is an assignment of a tuple of non-negative integers with $\mathbf{s}(v) = \sum_{i=1}^r s_i(v)$ together with an inclusion
$$\iota_s : \{1, \ldots, r\} \to \{1, \ldots, n\},$$
- (v) ϵ is an assignment to each $e \in E$ of a $\mathbf{T}$-invariant curve $\epsilon(e)$ of $\mathbb{G}(r,n)$ together with a covering number $\delta(e) \geq 1$,
- (vi) μ is a distribution of the markings to the vertices V.

The graph Γ is required to be connected. The two vertices incident to the edge $e \in E$ must correspond via ν to the two $\mathbf{T}$-fixed points incident to $\epsilon(e)$. The sum of γ over V together with $h^1(\Gamma)$ must equal g. The assignment s determines the splitting type of the subsheaf over the vertex v. The inclusion ι_s determines r trivial factors of $\mathbb{C}^n \otimes \mathcal{O}_C$

[9]Otherwise, the vertex is *degenerate*.

in which the subsheaf S injects. The inclusion ι_s must be compatible with $\nu(v)$. The sum of $\mathbf{s}(v)$ over V together with the sum of δ over E must equal d.

Unless v is a degenerate vertex satisfying

$$\gamma(v) = 0, \quad \mathrm{val}(v) = 2, \quad \mathbf{s}(v) = 0,$$

the stability condition

$$2\gamma(v) - 2 + \mathrm{val}(v) + \epsilon \cdot \mathbf{s}(v) > 0$$

holds for every strictly positive $\epsilon \in \mathbb{Q}$. The valence of v, as before, counts all incident edges and markings.

7.3.2. *Mixed pointed spaces.* The $\mathbf{T}$-fixed loci for the stable quotients are described in terms of mixed pointed spaces. Let $s = (s_1, \ldots, s_r)$ be a tuple of non-negative integers. Let $\overline{M}_{g,A|s}$ be the moduli space of genus g curves with markings

$$\{p_1, \ldots, p_A\} \ \cup \ \bigcup_{j=1}^{r} \{\widehat{p}_{j1}, \ldots, \widehat{p}_{js_j}\} \ \in C^{ns} \subset C$$

satisfying the conditions

(i) the points p_i are distinct,
(ii) the points $\widehat{p}_{jk}$ are distinct from the points p_i,

with stability given by the ampleness of

$$\omega_C(\sum_{i=1}^{A} p_i + \epsilon \sum_{j,k} \widehat{p}_{jk})$$

for every strictly positive $\epsilon \in \mathbb{Q}$. The conditions allow the points $\widehat{p}_{jk}$ and $\widehat{p}_{j'k'}$ to coincide. If

$$\mathbf{s} = \sum_{j=1}^{r} s_j,$$

then $\overline{M}_{g,A|s} = \overline{M}_{g,A|\mathbf{s}}$ defined in Section 4.1.

7.3.3. *Torus fixed quotients.* Fix a decorated graph $(\Gamma, \nu, \gamma, s, \epsilon, \delta, \mu)$ indexing a $\mathbf{T}$-fixed locus of the moduli space $\overline{Q}_{g,m}(\mathbb{G}(r,n), d)$. The corresponding $\mathbf{T}$-fixed locus is, up to automorphisms, the product of mixed pointed spaces

$$\prod_{v \in V} \overline{M}_{\gamma(v),\, \mathrm{val}(v)|s(v)}.$$

The corresponding $\mathbf{T}$-fixed stable quotients can be described explicitly. For each vertex v of the graph, pick a curve C_v in the mixed moduli space with markings

$$\{p_1, \ldots, p_{\mathrm{val}(v)}\} \cup \bigcup_{j=1}^{r} \{\widehat{p}_{j1}, \ldots, \widehat{p}_{js_j(v)}\}.$$

For each edge e, pick a rational curve C_e. A pointed curve C is obtained by gluing the curves C_v and C_e via the graph incidences, and distributing the markings on the domain via the assignment μ.

(i) On the component C_v, the stable quotient is given by the exact sequence

$$0 \to \oplus_{j=1}^{r} \mathcal{O}_{C_v}(-\sum_{k=1}^{s_j(v)} \widehat{p}_{jk}) \to \mathbb{C}^n \otimes \mathcal{O}_{C_v} \to Q \to 0.$$

The first inclusion is the composition of

$$\oplus_{j=1}^{r} \mathcal{O}_{C_v}(-\sum_{k=1}^{s_j(v)} \widehat{p}_{jk}) \to \mathbb{C}^r \otimes \mathcal{O}_{C_v}$$

with the r-plane $\mathbb{C}^r \otimes \mathcal{O}_{C_v} \to \mathbb{C}^n \otimes \mathcal{O}_{C_v}$ determined by i_s.

(ii) For each edge e, consider the degree δ_e covering of the $\mathbf{T}$-invariant curve $\epsilon(e)$ in the Grassmannian $\mathbb{G}(r, n)$:

$$f_e : C_e \to \epsilon(e)$$

ramified over the two torus fixed points. The stable quotient is obtained pulling back the tautological sequence of $\mathbb{G}(r, n)$ to C_e.

The gluing of stable quotients on different components is made possible by the compatibility of i_s and ν.

7.4. **Contributions.**

7.4.1. *Vertices for stable maps.* Consider the case of a nondegenerate vertex v occurring in a graph for stable maps. The vertex corresponds to the moduli space[10] $\overline{M}_{\gamma(v),\mathrm{val}(v)}$. The vertex contribution is computed in [10]

$$\mathsf{Cont}(v) = \frac{\mathsf{e}(\mathbb{E}^* \otimes T_{\nu(v)})}{\mathsf{e}(T_{\nu(v)})} \frac{1}{\prod_e \frac{w(e)}{\delta(e)} - \psi_e}. \tag{18}$$

[10]As usual we order all issues and quotient by the overcounting.

Here, e denotes the Euler class $T_{\nu(v)}$ is the $\mathbf{T}$-representation on the tangent space of $\mathbb{G}(r,n)$ at $\nu(v)$, and

$$\mathbb{E} \to \overline{M}_{\gamma(v),\mathrm{val}(v)},$$

is the Hodge bundle. Finally, the product in the denominator is over all half-edges incident to v. The factor $w(e)$ denotes the $\mathbf{T}$-weight of the tangent representation along the corresponding $\mathbf{T}$-fixed edge, and ψ_e denotes the cotangent line at the corresponding marking of $\overline{M}_{\gamma(v),\mathrm{val}(v)}$.

