

# THE $\kappa$ RING OF THE MODULI OF CURVES OF COMPACT TYPE: II

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ABSTRACT. The subalgebra of the tautological ring of the moduli of curves of compact type generated by the  $\kappa$  classes is studied. Relations, constructed via the virtual geometry of the moduli of stable maps, are used to prove universality results relating the  $\kappa$  rings in genus 0 to higher genus. Predictions for  $\kappa$  classes of the Gorenstein conjecture are proven.

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## 1. INTRODUCTION

1.1.  **$\kappa$  classes.** Let  $\overline{M}_{g,n}$  be the moduli space of genus  $g$ ,  $n$ -points stable curves. The  $\kappa$  classes in the Chow ring  $A^*(\overline{M}_{g,n})$  with  $\mathbb{Q}$ -coefficients are defined by the following construction. Let

$$\epsilon : \overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n}$$

be the universal curve viewed as the  $(n+1)$ -pointed space, let

$$\mathbb{L}_{n+1} \rightarrow \overline{M}_{g,n+1}$$

be the line bundle obtained from the cotangent space of the last marking, and let

$$\psi_{n+1} = c_1(\mathbb{L}_{n+1}) \in A^1(\overline{M}_{g,n+1})$$

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be the Chern class. The  $\kappa$  classes are

$$\kappa_i = \epsilon_*(\psi_{n+1}^{i+1}) \in A^i(\overline{M}_{g,n}), \quad i \geq 0 .$$

The simplest is  $\kappa_0$  which equals  $2g - 2 + n$  times the unit in  $A^0(\overline{M}_{g,n})$ .

The  $\kappa$  classes on the moduli space of curves of compact type

$$M_{g,n}^c \subset \overline{M}_{g,n}$$

are defined by restriction. The  $\kappa$  ring

$$\kappa^*(M_{g,n}^c) \subset A^*(M_{g,n}^c),$$

is the  $\mathbb{Q}$ -subalgebra generated by the  $\kappa$  classes. The  $\kappa$  rings are graded by degree.

By the results of [11],  $\kappa^*(M_{g,n}^c)$  is generated as a  $\mathbb{Q}$ -algebra by

$$\kappa_1, \kappa_2, \dots, \kappa_{g-1+\lfloor \frac{n}{2} \rfloor}.$$

Moreover, there are no relation of degree less than or equal to  $g - 1 + \lfloor \frac{n}{2} \rfloor$  if  $n > 0$ .

**1.2. Universality.** Let  $x_1, x_2, x_3, \dots$  be variables with  $x_i$  of degree  $i$ , and let

$$f \in \mathbb{Q}[x_1, x_2, x_3, \dots]$$

be *any* graded homogeneous polynomial. The following universality property was stated in [11].

**Theorem 1.** *If  $f(\kappa_i) = 0 \in \kappa^*(M_{0,n}^c)$ , then*

$$f(\kappa_i) = 0 \in \kappa^*(M_{g,n-2g}^c)$$

*for all genera  $g$  for which  $n - 2g \geq 0$ .*

By Theorem 1, the higher genus  $\kappa$  rings are canonically quotients of the genus 0 rings,

$$\kappa^*(M_{0,2g+n}^c) \xrightarrow{\iota_{g,n}} \kappa^*(M_{g,n}^c) \rightarrow 0.$$

Theorem 1 is our main result here.

