Midterm — Math 215 Fall 2008

You may use Rudin for the exam. In your answers, you may cite results proven in Rudin. No other source may be consulted.
You may not discuss the exam with anyone.
Remember to sign the pledge.

150 minutes, 5 questions

I. Consider the real numbers $\mathbb{R}$ with the standard Euclidean metric. For each of the three following functions

$$f, g, h : \mathbb{R} \to \mathbb{R},$$

find the set of points in the domain $\mathbb{R}$ at which the function is continuous.

(i) $f(x) = x^2 + \frac{1}{6}x + \frac{5}{6}$ for $x < 1$ and $f(x) = \frac{3}{7}x^7 + \frac{11}{7}$ for $x \geq 1$.

(ii) $g(x) = \lfloor \frac{3}{x^2+1} \rfloor$.

(iii) $h(x) = x^2 - 2x + 3$ if x is rational and $h(x) = x + 1$ if x is irrational.

For part (ii), recall $\lfloor x \rfloor$ denotes the integer part of x (the greatest integer less than or equal to x), so

$$\lfloor 6.9 \rfloor = 6, \quad \lfloor 7 \rfloor = 7, \quad \lfloor 7.1 \rfloor = 7 \quad .$$

Answer with proof — however, you may use without proof that a polynomial function is continuous.

II. Consider $\mathbb{R}$ with the standard Euclidean metric. Answer the following questions with proof.

(i) Does there exist a nonempty open set $U \subset \mathbb{R}$ which is compact?

(ii) Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ for which

$$f\Big((0,1)\Big) = [0,1] \quad ?$$

(iii) Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ which satisfies the following two properties (a) and (b)?

(a) For every compact set $C \subset \mathbb{R}$, the preimage $f^{-1}(C) \subset \mathbb{R}$ is compact.
(b) The limit of the infinite sequence
$$f(1),\ f(2),\ f(3),\ f(4),\ f(5),\ \ldots$$
exists in $\mathbb{R}$ and is equal to 1000.

In part (ii),
$$f\big((0,1)\big) = \{\ y \in \mathbb{R} \mid \exists x \in (0,1) \text{ such that } y = f(x)\ \}.$$
In part (iii),
$$f^{-1}(C) = \{\ x \in \mathbb{R} \mid f(x) \in C\ \}.$$

III. Determine (with proof) whether the following sequences and series converge.

(i) $\lim_{n\to\infty}(1+\frac{1}{n})^{\sqrt{n}}$

(ii) $\lim_{n\to\infty}(1+\frac{1}{n^2})^{n^2-n}$

(iii) $\sum_{n=1}^{\infty}(1+\frac{1}{n})^{-n^2}$

IV. For a nonzero rational number $r \in \mathbb{Q}$, the integer $v(r) \in \mathbb{Z}$ is defined uniquely by
$$r = 2^{v(r)} \cdot \frac{p}{q}$$
where p, q are odd integers. For example,
$$v\left(\frac{2}{3}\right) = 1,\quad v(8) = 3,\quad v\left(\frac{1}{12}\right) = -2,\quad v(-1) = 0.$$
The 2-adic metric d_2 on $\mathbb{Q}$ is defined by
$$d_2(x,y) = 2^{-v(x-y)},\quad x \neq y \in \mathbb{Q}$$
and $d_2(x,x) = 0$.

(i) Does the infinite sequence of points given by the reciprocals of odd integers

$$1,\ \frac{1}{3},\ \frac{1}{5},\ \frac{1}{7},\ \frac{1}{9},\ \frac{1}{11},\ \frac{1}{13},\ \ldots$$

converge in the metric space $(\mathbb{Q}, d_2)$? If so compute the limit.

(ii) Does the infinite sequence of points given by powers of 2,

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ \ldots$$

converge in the metric space $(\mathbb{Q}, d_2)$? If so compute the limit.

(iii) Does the sum $\sum_{n=0}^{\infty} 2^n$ converge in $(\mathbb{Q}, d_2)$? If so compute the sum.

By definition, the sum (iii) converges in $(\mathbb{Q}, d_2)$ if and only if the sequence of partial sums

$$1,\ 1+2,\ 1+2+4,\ 1+2+4+8,\ \ldots$$

converges in $(\mathbb{Q}, d_2)$.

Answer all questions with proof.

V. Let $\mathbb{R}$ be the real numbers, and let $\mathbb{C}$ be the complex numbers.

(i) Does there exist a power series $\sum_{n=0}^{\infty} c_n z^n$ with coefficients $c_n \in \mathbb{R}$ which diverges when

$$z = \frac{3}{5} + \frac{4}{5}i$$

but converges for all other values of $z \in \mathbb{C}$ on the unit circle $|z| = 1$?

(ii) Does there exist a power series $\sum_{n=0}^{\infty} c_n z^n$ with coefficients $c_n \in \mathbb{C}$ which diverges when

$$z = \frac{3}{5} + \frac{4}{5}i$$

but converges for all other values of $z \in \mathbb{C}$ on the unit circle $|z| = 1$?

(iii) Does there exist a power series $\sum_{n=0}^{\infty} c_n z^n$ with $c_n \in \mathbb{C}$ which converges for all values of $z \in \mathbb{C}$ on the unit circle except for the values $1, i, -1$ (at which the power series diverges) ?

Answer all questions with an example or a proof of the impossibility.