7.4.2. *Vertices for stable quotients.* Next, let v be a nondegenerate vertex occurring in a graph for stable quotients. For simplicity, assume

$$\iota_s(j) = j, \quad 1 \le j \le r.$$

The vertex corresponds to the moduli space[11] $\overline{M}_{\gamma(v),\mathrm{val}(v)|s(v)}$ where the subsheaf is given by

$$0 \to S = \oplus_{j=1}^{r} \mathcal{O}_C(-\sum_{k=1}^{s_j(v)} \widehat{p}_{jk}) \xrightarrow{\iota_s} \mathbb{C}^n \otimes \mathcal{O}_C \to Q \to 0.$$

The vertex contribution, determined by the *moving* part of $RHom(S,Q)$, is

$$\mathsf{Cont}(v) = \frac{\mathsf{e}(\mathrm{Ext}^1(S,Q)^{\mathrm{m}})}{\mathsf{e}(\mathrm{Ext}^0(S,Q)^{\mathrm{m}})} \frac{1}{\prod_e \frac{w(e)}{\delta(e)} - \psi_e} \quad . \tag{19}$$

Since the Ext spaces are not separately of constant rank, a better form is needed for (19).

Let $S_i \subset C$ be the divisor associated to points corresponding to s_i. By the results of Section 4.6, we see that (19) is equivalent to

$$\begin{aligned} \mathsf{Cont}(v) \quad = \quad & \frac{\mathsf{e}(\mathbb{E}^* \otimes T_{\nu(v)})}{\mathsf{e}(T_{\nu(v)})} \frac{1}{\prod_e \frac{w(e)}{\delta(e)} - \psi_e} \cdot \\ & \frac{1}{\prod_{i \ne j} \mathsf{e}(H^0(\mathcal{O}_C(S_i)|_{S_j}) \otimes [\mathsf{w}_j - \mathsf{w}_i])} \cdot \\ & \frac{1}{\prod_{i,j^*} \mathsf{e}(H^0(\mathcal{O}_C(S_i)|_{S_i}) \otimes [\mathsf{w}_{j^*} - \mathsf{w}_i]))} \end{aligned}$$

where the products in the last factors satisfy the following conditions

$$1 \le i \le r, \quad 1 \le j \le r, \quad r+1 \le j^* \le n \quad .$$

[11] Again, we order all issues and quotient by the overcounting.

The brackets $[\cdot]$ in the above expression denote the trivial line bundle with the specified weights.

While the vertex contributions for stable maps and stable quotients appear quite different, the genus dependent part of the integrand involving the Hodge bundle is the *same*. The differences all involve the local geometry of the points.

7.5. **Matching.** Under the map to $\overline{Q}_{g,m}(\mathbb{G}(1, \binom{n}{r}), d)$, the stable map side has many genus 0 tails which are collapsed. Similarly, the stable quotient side has many splitting types of the subbundle S which are collapsed. The differences in the localization formulas occur entirely in the nondegenerate vertices. For noncollapsed edges (not occurring in genus 0 tails of the stable map space) and noncollapsed degenerate vertices of valence 2, the edge and vertex contributions exactly coincide.

The crucial step in the argument is to notice Theorems 3 and 4 are a consequence of a universal calculation in a moduli space of pointed curves. In fact, the universal calculation is genus independent since the genus dependent integrand factors match.

In genus 0, a geometric argument can be given. Since

$$\overline{M}_{0,m}(\mathbb{G}(r,n), d) \text{ and } \overline{Q}_{0,m}(\mathbb{G}(r,n), d)$$

are nonsingular of expected dimension, the virtual class in both cases is the usual fundamental class. Moreover, since the moduli space are irreducible[12] and birational, Theorem 4 in the form

$$c_* \iota_{M*}\Big([\overline{M}_{0,m}(\mathbb{G}(r,n), d)]^{vir}\Big) = \iota_{Q*}\Big([\overline{Q}_{0,m}(\mathbb{G}(r,n), d)]^{vir}\Big). \tag{20}$$

is trivial. The image of $c \circ \iota_M$ simply coincides with the image of ι_Q.

Let $\xi \in \mathbb{G}(r,n)^{\mathbf{T}}$ be a fixed point and let

$$\iota(\xi) \in \mathbb{G}(1, \binom{n}{r})$$

be the image in the Plücker embedding. Consider the $\mathbf{T}$-fixed locus of $\overline{Q}_{0,m}(\mathbb{G}(1, \binom{n}{r}), d)$ which corresponds to a single vertex over $\iota(\xi)$ with no edges. By the discussion of Section 7.3, the associated $\mathbf{T}$-fixed locus is $\overline{M}_{0,m|d}/\mathbb{S}_d$.

Equality (20) implies a matching after $\mathbf{T}$-equivariant localization. In particular, there is a matching obtained for $\mathbf{T}$-equivariant residues on

[12] See [15, 32].

the locus $\overline{M}_{0,m|d}/\mathbb{S}_d$ over $\iota(\xi)$. The residue of

$$c_* \iota_{M*}\Big([\overline{M}_{0,m}(\mathbb{G}(r,n),d)]^{vir} \Big)$$

is a graph sum over all **T**-fixed point loci of $\overline{M}_{0,m}(\mathbb{G}(r,n),d)$ which contract to $\overline{M}_{0,m|d}/\mathbb{S}_d$ over $\iota(\xi)$. Similarly, the residue of

$$\iota_{Q*}\Big([\overline{Q}_{0,m}(\mathbb{G}(r,n),d)]^{vir} \Big)$$

is a splitting sum over all **T**-fixed point loci of $\overline{Q}_{0,m}(\mathbb{G}(r,n),d)$ which collapse to $\overline{M}_{0,m|d}/\mathbb{S}_d$ over $\iota(\xi)$.

While the equality of residues holds on $\overline{M}_{0,m|d}/\mathbb{S}_d$, we can canonically lift both sides to symmetric polynomials in $\widehat{\psi}_j$ and D_J on $\overline{M}_{0,m|d}$. On the left side, we use the contribution formulas of Section 7.4.1 and Lemma 3 to obtain

$$L_{d,\xi}(\widehat{\psi}_j, D_J).$$

On the right side we use the contribution formulas of Section 7.4.2 to obtain

$$R_{d,\xi}(\widehat{\psi}_j, D_J).$$

The symmetry of $L_{d,\xi}$ and $R_{d,\xi}$ is with respect the points $\widehat{p}_1, \ldots, \widehat{p}_d$. We may take the symmetric polynomials $L_{d,\xi}$ and $R_{d,\xi}$ to be in the canonical form of Section 4.4. The polynomials $L^C_{d,\xi}$ and $R^C_{d,\xi}$ are independent of m.