1.3. **Bases.** Let  $P(d)$  be the set of partitions of  $d$ , and let

$$P(d, k) \subset P(d)$$

be the set of partitions of  $d$  into at most  $k$  parts. Let  $|P(d, k)|$  be the cardinality. To a partition<sup>1</sup>

$$\mathbf{p} = (p_1, \dots, p_\ell) \in P(d, k),$$

we associate a  $\kappa$  monomial by

$$\kappa_{\mathbf{p}} = \kappa_{p_1} \cdots \kappa_{p_\ell} \in \kappa^d(M_{g,n}^c).$$

In [11], two basic facts about the  $\kappa$  rings of the moduli space of curves of compact type are derived from Theorem 1:

- the canonical quotient,

$$\kappa^*(M_{0,2g+n}^c) \xrightarrow{lg,n} \kappa^*(M_{g,n}^c) \rightarrow 0$$

is an isomorphism for  $n > 0$ ,

- a  $\mathbb{Q}$ -basis of  $\kappa^d(M_{g,n}^c)$  is given by

$$\{ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P(d, 2g - 2 + n - d) \}$$

for  $n > 0$ .

The main tools used in [11] are the virtual geometry of the moduli space of stable quotients [9] and the intersection theory of strata classes in the tautological ring  $R^*(M_{g,n}^c)$ .

By Theorem 5 of [11], proven unconditionally,

$$\dim_{\mathbb{Q}} \kappa^d(M_{0,n}^c) = |P(d, n - 2 - d)|.$$

Hence, Theorem 1 is a consequence of the following result.

**Proposition 1.** *The space of relations among  $\kappa$  monomials of degree  $d$  valid in all the rings*

$$\{ \kappa^*(M_{g,n}^c) \mid 2g - 2 + n = \zeta \}$$

*is of rank at least  $|P(d)| - |P(d, \zeta - d)|$ .*

Proposition 1 is proven in Sections 2 - 4 by constructing universal relations in  $\kappa^*(M_{g,n}^c)$  via the virtual geometry of the moduli space of stable maps. The interplay between stable quotients and stable maps is an interesting aspect of the study of  $\kappa^*(M_{g,n}^c)$ .

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<sup>1</sup>The parts of  $\mathbf{p}$  are positive and satisfy  $p_1 \geq \dots \geq p_\ell$ .

1.4. **Gorenstein conjecture.** The rank  $g$  Hodge bundle over the moduli space of curves

$$\mathbb{E} \rightarrow \overline{M}_{g,n}$$

has fiber  $H^0(C, \omega_C)$  over  $[C, p_1, \dots, p_n]$ . Let

$$\lambda_k = c_k(\mathbb{E})$$

be the Chern classes. Since  $\lambda_g$  vanishes when restricted to

$$\delta_0 = \overline{M}_{g,n} \setminus M_{g,n}^c,$$

we obtain a well-defined evaluation

$$\phi : A^*(M_{g,n}^c) \rightarrow \mathbb{Q}$$

given by integration

$$\phi(\gamma) = \int_{\overline{M}_{g,n}} \overline{\gamma} \cdot \lambda_g,$$

where  $\overline{\gamma}$  is any lift of  $\gamma \in A^*(M_{g,n}^c)$  to  $A^*(\overline{M}_{g,n})$ .

The tautological rings  $R^*(M_{g,n}^c) \subset A^*(M_{g,n}^c)$  have been conjectured in [4, 10] to be Gorenstein algebras with socle in degree  $2g - 3 + n$ ,

$$\phi : R^{2g-3+n}(M_{g,n}^c) \xrightarrow{\sim} \mathbb{Q}.$$

As a consequence of Theorem 1 and the intersection calculations of [11], we obtain the following result.

**Theorem 2.** *If  $n > 0$  and  $\xi \in \kappa^d(M_{g,n}^c) \neq 0$ , the linear function*

$$L_\xi : R^{2g-3+n-d}(M_{g,n}^c) \rightarrow \mathbb{Q}$$

*defined by the socle evaluation*

$$L_\xi(\gamma) = \phi(\gamma \cdot \xi)$$

*is non-trivial.*

Theorem 2, discussed in Section 5.1, may be viewed as significant evidence for the Gorenstein conjecture for all  $M_{g,n}^c$  with  $n > 0$ .

**1.5. Acknowledgments.** The results here on the  $\kappa$  rings were motivated by the study of stable quotients in [9]. Discussions with A. Marian and D. Oprea were very helpful. The methods developed with C. Faber in [5] played an important role.

