Final — Math 215 Fall 2008

You may use Rudin for the exam. In your answers, you may cite
results proven in Rudin. No other source may be consulted.
You may not discuss the exam with anyone.
Remember to sign the pledge.

The exam must be returned by noon on Friday January 16th to my
mailbox in the department office on the 3^{rd} floor of Fine Hall.

24 hours, 8 questions

I. Let $\mathbb{R}$ be the real numbers with the standard Euclidean metric. Let (M, d) be a metric space, and let

$$f : \mathbb{R} \to M$$

be a function satisfying

$$|x - y| \leq d(f(x), f(y)) \tag{1}$$

for all $x, y \in \mathbb{R}$. Let $\mathrm{Im}(f) \subset M$ be the image of f.

(i) Must f be 1-to-1 onto $\mathrm{Im}(f)$?

(ii) Must f be continuous?

(iii) If f is a continuous function that satisfies (1), must

$$\mathrm{Im}(f) \subset M$$

be a closed subset?

Recall the image denotes the set

$$\mathrm{Im}(f) = \{\ m \in M \mid \exists x \in \mathbb{R} \ \text{ such that } \ m = f(x)\ \}\ .$$

Answer all questions with proof or counterexample.

II. Let $\sqrt{2} = 1.4142135\ldots \in \mathbb{R}$ be the positive square root of 2. Consider the harmonic sequence

$$a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3},\ \ldots \tag{2}$$

with $a_k = \frac{1}{k}$. Answer the following questions with proof.

(i) Does there exists a *finite* subsequence

$$a_{n_1}, a_{n_2}, \ldots, a_{n_r}$$

of the harmonic sequence (2) such that

$$\sqrt{2} = \sum_{i=1}^{r} a_{n_i} \quad ?$$

(ii) Does there exist an infinite subsequence

$$a_{n_1}, a_{n_2}, a_{n_3},\ \ldots$$

of the harmonic sequence such that

$$\sqrt{2} = \sum_{i=1}^{\infty} a_{n_i} \quad ?$$

(iii) Does there exist an infinite subsequence

$$a_{n_1}, a_{n_2}, a_{n_3},\ \ldots$$

of the harmonic sequence such that

$$\sqrt{2} = \frac{1}{2}\sum_{i=1}^{\infty} \frac{a_{n_i}}{(n_i - 1)!} \quad ?$$

Recall a subsequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ satisfies the condition $n_i < n_{i+1}$ for all i.

III. Determine (with proof) whether the following sequences and series converge.

(i) $\lim_{n\to\infty} \frac{\sin^2(\frac{1}{n})}{1-\cos(\frac{1}{n})}$

(ii) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(iii) $\sum_{n=1}^{\infty} \sin^2(\frac{1}{n})$

IV. Let $I = [0,1] \subset \mathbb{R}$ be the closed unit interval. Let

$$g : I \to \mathbb{R}$$

be the constant function $g(x) = 0$ for all $x \in I$. Let

$$\{\ f_n : I \to \mathbb{R}\ \}$$

be a sequence of continuous functions which converge pointwise to g on I.

(i) Must the sequence of functions $\{f_n\}$ converge uniformly to g on I?

(ii) Consider the sequence of real numbers

$$\int_0^1 f_1(x)dx,\quad \int_0^1 f_2(x)dx,\quad \int_0^1 f_3(x)dx,\ \ldots\ .$$

Must $\lim_{n\to\infty} \left(\int_0^1 f_n(x)dx \right) = 0$?

(iii) If all the functions f_n are differentiable, must the sequence of derivatives $\{f'_n\}$ converge pointwise to g on I?

Answer all questions with proof or counterexample.

V. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with

$$f(0) = 0, \quad f'(0) = 2,$$

and $|f''(x)| \leq 2$ for all $x \in \mathbb{R}$.

(i) Let $F_n : \mathbb{R} \to \mathbb{R}$ be defined inductively by

$$F_1 = f, \quad F_n = f \circ F_{n-1}\ .$$

In other words, $F_2(x) = f(f(x))$, $F_3(x) = f(f(f(x)))$ and so on. Calculate

$$\frac{dF_9}{dx}(0)\ .$$

(ii) Is $f(1) = \pi$ possible? Here,

$$\pi = 3.141592653589\ldots$$

as usual.

(iii) If the value $f(1) = 1$ is specified, is there enough information to calculate $f''(1/2)$?

Answer all questions with proof.

VI. Let (X, d) be a metric space and let $f : X \to X$ be a function satisfying

$$d(x, y) > d(f(x), f(y))$$

for all points $x \neq y \in X$.

(i) Must f be continous?

(ii) A *fixed point* of f is a point $p \in X$ such that $f(p) = p$. Find an example where f has no fixed points on X.

(iii) If X is compact, must f have a fixed point?

(iv) If X is complete, must f have a fixed point?

Recall X is complete if and only if every Cauchy sequence converges.

Answer all questions with proof.

VII. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\begin{aligned} f(x) &= 1 \qquad \text{for } x < 0 \ , \\ f(0) &= 2 \ , \\ f(x) &= 3 \qquad \text{for } x > 0 \ . \end{aligned}$$

(i) Define the function $g : [-1,1] \to \mathbb{R}$ by

$$g(z) = \int_{-1}^{z} f(x)dx \ .$$

Is g differentiable at $z = 0$? If so calculate $g'(0)$.

(ii) Define the function $h : [-1,1] \to \mathbb{R}$ by

$$h(z) = \int_{-1}^{z^2} f(x)dx \ .$$

Is h differentiable at $z = 0$? If so calculate $h'(0)$.

VIII. A polynomial $p(x)$ with real coefficients is *even* if only even powers of x occur,

$$p(x) = c_0 + c_2x^2 + c_4x^4 + \ldots + c_{2n}x^{2n}, \quad c_i \in \mathbb{R} \ .$$

(i) Let $f : [0,2] \to \mathbb{R}$ be a continuous function. Does there always exist a sequence

$$p_1(x), \ p_2(x), \ p_3(x), \ \ldots$$

of even polynomials which converges to f uniformly on the interval $[0,2]$?

(ii) Let $g : [-1,2] \to \mathbb{R}$ be a continuous function. Does there always exist a sequence

$$p_1(x), \ p_2(x), \ p_3(x), \ \ldots$$

of even polynomials which converges to g uniformly on the interval $[-1,2]$?

(iii) Find (with proof) a sequence of even polynomials which converges uniformly to $\cos(x)$ on the interval $[-1,2]$.