We know the push-forwards of $L^C_{d,\xi}$ and $R^C_{d,\xi}$ to $\overline{M}_{0,m|d}/\mathbb{S}_d$ are equal by the matching of residues in genus 0. By the independence result of Section 7.6, we conclude the much stronger equality

$$L^C_{d,\xi} = R^C_{d,\xi} \tag{21}$$

as abstract *polynomials.*

The equality (21) as polynomials implies Theorems 3 and 4 for arbitrary genus since the cotangent calculus is genus independent. □

7.6. **Independence.**

7.6.1. *Polynomials.* Consider variables $\widehat{\psi}_1, \ldots, \widehat{\psi}_d$ and

$$\{\ D_J \mid J \subset \{1, \ldots, d\},\ |J| \geq 2\ \}$$

for fixed $d \geq 0$. Given a polynomial $P(\widehat{\psi}_j, D_J)$, we obtain a canonical form P^C in the sense of Section 4.4.

We view P^C in two different ways. First, P^C yields a class

$$P^C = P(\widehat{\psi}_j, D_J) \in A^*(\overline{M}_{0,m|d}, \mathbb{Q}) \tag{22}$$

for every m. We will always take $m \geq 3$ to avoid unstable cases. Second, P^C is an abstract polynomial. If P^C always vanishes in the first sense (22), then we will show that P^C vanishes as an abstract polynomial.

If $P(\widehat{\psi}_j, D_J)$ is symmetric with respect to the $\mathbb{S}_d$-action on the variables, then P^C is also symmetric. The class (22) lies in the $\mathbb{S}_d$-invariant sector,

$$P^C \in A^*(\overline{M}_{0,m|d}, \mathbb{Q})^{\mathbb{S}_d} = A^*(\overline{M}_{0,m|d}/\mathbb{S}_d, \mathbb{Q}). \tag{23}$$

Hence, for symmetric P, only the vanishing in $A^*(\overline{M}_{0,m|d}/\mathbb{S}_d, \mathbb{Q})$ will be required to show P^C vanishes as an abstract polynomial.

7.6.2. *Partitions.* Fix a codimension k. Let

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_\ell)$$

be a set partition of $\{1, \ldots, d\}$ with $\ell \geq d - k$ nonempty parts,

$$\cup_{i=1}^{\ell} \mathcal{P}_i = \{1, \ldots, d\}.$$

The parts $\mathcal{P}_i$ are ordered by lexicographical ordering.[13] Let

$$\tau = (t_1, \ldots, t_\ell), \quad \sum_{i=1}^{\ell} t_i = k - d + \ell$$

be an ordered partition of $k - d + \ell$. The parts t_i *are* allowed to be 0.

Let $\mathsf{P}[d,k]$ be the set of all such pairs $[\mathcal{P}, \tau]$. Let $\mathbf{V}[d,k]$ be a $\mathbb{Q}$-vector space with basis given by the elements of $\mathsf{P}[d,k]$. To each pair $[\mathcal{P}, \tau] \in \mathsf{P}[d,k]$ and integer $m \geq 3$, we associate the class

$$X_m[\mathcal{P}, \tau] = \widehat{\psi}_{\mathcal{P}_1}^{t_1} \ldots \widehat{\psi}_{\mathcal{P}_\ell}^{t_\ell} \cdot D_{\mathcal{P}_1} \cdots D_{\mathcal{P}_\ell} \quad \in A^k(\overline{M}_{0,m|d}) \ .$$

7.6.3. *Pairing.* Let $\mathbf{W}_m$ the $\mathbb{Q}$-vector space with basis given by the symbols $[\mathcal{S}, \mu]$ where $\mathcal{S} \subset \overline{M}_{0,m|d}$ is a stratum of dimension $s \geq k$ and μ is a monomial in the variables

$$\psi_1, \ldots, \psi_m$$

of degree $s - k$.

[13] The choice of ordering will not play a role in the argument.

There is a canonical Poincaré pairing

$$I : \mathbf{V}[d,k] \times \mathbf{W}_m \to \mathbb{Q}$$

defined on the bases by

$$I([\mathcal{P},\tau],[\mathcal{S},\mu]) = \int_{\mathcal{S}} X_m[\mathcal{P},\tau] \cup \mu(\psi_1,\ldots,\psi_m).$$

Lemma 5. *A vector $v \in \mathbf{V}[d,k]$ is 0 if and only if v is in the null space of all the pairings I for $m \geq 3$.*

Proof. To $[\mathcal{P},\tau] \in \mathsf{P}[d,k]$, we associate $Y_m[\mathcal{P},\tau] \in \mathbf{W}_m$, where

$$m = 4 + k - d + 2\ell$$

and ℓ is the length of $\mathcal{P}$, by the following construction.

Let $\mathcal{S} \subset \overline{M}_{0,m|d}$ be the stratum consisting of a chain of $\ell + 2$ rational curves attached head to tail,

$$R_0 - R_1 - R_2 - \ldots - R_\ell - R_{\ell+1},$$

with the m markings $p_1,\ldots,p_m$ distributed by the rules:

(i) R_0 and $R_{\ell+1}$ each carry exactly 2 markings,
(ii) For $1 \leq i \leq \ell$, R_i carries $t_i + 1$ markings,
(iii) the marking are distributed in order from left to right.

We write r_i for the minimal label of the markings on R_i. The d markings $\widehat{p}_1,\ldots,\widehat{p}_d$ are distributed by the rules

(iv) R_0 and $R_{\ell+1}$ each carry 0 markings
(v) For $1 \leq i \leq \ell$, R_i carries the markings corresponding to $\mathcal{P}_i$.