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## 2. $\kappa$ AND $\psi$

**2.1.  $\psi$  classes.** Consider the cotangent line classes

$$\psi_{n+1}, \dots, \psi_{n+\ell} \in A^1(M_{g,n+\ell}^c)$$

at the last  $\ell$  marked points. Let

$$\epsilon^c : M_{g,n+\ell}^c \rightarrow M_{g,n}^c$$

be the proper forgetful map. For each partition  $\mathbf{p} \in P(d)$  of length  $\ell$ , we associate the class

$$\epsilon_*^c (\psi_{n+1}^{1+p_1} \cdots \psi_{n+\ell}^{1+p_\ell}) \in A^d(M_{g,n}^c) .$$

The relation between the above push-forwards of  $\psi$  monomials and the  $\kappa$  classes is easily obtained. For  $\mathbf{p} = (d)$ , we have

$$\epsilon_*^c (\psi_{n+1}^{1+d}) = \kappa_d$$

by definition. The standard cotangent line comparison formulas yield the length 2 case,

$$\epsilon_*^c (\psi_{n+1}^{1+p_1} \psi_{n+2}^{1+p_2}) = \kappa_{p_1} \kappa_{p_2} + \kappa_{p_1+p_2} .$$

The full formula, due to Faber, is

$$(1) \quad \epsilon_*^c (\psi_{n+1}^{1+p_1} \cdots \psi_{n+\ell}^{1+p_\ell}) = \sum_{\sigma \in S_\ell} \kappa_{\sigma(\mathbf{p})} ,$$

where the sum is over the symmetric group  $S_\ell$ . For  $\sigma \in S_\ell$ , let

$$\sigma = \gamma_1 \cdots \gamma_r$$

be the canonical cycle decomposition (including the 1-cycles), and let  $\sigma(\mathbf{p})_i$  be the sum of the parts of  $\mathbf{p}$  with indices in the cycle  $\gamma_i$ . Then,

$$\kappa_{\sigma(\mathbf{p})} = \kappa_{\sigma(\mathbf{p})_1} \cdots \kappa_{\sigma(\mathbf{p})_r} .$$

A discussion of (1) can be found in [1].

**Lemma 1.** *The sets of classes in  $A^d(M_{g,n}^c)$  defined by*

$$\{ \epsilon_*^c (\psi_{n+1}^{1+p_1} \cdots \psi_{n+\ell}^{1+p_\ell}) \mid \mathbf{p} \in P(d) \} \quad \text{and} \quad \{ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P(d) \}$$

*are related by an invertible linear transformation independent of  $g$  and  $n$ .*

*Proof.* Formula (1) defines a universal transformation independent of  $g$  and  $n$ . Since the transformation is triangular in the partial ordering of  $P(d)$  by length (with 1's on the diagonal), the invertibility is clear.  $\square$

**2.2. Bracket classes.** Let  $\mathbf{p} \in P(d)$  be a partition of length  $\ell$ . Let

$$(2) \quad \langle \mathbf{p} \rangle = \epsilon_*^c \left[ \prod_{i=1}^{\ell} \frac{1}{1 - p_i \psi_{n+i}} \right]^{\ell+d} \in A^d(M_{g,n}^c).$$

The superscript in the inhomogeneous expression  $\left[ \prod_{i=1}^{\ell} \frac{1}{1 - p_i \psi_{n+i}} \right]^{\ell+d}$  indicates the summand in  $A^{\ell+d}(M_{g,n+\ell}^c)$ .

We can easily expand definition (2) to express the class  $\langle \mathbf{p} \rangle$  linearly in terms of the classes

$$\{ \epsilon_*^c (\psi_{n+1}^{1+p_1} \cdots \psi_{n+\ell}^{1+p_\ell}) \mid \mathbf{p} \in P(d) \}.$$

Since the string and dilation equation must be used to remove the  $\psi_{n+i}^0$  and  $\psi_{n+i}^1$  factors, the transformation depends upon  $g$  and  $n$  only through  $2g - 2 + n$ .

