Quiz — Math 215 Fall 2008

You may use Rudin for the quiz. In your answers, you may cite results proven in Rudin. No other source may be consulted. You may not discuss the quiz with anyone.

75 minutes, 3 questions

I. Let $\mathbb{Q} \subset \mathbb{R}$ be the subset of rational numbers in the real numbers with the standard metric.

(i) A closed interval is a subset of $\mathbb{R}$ of the form

$$I_{a,b} = \{r \in \mathbb{R} \mid a \leq r \leq b\}$$

for real numbers $a < b$.

If $C, D \subset \mathbb{R}$ are closed intervals and

$$C \cap \mathbb{Q} = D \cap \mathbb{Q},$$

must $C = D$?

(ii) If $U, V \subset \mathbb{R}$ are open sets and

$$U \cap \mathbb{Q} = V \cap \mathbb{Q},$$

must $U = V$?

Answer all questions with proof.

II. Which of the following sets are countable? Answer with proof (that $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable need not be proved).

(i) The subset $X \subset \mathbb{R}^2$ defined by

$$X = \{ (a, b) \mid a \in \mathbb{R},\ b \in \mathbb{R},\ a + b \in \mathbb{Q} \}.$$

(ii) The subset $Y \subset \mathbb{R}^2$ defined by

$$Y = \{ (a, b) \mid a \in \mathbb{R},\ b \in \mathbb{R},\ a + b \in \mathbb{Q},\ ab \in \mathbb{Q} \}.$$

(iii) The subset $S \subset \mathbb{R}$ of real numbers $0 < s < 1$ with decimal expansions

$$s = 0.d_1d_2d_3d_4d_5d_6d_7d_8 \ \ldots$$

with digits satisfying $d_i \geq d_{i+1}$ for all i. For example,

$$0.97777665555555555555555555555555\ldots$$

ending in infinitely many 5's is in S, but

$$\pi/10 = 0.31415926535897932384 6\ldots$$

is not in S.

III. Suppose $E \subset \mathbb{R}$ is a compact set with a countable (infinite) number of elements. For each question below, provide an example or prove the impossibility.

(i) Can the set of limit points of E in $\mathbb{R}$ be empty?

(ii) Can the set of limit points of E in $\mathbb{R}$ have exactly 5 elements?

(iii) Can the set of limit points of E in $\mathbb{R}$ contain the set $\mathbb{Q} \cap (0, 1)$?

(iv) Can the set of limit points of E in $\mathbb{R}$ contain all the integers $\mathbb{Z}$?