The dimension of $\mathcal{S}$ is easily calculated,

$$\begin{aligned}
\dim(\mathcal{S}) &= \dim(\overline{M}_{0,m|d}) - \ell - 1 \\
&= 4 + k - d + 2\ell + d - 3 - \ell - 1 \\
&= \ell + k.
\end{aligned}$$

The associated element of $\mathbf{W}_m$ is defined by

$$Y_m[\mathcal{P},\tau] = [\mathcal{S}, \psi_{r_1}\cdots\psi_{r_\ell}].$$

The next step is to find when the pairing

$$I([\mathcal{P},\tau], Y_{m'}[\mathcal{P}',\tau']) \tag{24}$$

is nontrivial for

$$[\mathcal{P},\tau],\ [\mathcal{P}',\tau'] \in \mathsf{P}[d,k],$$

with $m' = 4 + k - d + 2\ell'$. By definition, the pairing (24) equals

$$(25)\quad \int_{\mathcal{S}'} X_{m'}[\mathcal{P},\tau] \cup \psi_{r'_1} \cdots \psi_{r'_{\ell'}} = \int_{\mathcal{S}'} \psi_{r'_1} \cdots \psi_{r'_{\ell'}} \cdot \widehat{\psi}^{t_1}_{\mathcal{P}_1} \ldots \widehat{\psi}^{t_\ell}_{\mathcal{P}_\ell} \cdot D_{\mathcal{P}_1} \cdots D_{\mathcal{P}_\ell}$$

The integral (25) is calculated by distributing the diagonal points corresponding to $D_{\mathcal{P}_j}$ to the components R'_i of curves in $\mathcal{S}'$ in all possible ways. Note that unless there is at least one diagonal $D_{\mathcal{P}_j}$ distributed to *each* R'_i for $1 \le i \le \ell'$, the contribution to the integral (25) vanishes. Hence, nonvanishing implies $\ell \ge \ell'$.

If $\ell = \ell'$, then the distribution rule (v) implies the set theoretic intersection

$$\mathcal{S}' \cap D_{\mathcal{P}_1} \cap \cdots \cap D_{\mathcal{P}_\ell}$$

is empty unless $\mathcal{P} = \mathcal{P}'$. If $\mathcal{P} = \mathcal{P}'$, the only nonvanishing diagonal distribution is given by sending $D_{\mathcal{P}_i}$ to R'_i. The integral (25) is easily seen to be nonzero then if and only if $\tau = \tau'$. Indeed, the contribution of R'_i to the integral is

$$\int_{\overline{M}_{0,t'_i+3||\mathcal{P}_i|}} \psi_{r'_i}\, \widehat{\psi}^{t_i}_{\mathcal{P}_i} \cdot D_{\mathcal{P}_i} = \int_{\overline{M}_{0,t'_i+4}} \psi_{r'_i}\, \psi^{t_i}_{t'_i+4} = \begin{cases} t_i + 1 & \text{if } t_i = t'_i \\ 0 & \text{otherwise} \end{cases}.$$

The linear functions on $\mathbf{V}[d,k]$ determined by $I(\cdot, Y_{m'}[\mathcal{P}',\tau'])$ are block lower-triangular with respect to the partial ordering by the length of the set partition. Moreover, the diagonal blocks are themselves diagonal with nonzero entries. □

Following the notation of Section 7.6.1, Lemma 5 proves that if

$$P^C \in A^*(\overline{M}_{0,m|d}, \mathbb{Q})$$

vanishes for all $m \ge 3$, then P^C vanishes as an abstract polynomial. The proofs of Theorems 3 and 4 are therefore complete.

8. Tautological relations

8.1. **Tautological classes.** Let $g \ge 2$. The tautological ring of the moduli space of curves

$$R^*(M_g) \subset A^*(M_g, \mathbb{Q})$$

is generated by the classes

$$\kappa_i = \epsilon_*(\psi_1^{i+1}), \qquad M_{g,1} \xrightarrow{\epsilon} M_g.$$

Here, $\kappa_0 = 2g-2$ is a multiple of the unit class. A conjectural description of $R^*(M_g)$ is presented in [5]. The basic vanishing result,

$$R^i(M_g) = 0$$

for $i > g-2$, has been proven by Looijenga [21].

8.2. **Relations.** Let $g \geq 2$ and $d \geq 0$. The moduli space

$$M_{g,0|d} \xrightarrow{\epsilon} M_g$$

is simply the d-fold product of the universal curve over M_g. Given an element

$$[C, \widehat{p}_1, \ldots, \widehat{p}_d] \in M_{g,0|d} \ ,$$

there is a canonically associated stable quotient

$$0 \to \mathcal{O}_C(-\sum_{j=1}^{d} \widehat{p}_j) \to \mathcal{O}_C \to Q \to 0. \tag{26}$$

Consider the universal curve

$$\pi : U \to M_{g,0|d}$$

with universal quotient sequence

$$0 \to S_U \to \mathcal{O}_U \to Q_U \to 0$$

obtained from (26). Let

$$\mathbb{F}_d = -R\pi_*(S_U^*) \in K(M_{g,0|d})$$

be the class in K-theory. For example,

$$\mathbb{F}_0 = \mathbb{E}^* - \mathbb{C}$$

is the dual of the Hodge bundle minus a rank 1 trivial bundle.

By Riemann-Roch, the rank of $\mathbb{F}_d$ is

$$r_g(d) = g - d - 1.$$

However, $\mathbb{F}_d$ is not always represented by a bundle. By the derivation of Section 4.6,

$$\mathbb{F}_d = \mathbb{E}^* - \mathbb{B}_d - \mathbb{C}, \tag{27}$$

where $\mathbb{B}_d$ has fiber $H^0(C, \mathcal{O}_C(\sum_{j=1}^d \widehat{p}_j)|_{\sum_{j=1}^d \widehat{p}_j})$ over $[C, \widehat{p}_1, \ldots, \widehat{p}_d]$. Alternatively, $\mathbb{B}_d$ is the ϵ-relative tangent bundle.

Theorem 5. *For every integer $k > 0$,*

$$\epsilon_* \left(c_{r_g(d)+2k}(\mathbb{F}_d) \right) = 0 \ \in R^*(M_g).$$

Since the morphism ϵ has fibers of dimension d,

$$\epsilon_* \left(c_{r_g(d)+2k}(\mathbb{F}_d) \right) \in R^{g-2d-1+2k}(M_g).$$

By Looijenga's vanishing, Theorem 5 is only nontrivial when

$$0 \leq 2d - 2k - 1 \leq g - 2.$$

The vanishing of Theorem 5 does not naively extend. We calculate

$$\epsilon_* \left(c_{r_g(1)+1}(\mathbb{F}_1) \right) = \kappa_{g-2} - \lambda_1 \kappa_{g-3} + \ldots + (-1)^{g-2} \kappa_0 \lambda_{g-2} \tag{28}$$

in $R^{g-2}(M_g)$ by (27). However, the class (28) is known not to vanish by the pairing with $\lambda_g \lambda_{g-1}$ calculated in [26].

Theorem 5 directly yields relations among the generators κ_i of $R^*(M_g)$ by the standard ϵ push-forward rules [5]. The construction is more subtle than the method of [5] as the relations only hold after push-forward. An advantage is that the boundary terms of the relations here can easily be calculated.