**Lemma 2.** *The sets of classes in  $A^d(M_{g,n}^c)$  defined by*

$$\{ \langle \mathbf{p} \rangle \mid \mathbf{p} \in P(d) \} \quad \text{and} \quad \{ \epsilon_*^c (\psi_{n+1}^{1+p_1} \cdots \psi_{n+\ell}^{1+p_\ell}) \mid \mathbf{p} \in P(d) \}$$

*are related by an invertible linear transformation depending only upon  $2g - 2 + n$ .*

*Proof.* Only the invertibility remains to be established. The result exactly follows from the proof of Proposition 3 in [5].  $\square$

By Lemma 1 and 2, the bracket classes lie in the  $\kappa$  ring,

$$\langle \mathbf{p} \rangle \in \kappa^d(M_{g,n}^c).$$

We will prove Proposition 1 in the following equivalent form.

**Proposition 2.** *The space of relations among the classes*

$$\{ \langle \mathbf{p} \rangle \mid \mathbf{p} \in P(d) \}$$

*valid in all the rings*

$$\{ \kappa^*(M_{g,n}^c) \mid 2g - 2 + n = \zeta \}$$

*is of rank at least  $|P(d)| - |P(d, \zeta - d)|$ .*

### 3. RELATIONS VIA STABLE MAPS

**3.1. Moduli of stable maps.** Let  $\overline{M}_{g,n+m}(\mathbb{P}^1, d)$  denote the moduli of stable maps<sup>2</sup> to  $\mathbb{P}^1$  of degree  $d$ , and let

$$\nu : \overline{M}_{g,n+m}(\mathbb{P}^1, d) \rightarrow \overline{M}_{g,n}$$

be the morphism forgetting the map and the last  $m$  markings. The moduli space

$$M_{g,n+m}^c(\mathbb{P}^1, d) \subset \overline{M}_{g,n+m}(\mathbb{P}^1, d)$$

is defined by requiring the domain curve to be of compact type. The restriction

$$\nu^c : M_{g,n+m}^c(\mathbb{P}^1, d) \rightarrow M_{g,n}^c$$

is proper and equivariant with respect to the symmetries of  $\mathbb{P}^1$ .

We will find relations in  $A^*(M_{g,n}^c)$  by localizing  $\nu^c$  push-forwards which vanish geometrically. A complete analysis in the socle  $A^{2g-3}(M_g^c)$  was carried out in [5], but much more will be required for Theorem 1. While the relations in  $A^*(M_{g,n}^c)$  via stable quotients [11] are more elegantly expressed, the ranks of the relations via stable maps appear easier to compute.

### 3.2. Relations.

**3.2.1. Indexing.** Let  $d \leq 2g - 3 + n$ , and let

$$\delta = 2g - 3 + n - d.$$

We will construct a series of relations  $I(g, d, \alpha)$  in  $A^d(M_{g,n}^c)$  where

$$\alpha = (\alpha_1, \dots, \alpha_m)$$

is a (non-empty) vector of non-negative integers satisfying two conditions:

$$(i) \quad |\alpha| = \sum_{i=1}^m \alpha_i \leq d - 2 - \delta,$$

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<sup>2</sup>Stable maps were defined in [8], see [6] for an introduction.

(ii)  $\alpha_i > 0$  for  $i > 1$ .

By condition (i),  $d - 2 - \delta \geq 0$  so

$$d > g - 1 + \lfloor \frac{n}{2} \rfloor .$$

Condition (ii) implies  $\alpha_1$  is the only integer permitted to vanish. The relation  $I(g, d, \alpha)$  will be a variant of the equations considered in [5].