8.3. **Example.** The Chern classes of $\mathbb{F}_d$ can be easily computed. Recall the divisor $D_{i,j}$ where the markings $\widehat{p}_i$ and $\widehat{p}_j$ coincide. Set

$$\Delta_i = D_{1,i} + \ldots + D_{i-1,i},$$

with the convention $\Delta_1 = 0$. Over $[C, \widehat{p}_1, \ldots, \widehat{p}_d]$, the virtual bundle $\mathbb{F}_d$ is the formal difference

$$H^1(\mathcal{O}_C(\widehat{p}_1 + \ldots + \widehat{p}_d)) - H^0(\mathcal{O}_C(\widehat{p}_1 + \ldots + \widehat{p}_d)).$$

Taking the cohomology of the exact sequence

$$0 \to \mathcal{O}_C(\widehat{p}_1 + \ldots + \widehat{p}_{d-1}) \to \mathcal{O}_C(\widehat{p}_1 + \ldots + \widehat{p}_d) \to \mathcal{O}_C(\widehat{p}_1 + \ldots + \widehat{p}_d)|_{\widehat{p}_d} \to 0,$$

we find

$$c(\mathbb{F}_d) = \frac{c(\mathbb{F}_{d-1})}{1 + \Delta_d - \widehat{\psi}_d}.$$

Inductively, we obtain

$$c(\mathbb{F}_d) = \frac{c(\mathbb{E}^*)}{(1 + \Delta_1 - \widehat{\psi}_1) \cdots (1 + \Delta_d - \widehat{\psi}_d)}. \tag{29}$$

In the $d = 2$ and $k = 1$ case, Theorem 5 gives the vanishing of the class

$$\epsilon_* \, c_{g-1}(\mathbb{F}_d) = \epsilon_* \left[\frac{c(\mathbb{E}^*)}{(1-\widehat{\psi}_1)(1+\Delta-\widehat{\psi}_2)} \right]^{g-1},$$

where Δ is the divisor of coincident markings on $M_{g,0|2}$. The superscript indicates the degree $g-1$ part of the bracketed expression. Expanding, we obtain

$$\sum_i (-1)^i \lambda_{g-1-i} \sum_{i_1+i_2=i} \epsilon_* \left(\widehat{\psi}_1^{i_1} \, (\widehat{\psi}_2 - \Delta)^{i_2} \right) = 0. \tag{30}$$

We have

$$\epsilon_* \left(\widehat{\psi}_1^{i_1} \, (\widehat{\psi}_2 - \Delta)^{i_2} \right) = \sum_m (-1)^{i_2-m} \binom{i_2}{m} \epsilon_* \left(\widehat{\psi}_1^{i_1} \, \widehat{\psi}_2^{m} \, \Delta^{i_2-m} \right).$$

Using

$$\Delta^2 = -\widehat{\psi}_1 \, \Delta = -\widehat{\psi}_2 \, \Delta$$

and the ϵ-calculus rules in [5], we rewrite the last expression as

$$-\sum_{m \neq i_2} \binom{i_2}{m} \epsilon_*(\widehat{\psi}_1^{i_1+i_2-1} \Delta) + \epsilon_*(\widehat{\psi}_1^{i_1} \, \widehat{\psi}_2^{i_2}) = -(2^{i_2}-1)\kappa_{i_1+i_2-2} + \kappa_{i_1-1} \, \kappa_{i_2-1}.$$

After summing over i_1, i_2 in (30), we arrive at the relation

$$\sum_{i=2}^{g-1} (-1)^i \lambda_{g-1-i} \left(\left(\sum_{i_1+i_2=i} \kappa_{i_1-1}\kappa_{i_2-1} \right) - (2^{i+1} - i - 2)\kappa_{i-2} \right) = 0 \tag{31}$$

in $R^{g-3}(M_g)$.

The λ classes can be expressed in terms of the κ classes by Mumford's Chern character calculation

$$\mathrm{ch}_{2\ell}(\mathbb{E}) = 0, \qquad \mathrm{ch}_{2\ell-1}(\mathbb{E}) = \frac{B_{2\ell}}{(2\ell)!} \kappa_{2\ell-1}$$

for $\ell > 0$. From (31), we obtain a relation involving only the tautological generators κ_i. To illustrate, in genus 6, we obtain the relation

$$25\kappa_1^3 + 15912\kappa_3 - 1080\kappa_1\kappa_2 = 0,$$

which is consistent with the presentation of $R^*(M_6)$ in [5].

8.4. **Brill-Noether construction.** The $k=1$ case of Theorem 5 for positive $d \leq g$ admits an alternative derivation via Brill-Noether theory.[14]

To start, consider the rank d bundle,

$$\mathbb{W}_d \to M_{g,0|d} \ ,$$

with fiber $H^0(C, \omega_C|_{\sum_{j=1}^d \widehat{p}_j})$ over $[C, \widehat{p}_1, \ldots, \widehat{p}_d]$. There is a canonical map of vector bundles on $M_{g,0|d}$,

$$\rho : \mathbb{E} \to \mathbb{W}_d \ ,$$

defined by the restriction $H^0(C, \omega_C) \to H^0(C, \omega_C|_{\sum_{j=1}^d \widehat{p}_j})$. After dualizing, we obtain

$$\rho^* : \mathbb{W}_d^* \to \mathbb{E}^* \ .$$

If ρ^* fails to have maximal rank at $[C, \widehat{p}_1, \ldots, \widehat{p}_d] \in M_{g,0|d}$, then the divisor $\widehat{p}_1 + \ldots + \widehat{p}_d$ must move in a nontrivial linear series. The degeneracy locus of ρ^* precisely defines the Brill-Noether variety [1]

$$G^1_d \subset M_{g,0|d} \ ,$$

well-known to be of expected codimension $g-d+1$. Since

$$\epsilon : G^1_d \to M_g$$

has positive dimensional fibers, certainly

$$\epsilon_*[G^1_d] = 0 \ \in A^*(M_g)$$

By the Porteous formula [7],

$$[G^1_d] = c_{g-d+1}(\mathbb{E}^* - \mathbb{W}_d^*) \ .$$

Hence, we obtain the relation

$$\epsilon_* \left(c_{g-d+1}(\mathbb{E}^* - \mathbb{W}_d^*) \right) = 0 \ \in R^*(M_g) \ . \tag{32}$$

Lemma 6. $\mathbb{W}_d \stackrel{\sim}{=} \mathbb{B}_d^*$.

Proof. Let $\widehat{P} \subset C$ denote the divisor $\widehat{p}_1 + \ldots + \widehat{p}_d$. The fiber of $\mathbb{W}_d$ over $[C, \widehat{p}_1, \ldots, \widehat{p}_d]$ is

$$\mathrm{Ext}^0(\mathcal{O}_C, \omega_C|_{\widehat{P}}) \stackrel{\sim}{=} \mathrm{Ext}^1(\mathcal{O}_{\widehat{P}}, \mathcal{O}_C)^*$$

[14] The Brill-Noether connection was suggested by C. Faber who recognized equation (31).

by Serre duality. Let

$$I = [\mathcal{O}_C(-\widehat{P}) \to \mathcal{O}_C]$$

denote the complex of line bundles in grade -1 and 0. Since I is quasi-isomorphic to $\mathcal{O}_{\widehat{P}}$, we find

$$\mathrm{Ext}^1(I, \mathcal{O}_C) \stackrel{\sim}{=} \mathrm{Ext}^1(\mathcal{O}_{\widehat{P}}, \mathcal{O}_C)$$

On the other hand, we have

$$I^* = [\mathcal{O}_C \to \mathcal{O}_C(\widehat{P})] \ \text{ and } \ \mathrm{Ext}^1(\mathcal{O}_C, I^*) \stackrel{\sim}{=} \mathrm{Ext}^0(\mathcal{O}_C, \mathcal{O}_{\widehat{P}}(\widehat{P})).$$

We have hence found a canonical isomorphism

$$\mathrm{Ext}^1(\mathcal{O}_{\widehat{P}}, \mathcal{O}_C) \stackrel{\sim}{=} \mathrm{Ext}^0(\mathcal{O}_C, \mathcal{O}_{\widehat{P}}(\widehat{P}))$$

where the latter space is the fiber of $\mathbb{B}_d$ □

The $k = 1$ case of Theorem 5 concerns the class

$$\begin{aligned} c_{g-d+1}(\mathbb{F}_d) &= c_{g-d+1}(\mathbb{E}^* - \mathbb{B}_d - \mathbb{C}) \\ &= c_{g-d+1}(\mathbb{E}^* - \mathbb{B}_d) \\ &= c_{g-d+1}(\mathbb{E}^* - \mathbb{W}_d^*). \end{aligned}$$

Hence, the vanishing

$$\epsilon_*(c_{g-d+1}(\mathbb{F}_d)) = 0$$

of Theorem 5 exactly coincides with the Brill-Noether vanishing (32).

Theorem 5 may be viewed as a generalization of Brill-Noether vanishing obtained from the virtual geometry of the moduli of stable quotients.

8.5. **Proof of Theorem 5.** Consider the proper morphism

$$\nu : Q_g(\mathbb{P}^1, d) \to M_g.$$

The universal curve

$$\pi : U \to Q_g(\mathbb{P}^1, d)$$

carries the basic divisor classes

$$s = c_1(S_U^*), \qquad \omega = c_1(\omega_\pi)$$

obtained from the universal subsheaf S_U and the π-relative dualizing sheaf. The class

$$\nu_* \left(\pi_*(s^a \omega^b) \cdot 0^c \cap [Q_g(\mathbb{P}^1, d)]^{vir} \right) \in A^*(M_g, \mathbb{Q}), \tag{33}$$

where 0 is first Chern class of the trivial bundle, certainly vanishes if $c > 0$. Theorem 5 is proven by calculating (33) by localization. We will find Theorem 5 is a subset of a richer family of relations.

Let the 1-dimensional torus $\mathbb{C}^*$ act on a 2-dimensional vector space $V \cong \mathbb{C}^2$ with diagonal weights $[0, 1]$. The $\mathbb{C}^*$-action lifts canonically to the following spaces and sheaves:

$$\mathbb{P}(V), \quad Q_g(\mathbb{P}(V), d), \quad U, \quad S_U, \quad \text{and} \quad \omega_\pi.$$

We lift the $\mathbb{C}^*$-action to a rank 1 trivial bundle on $Q_g(\mathbb{P}(V), d)$ by specifying fiber weight 1. The choices determine a $\mathbb{C}^*$-lift of the class

$$\pi_*(s^a \cdot \omega^b) \cdot 0^c \cap [Q_g(\mathbb{P}(V), d)]^{vir} \in A_{2d+2g-1-a-b-c}(Q_g(\mathbb{P}(V), d), \mathbb{Q}).$$

The push-forward (33) is determined by the virtual localization formula [10]. There are only two $\mathbb{C}^*$-fixed loci. The first corresponds to a vertex lying over $0 \in \mathbb{P}(V)$. The locus is isomorphic to

$$M_{g,0|d} \;/\; \mathbb{S}_d$$

and the associated subsheaf (26) lies in the first factor of $V \otimes \mathcal{O}_C$ when considered as a stable quotient in the moduli space $Q_g(\mathbb{P}(V), d)$. Similarly, the second fixed locus corresponds to a vertex lying over $\infty \in \mathbb{P}(V)$.

The localization contribution of the first locus to (33) is

$$\frac{1}{d!}\epsilon_* \left(\pi_*(s^a \omega^b) \cdot c_{g-d-1+c}(\mathbb{F}_d) \right)$$

where s and ω are the corresponding classes on the universal curve over $M_{g,0|d}$. Let $c_-(\mathbb{F}_d)$ denote the total Chern class of $\mathbb{F}_d$ evaluated at -1. The localization contribution of the second locus is

$$\frac{(-1)^{g-d-1}}{d!}\epsilon_* \Big[\pi_* \left((s-1)^a \omega^b \right) \cdot c_-(\mathbb{F}_d) \Big]^{g-d-2+a+b+c}$$

where $[\gamma]^k$ is the part of γ in $A^k(M_{g,0|d}, \mathbb{Q})$.

Both localization contributions are found by straightforward expansion of the vertex formulas of Section 7.4.2. Summing the contributions yields the following result.

Proposition 5. *Let $c > 0$. Then*

$$\epsilon_*\Big(\pi_*(s^a\omega^b)\cdot c_{g-d-1+c}(\mathbb{F}_d)+ (-1)^{g-d-1}\Big[\pi_*\left((s-1)^a\omega^b\right)\cdot c_-(\mathbb{F}_d)\Big]^{g-d-2+a+b+c}\Big)=0$$

in $R^*(M_g)$.