3.2.2. *Formulas.* Let  $\Gamma$  denote the data type

$$(3) \quad (p_1, \dots, p_m) \cup \{p_{m+1}, \dots, p_\ell\},$$

satisfying

$$p_i > 0, \quad \sum_{i=1}^{\ell} p_i = d.$$

The first part of  $\Gamma$  is an ordered  $m$ -tuple  $(p_1, \dots, p_m)$ . The second part  $\{p_{m+1}, \dots, p_\ell\}$  is an unordered set. Let  $\text{Aut}(\{p_{m+1}, \dots, p_\ell\})$  be the group which permutes equal parts. The group of automorphisms  $\text{Aut}(\Gamma)$  equals  $\text{Aut}(\{p_{m+1}, \dots, p_\ell\})$ .

**Theorem 3.** *For all  $\alpha$  satisfying (i-ii),*

$$\sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{i=1}^m p_i^{-\alpha_i} \prod_{i=m+1}^{\ell} (-p_i)^{-1} \prod_{j=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \langle p_1, \dots, p_\ell \rangle = 0 \in A^d(M_{g,n}^c),$$

where the sum is over all  $\Gamma$  of type (3).

The bracket  $\langle p_1, \dots, p_\ell \rangle \in A^d(M_{g,n}^c)$  denotes the class associated to the partition defined by the union of all the parts  $p_i$  of  $\Gamma$ .

### 3.3. Proof of Theorem 3.

3.3.1. *Torus actions.* The first step is to define the appropriate torus actions. Let

$$\mathbb{P}^1 = \mathbb{P}(V)$$

where  $V = \mathbb{C} \oplus \mathbb{C}$ . Let  $\mathbb{C}^*$  act diagonally on  $V$ :

$$(4) \quad \xi \cdot (v_1, v_2) = (v_1, \xi \cdot v_2).$$

Let  $\mathbf{p}_1, \mathbf{p}_2$  be the fixed points  $[1, 0], [0, 1]$  of the corresponding action on  $\mathbb{P}(V)$ . An equivariant lifting of  $\mathbb{C}^*$  to a line bundle  $L$  over  $\mathbb{P}(V)$  is

uniquely determined by the weights  $[l_1, l_2]$  of the fiber representations at the fixed points

$$L_1 = L|_{\mathfrak{p}_1}, \quad L_2 = L|_{\mathfrak{p}_2}.$$

The canonical lifting of  $\mathbb{C}^*$  to the tangent bundle  $T_{\mathbb{P}}$  has weights  $[1, -1]$ . We will utilize the equivariant liftings of  $\mathbb{C}^*$  to  $\mathcal{O}_{\mathbb{P}(V)}(1)$  and  $\mathcal{O}_{\mathbb{P}(V)}(-1)$  with weights  $[1, 0]$ ,  $[0, 1]$  respectively.

Over the moduli space of stable maps  $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$ , we have

$$\pi : U \rightarrow \overline{M}_{g,n+m}(\mathbb{P}(V), d), \quad \mu : U \rightarrow \mathbb{P}(V)$$

where  $U$  is the universal curve and  $\mu$  is the universal map. The representation (4) canonically induces  $\mathbb{C}^*$ -actions on  $U$  and  $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$  compatible with the maps  $\pi$  and  $\mu$ . The  $\mathbb{C}^*$ -equivariant virtual class

$$[\overline{M}_{g,n+m}(\mathbb{P}(V), d)]^{vir} \in A_{2g+2d-2+n+m}^{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V), d))$$

will play an important role.

**3.3.2. Equivariant classes.** Three types of equivariant Chow classes on  $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$  will be considered here:

- The linearization  $[0, 1]$  on  $\mathcal{O}_{\mathbb{P}(V)}(-1)$  defines an  $\mathbb{C}^*$ -action on the rank  $d + g - 1$  bundle

$$\mathbb{R} = R^1\pi_*(\mu^*\mathcal{O}_{\mathbb{P}(V)}(-1))$$

on  $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$ . Let

$$c_{top}(\mathbb{R}) \in A_{\mathbb{C}^*}^{g+d-1}(\overline{M}_{g,n+m}(\mathbb{P}(V), d))$$

be the top Chern class.