If $a = 0$ and $b = 1$, the relation of Proposition 5 specializes to Theorem 5 for even $c = 2k$. □

Question 1. *Do the relations obtained from Proposition 5 generate all the relations among the classes κ_i in $R^*(M_g)$?*

8.6. **Further examples.** Let $\sigma_i \in A^1(U,\mathbb{Q})$ be the class of the i^{th} section of the universal curve

$$\pi : U \to M_{g,0|d}\ .$$

The class $s = c_1(S_U^*)$ of Proposition 5 is

$$s = \sigma_1 + \ldots + \sigma_d \ \in A^1(U,\mathbb{Q}).$$

We calculate

$$\begin{aligned} \pi_*(s) &= d \\ \pi_*(\omega) &= 2g-2 \\ \pi_*(s\ \omega) &= \widehat{\psi}_1 + \ldots + \widehat{\psi}_d \\ \pi_*(s^2) &= -(\widehat{\psi}_1 + \ldots + \widehat{\psi}_d) + 2\Delta \end{aligned}$$

in $A^*(M_{g,0|d},\mathbb{Q})$, where

$$\Delta = \sum_{i<j} D_{i,j} \ \in A^1(M_{g,0|d},\mathbb{Q})$$

is the symmetric diagonal. The push-forwards $\pi_*(s^a\omega^b)$ are all easily obtained.

Using the above π_* calculations, the $a = 1$, $b = 1$, $c = 2k$ case of Proposition 5 yields

$$\epsilon_*\Big(2(\widehat{\psi}_1 + \ldots + \widehat{\psi}_d)\cdot c_{r_g(d)+2k}(\mathbb{F}_d) + (2g-2)\ c_{r_g(d)+2k+1}(\mathbb{F}_d)\Big) = 0.$$

The $a = 2$, $b = 0$, $c = 2k$ case yields

$$\epsilon_*\Big(-2(\widehat{\psi}_1 + \ldots + \widehat{\psi}_d - 2\Delta)\cdot c_{r_g(d)+2k}(\mathbb{F}_d) + 2d\cdot c_{r_g(d)+2k+1}(\mathbb{F}_d)\Big) = 0.$$

Summation yields a third relation,

$$\epsilon_*\Big(2\Delta \cdot c_{r_g(d)+2k}(\mathbb{F}_d) + (d+g-1)\cdot c_{r_g(d)+2k+1}(\mathbb{F}_d)\Big) = 0.$$

The relations of Proposition 5 include the classes $c_{r_g(d)+2k+1}(\mathbb{F}_d)$ omitted in Theorem 5.

9. Calabi-Yau geometry

The moduli of stable quotients may be used to define counting invariants in the local Calabi-Yau geometries. For example consider the conifold, the total space of

$$\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\to\mathbb{P}^1.$$

Just as in Gromov-Witten theory, we define

$$N_{g,d} = \frac{1}{d^2}\int_{[\overline{Q}_{g,2}(\mathbb{P}^1,d)]^{vir}} \mathsf{e}(R^1\pi_*(S_U)\oplus R^1\pi_*(S_U))\cup \text{ev}_1^*(H)\cdot \text{ev}_2^*(H) \tag{34}$$

where S_U is the universal subsheaf on the universal curve

$$\pi : U \to \overline{Q}_{g,2}(\mathbb{P}^1, d)$$

and $H \in H^2(\mathbb{P}^1,\mathbb{Q})$ is the hyperplane class. The two point insertions are required for stability in genus 0. Let

$$F(t) = \sum_{g\geq 1} N_{g,1}t^{2g}.$$

Proposition 6. *The local invariants $N_{g,d}$ are determined by the following two equations,*

$$N_{g,d} = d^{2g-3}N_{g,1},$$

$$F(t) = \left(\frac{t/2}{\sin(t/2)}\right)^2.$$

Proof. We compute the integral $N_{g,d}$ by localization. Let $\mathbb{C}^*$ act on the vector space $V \stackrel{\sim}{=} \mathbb{C}^2$ with diagonal weights $[0,1]$. The $\mathbb{C}^*$-action lifts canonically to $\overline{Q}_{g,2}(\mathbb{P}(V),d)$ and S_U. For the first S_U in the integrand (34), we use the canonical lifting of $\mathbb{C}^*$. For the second S_U, we tensor by a trivial line bundle with fiber weights -1 over the two $\mathbb{C}^*$-fixed points of $\mathbb{P}(V)$. The classes H are lifted to the distinct $\mathbb{C}^*$-fixed points on $\mathbb{P}(V)$.

The above choice of $\mathbb{C}^*$-action on the integrand exactly parallels the choice of $\mathbb{C}^*$-action taken in [6] for the analogous Gromov-Witten calculation. The vanishing obtained in [6] also applies for the stable quotient calculation here. The only loci with non-vanishing contribution to the localization sum consist of two vertices of genera

$$g_1 + g_2 = g$$

connected by a single edge of degree d. The moduli spaces at these vertices are $\overline{M}_{g_i,2|0}$ where

(i) the first two points are the respective node and marking,
(ii) there are no markings after the bar by vanishing.

We find that the only non-vanishing contributions occur on $\mathbb{C}^*$-fixed loci where the moduli of stable quotients and the moduli of stable maps are isomorphic. Moreover, on these loci, the bundle $R^1\pi_*(S_U)$ agrees with the analogous Gromov-Witten bundle. Hence, the stable quotient integral $N_{g,d}$ is equal to the Gromov-Witten calculation of the conifold [6]. □

The matching is somewhat of a surprise. While the virtual classes of the stable quotient and stable maps spaces to $\mathbb{P}^1$ are related by Theorem 3, the bundles in the respective integrands for the conifold geometry are *not* compatible. However, the differences happen away from the non-vanishing loci.

If $g \geq 1$, no point insertions are required for stability. The associated conifold integral is more subtle to calculate, but the same result is obtained. We leave the details to the reader.[15]

Proposition 7. *For $g \geq 1$,*

$$N_{g,d} = \int_{[\overline{Q}_{g,0}(\mathbb{P}^1,d)]^{vir}} \mathsf{e}(R^1\pi_*(S_U) \oplus R^1\pi_*(S_U)).$$

There are many other well-defined local toric Calabi-Yau geometries to consider for stable quotients both in dimension 3 and higher [16, 29]. The simplest is local $\mathbb{P}^2$.

[15] The vanishing, as before, matches the $\mathbb{C}^*$-fixed point loci of the stable quotients and stable maps spaces. However, the two which correspond to a single vertex of genus g are now not obviously equal. The match for these is obtained by redoing the pointed integral (34) with both H classes in the integrand taken to lie over the *same* $\mathbb{C}^*$-fixed point.