- For each marking  $i$ , let  $\psi_i \in A_{\mathbb{C}^*}^1(\overline{M}_{g,n+m}(\mathbb{P}(V), d))$  denote the first Chern class of the canonically linearized cotangent line corresponding to  $i$ .
- Denote the  $i^{th}$  evaluation morphism by

$$\text{ev}_i : \overline{M}_{g,n+m}(\mathbb{P}(V), d) \rightarrow \mathbb{P}(V).$$

With  $\mathbb{C}^*$ -linearization  $[1, 0]$  on  $\mathcal{O}_{\mathbb{P}(V)}(1)$ , let

$$\rho_i = c_1(\text{ev}_i^*\mathcal{O}_{\mathbb{P}(V)}(1)) \in A_{\mathbb{C}^*}^1(\overline{M}_{g,n+m}(\mathbb{P}(V), d)).$$

With  $\mathbb{C}^*$ -linearization  $[0, -1]$  on  $\mathcal{O}_{\mathbb{P}(V)}(1)$ , let

$$\tilde{\rho}_i = c_1(\text{ev}_i^*\mathcal{O}_{\mathbb{P}(V)}(1)) \in A_{\mathbb{C}^*}^1(\overline{M}_{g,n+m}(\mathbb{P}(V), d)).$$

In the non-equivariant limit,  $\rho_i^2 = 0$ . Our notation here closely follows [5].

3.3.3. *Vanishing integrals.* The forgetful morphism

$$\nu : \overline{M}_{g,n+m}(\mathbb{P}(V), d) \rightarrow \overline{M}_{g,n}$$

is  $\mathbb{C}^*$ -equivariant with respect to the trivial action on  $\overline{M}_{g,n}$ . As in Section 3.2.1, let

$$d \leq 2g - 3 + n, \quad \delta = 2g - 3 + n - d,$$

and let  $\alpha = (\alpha_1, \dots, \alpha_m)$  satisfy

- (i)  $|\alpha| = \sum_{i=1}^m \alpha_i \leq d - 2 - \delta$ ,
- (ii)  $\alpha_i > 0$  for  $i > 1$ .

Let  $I(g, d, \alpha)$  be the  $\mathbb{C}^*$ -equivariant push-forward

$$\nu_* \left( \rho_{n+1}^{d-1-\delta-|\alpha|} \prod_{i=1}^m \rho_{n+i} \psi_{n+i}^{\alpha_i} \prod_{j=1}^n \tilde{\rho}_j c_{top}(\mathbb{R}) \cap [\overline{M}_{g,n+m}(\mathbb{P}(V), d)]^{vir} \right).$$

The degree of the class

$$\rho_{n+1}^{d-1-\delta-|\alpha|} \prod_{i=1}^m \rho_{n+i} \psi_{n+i}^{\alpha_i} \prod_{j=1}^n \tilde{\rho}_j c_{top}(\mathbb{R})$$

is easily computed to be

$$\begin{aligned} d - 1 - \delta - |\alpha| + m + |\alpha| + n + d + g - 1 &= \\ g + 2d - 2 + n + m - \delta. \end{aligned}$$

Since the cycle dimension of the virtual class is  $2g + 2d - 2 + n + m$ , the push-forward  $I(g, d, \alpha)$  has cycle dimension

$$\begin{aligned} 2g + 2d - 2 + n + m - (g + 2d - 2 + n + m - \delta) &= g + \delta \\ &= 3g - 3 + n - d. \end{aligned}$$

Equivalently,  $I(g, d, \alpha) \in A_{\mathbb{C}^*}^d(\overline{M}_{g,n})$ . Since the class  $\rho_{n+1}$  appears with exponent

$$d - \delta - |\alpha| \geq 2,$$

$I(g, d, \alpha)$  vanishes in the non-equivariant limit.

3.3.4. *Localization terms.* The virtual localization formula of [7] calculates  $I(g, d, \alpha)$  in terms of tautological classes on the moduli space  $\overline{M}_{g,n}$ . To prove Theorem 3, we will calculate the restriction of the localization formula to  $M_{g,n}^c$ .

The localization formula expresses  $I(g, d, \alpha)$  as a sum over connected decorated graphs  $\Gamma$  indexing the  $\mathbb{C}^*$ -fixed loci of  $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$ . The vertices of the graphs lie over the fixed points  $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{P}(V)$  and are labelled with genera (which sum over the graph to  $g - h^1(\Gamma)$ ). The edges of the graphs lie over  $\mathbb{P}^1$  and are labelled with degrees (which sum over the graph to  $d$ ). Finally, the graphs carry  $n + m$  markings on the vertices. The valence  $\text{val}(v)$  of a vertex  $v \in \Gamma$  counts both the incident edges and markings. The edge valence of  $v$  counts only the incident edges.

Only a very restricted subset of graphs will yield non-vanishing contributions to  $I(g, d, \alpha)$  in the non-equivariant limit. If a graph  $\Gamma$  contains a vertex lying over  $\mathbf{p}_1$  of edge valence greater than 1, then the contribution of  $\Gamma$  to vanishes by our choice of linearization on the bundle  $\mathbb{R}$ . A vertex over  $\mathbf{p}_1$  of edge valence greater than 1 yields a trivial Chern root of  $\mathbb{R}$  (with trivial weight 0) in the numerator of the localization formula to force the vanishing.

By the above vanishing, only *comb* graphs  $\Gamma$  contribute to  $I(g, d, \alpha)$ . Comb graphs contain  $\ell \leq d$  vertices lying over  $\mathbf{p}_1$  each connected by a distinct edge to a unique vertex lying over  $\mathbf{p}_2$ .

If  $\Gamma$  contains a vertex over  $\mathbf{p}_1$  of positive genus, then the restriction to  $M_{g,n}^c$  of the contribution of  $\Gamma$  to  $I(g, d, \alpha)$  vanishes by the following argument. Let  $v$  be a genus  $g(v) > 0$  vertex lying over  $\mathbf{p}_1$ . The integrand term  $c_{\text{top}}(\mathbb{R})$  yields a factor  $c_{g(v)}(\mathbb{E}^*)$  with trivial  $\mathbb{C}^*$ -weight on the genus  $g(v)$  moduli space corresponding to the vertex  $v$ . Since

$$\lambda_{g(v)}|_{M_{g(v), \text{val}(v)}^c} = 0$$

by [12], the required vanishing holds.

The linearizations of the classes  $\rho_i$  and  $\tilde{\rho}_j$  place restrictions on the marking distribution. Since the class  $\tilde{\rho}_j$  is obtained from  $\mathcal{O}_{\mathbb{P}(V)}(1)$  with linearization  $[0, -1]$ , the first  $n$  markings must lie on the unique vertex over over  $\mathbf{p}_2$ . Since the class  $\rho_i$  is obtained from  $\mathcal{O}_{\mathbb{P}(V)}(1)$  with linearization  $[1, 0]$ , the last  $m$  markings must lie on vertices over  $\mathbf{p}_1$ .