Question 2. *What is the answer for the stable quotient theory for*

$$\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2 \quad ?$$

10. Other targets

10.1. **Virtual classes.** Let $X \subset \mathbb{P}^n$ be a projective variety. There is a naturally associated substack

$$\overline{Q}_{g,m}(X,d) \subset \overline{Q}_{g,m}(\mathbb{P}^n,d) \tag{35}$$

defined by the following principle. Let $I \subset \mathbb{C}[z_0,\ldots,z_n]$ be the homogeneous ideal of X. Given an element

$$(C,\ p_1,\ldots,p_m,\ 0 \to S \to \mathbb{C}^{n+1} \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0) \tag{36}$$

of $\overline{Q}_{g,m}(\mathbb{P}^n,d)$, consider the dual

$$\mathbb{C}^{n+1} \otimes \mathcal{O}_C \xrightarrow{q^*} S^*$$

as a line bundle with $n+1$ sections $s_0,\ldots,s_n$. The stable quotient (36) lies in $\overline{Q}_{g,m}(X,d)$ if for every homogeneous degree k polynomial $f_k \in I$,

$$f_k(s_0,\ldots,s_n) = 0 \in H^0(C, S^{k*}). \tag{37}$$

Condition (37) is certainly well-defined in families and determines a Deligne-Mumford substack. Local equations for the substack (35) can easily be found.

Question 3. *If X is nonsingular, does $\overline{Q}_{g,m}(X,d)$ carry a canonical 2-term perfect obstruction theory?*

The moduli space $\overline{Q}_{g,m}(X,d)$ depends upon the projective embedding of X. If $\overline{Q}_{g,m}(X,d)$ does carry a virtual class, the theory will almost certainly differ somewhat from the Gromov-Witten counts.

If $X \subset \mathbb{P}^n$ is nonsingular complete intersection, more definite claims can be made. For simplicity, assume X is a hypersurface defined by a degree k equation F. Given an element

$$(C,\ p_1,\ldots,p_m,\ 0 \to S \to \mathbb{C}^{n+1} \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0)$$

of $\overline{Q}_{g,m}(X,d)$, the pull-back to C of the tangent bundle to X may be viewed as the complex

$$S^* \otimes Q \xrightarrow{dF} S^{k*} \tag{38}$$

defined by differentiation of the section F on the zero locus. We speculate an obstruction theory on $\overline{Q}_{g,m}(X,d)$ can be defined by the hypercohomology of the sequence (38). The 2-term condition follows from the fact that the map dF has cokernel with dimension 0 support. Many details have to be checked here.

10.2. **Elliptic invariants.** An interesting example to consider is the moduli space $\overline{Q}_{1,0}(X_{n+1} \subset \mathbb{P}^n, d)$ of stable quotients associated to the Calabi-Yau hypersurfaces $X_{n+1} \subset \mathbb{P}^n$.

By Proposition 1, $\overline{Q}_{1,0}(\mathbb{P}^n, d)$ is a nonsingular space of expected dimension $(n+1)d$. As before, let S_U be the universal subsheaf on the universal curve

$$\pi : U \to \overline{Q}_{1,0}(\mathbb{P}^n, d).$$

Since S_U is locally free of rank 1, S_U is a line bundle. By the vanishing used in the proof of Proposition 1,

$$\pi_* S_U^{*(n+1)} \to \overline{Q}_{1,0}(\mathbb{P}^n, d)$$

is locally free of rank $(n+1)d$.

We define the genus 1 stable quotient invariants of $X_{n+1} \subset \mathbb{P}^n$ by the integral

$$N_{1,d}^{X_{n+1}} = \int_{\overline{Q}_{1,0}(\mathbb{P}^n,d)} \mathsf{e}\left(\pi_* S_U^{*(n+1)}\right). \tag{39}$$

The definition of $N_{1,d}^{X_{n+1}}$ is compatible with the discussion of the virtual classes of hypersurfaces in Section 10.1.

The genus 1 Gromov-Witten theory of hypersurfaces has recently been solved by Zinger [35]. Substantial work is required to convert the Gromov-Witten calculation to an Euler class on a space of genus 1 maps to projective space. The stable quotient invariants are immediately given by such an Euler class. There is no obstruction to calculating (39) by localization.

Question 4. *What is the relationship between the stable quotient and stable map invariants in genus 1 for Calabi-Yau hypersurfaces?*

10.3. **Variants.** There are several variants which can be immediately considered. Let X be a nonsingular projective variety with an ample line bundle L. The stable quotient construction can be carried out over the moduli space of stable maps $\overline{M}_{g,m}(X,\beta)$ instead of the moduli space of curves $\overline{M}_{g,m}$. An object then consists of three pieces of data:

(i) a genus g, m-pointed, quasi-stable curve $(C, p_1, \ldots, p_m)$,
(ii) a map $f : C \to X$ representing class $\beta \in H_2(X, \mathbb{Z})$,
(iii) and a quasi-stable quotient sequence

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_C \to Q \to 0.$$

Stability is defined by the ampleness of

$$\omega_C(p_1 \ldots + p_m) \otimes f^*(L^3) \otimes (\wedge^r S^*)^{\otimes \epsilon}$$

on C for every strictly positive $\epsilon \in \mathbb{Q}$. We leave the details to the reader. The moduli space is independent of the choice of L.

The moduli space carries a 2-term obstruction theory and a virtual class. The corresponding descendent theory is equivalent to the Gromov-Witten theory of $X \times \mathbb{G}(r,n)$ by straightforward modification of the arguments used to prove Theorem 4.

There is no reason to restrict to the trivial bundle in (iii) above. We may fix a rank n vector bundle

$$B \to X$$

and replace the quasi-stable quotient sequence by

$$0 \to S \to f^*(B) \to Q \to 0.$$

The corresponding theory is perhaps equivalent to the Gromov-Witten theory of the Grassmannian bundle over X associated to B. As B may not split, a torus action may not be available. The strategy of the proof of Theorem 4 does not directly apply.

A stranger replacement of the trivial bundle can be made even when X is a point. We may choose the quotient sequence to be

$$0 \to S \to H^0(C, \omega_C) \otimes \mathcal{O}_C \to Q \to 0.$$

The middle term is essentially the pull-back of the Hodge bundle from the moduli space of curves.

Question 5. *What do integrals over the moduli of stable Hodge quotients correspond to in Gromov-Witten theory?*

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