Finally, we claim the last  $m$  markings of  $\Gamma$  must lie on distinct vertices over  $\mathbf{p}_1$  for nonvanishing contribution to  $I(g, d, \alpha)$ . Let  $v$  be a vertex over  $\mathbf{p}_1$  (with  $g(v) = 0$ ). If  $v$  carries at least two markings, the fixed locus corresponding to  $\Gamma$  contains a product factor  $\overline{M}_{0, r+1}$  where  $r$  is the number of markings incident to  $v$ . The classes  $\psi_{n+i}^{\alpha_i}$  carry trivial  $\mathbb{C}^*$ -weight. Moreover, as each  $\alpha_i > 0$  for  $i > 1$ , we see the sum of the  $\alpha_i$  as  $i$  ranges over the set of markings incident to  $v$  is at least  $r - 1$ . Since the sum exceeds the dimension of  $\overline{M}_{0, r+1}$ , the graph contribution to  $I(g, d, \alpha)$  vanishes.

The proof of the main result about the localization terms for  $I(g, d, \alpha)$  is now complete.

**Proposition 3.** *The restriction of  $I(g, d, \alpha)$  to  $M_{g, n}^c$  is expressed via the virtual localization formula as a sum over genus  $g$ , degree  $d$ , marked comb graphs  $\Gamma$  satisfying:*

- (i) *all vertices over  $\mathbf{p}_1$  are of genus 0,*
- (ii) *the unique vertex over  $\mathbf{p}_2$  carries all of the first  $n$  markings,*
- (iii) *the last  $m$  markings all lie over  $\mathbf{p}_1$ ,*
- (iv) *each vertex over  $\mathbf{p}_1$  carries at most 1 of the last  $m$  markings.*

3.3.5. *Formulas.* The precise contributions of allowable graphs  $\Gamma$  to the non-equivariant limit of  $I(g, d, \alpha)$  are now calculated.

Let  $\Gamma$  be a genus  $g$ , degree  $d$ , comb graph with  $n + m$  markings satisfying conditions (i-iv) of Proposition 3. By condition (iv),  $\Gamma$  must have  $\ell \geq m$  edges.  $\Gamma$  may be described uniquely by the data

$$(5) \quad (p_1, \dots, p_m) \cup \{p_{m+1}, \dots, p_\ell\},$$

satisfying:

$$p_i > 0, \quad \sum_{i=1}^{\ell} p_i = d.$$

The elements of the ordered  $m$ -tuple  $(p_1, \dots, p_m)$  correspond to the degree assignments of the edges incident to the vertices marked by the last  $m$  markings. The elements of the unordered partition  $\{p_{m+1}, \dots, p_\ell\}$  correspond to the degrees of edges incident to the unmarked vertices over  $\mathbf{p}_1$ . The group of graph automorphisms is

$$\text{Aut}(\Gamma) = \text{Aut}(\{p_{m+1}, \dots, p_\ell\}) .$$

By a direct application of the virtual localization formula of [7], we find the contribution of the graph (5) to the normalized<sup>3</sup> push-forward

$$(-1)^{g+1+|\alpha|+n+m} \cdot I(g, d, \alpha)$$

equals

$$\frac{1}{|\text{Aut}(\Gamma)|} \prod_{i=1}^m p_i^{-\alpha_i} \prod_{i=m+1}^{\ell} (-p_i)^{-1} \prod_{i=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \langle p_1, \dots, p_{\ell} \rangle .$$

Hence, the vanishing of  $I(g, d, \alpha)$  yields the relation

$$\sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{i=1}^m p_i^{-\alpha_i} \prod_{i=m+1}^{\ell} (-p_i)^{-1} \prod_{i=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \langle p_1, \dots, p_{\ell} \rangle = 0 ,$$

where the sum is over all graphs (5).  $\square$

**Question 1.** *Are the relations of Theorem 3 equivalent to relations constructed in Section 3 of [9] via stable quotients?*

#### 4. RANK ANALYSIS

**4.1. Matrix of relations.** Theorem 3 yields relations in  $\kappa^d(M_{g,n}^c)$ , indexed by  $\alpha = (\alpha_1, \dots, \alpha_m)$  satisfying conditions (i-ii) of Section 3.2.1 with

$$\delta = 2g - 3 + n - d \geq 0.$$

We rewrite the relation obtained from the vanishing of  $I(g, d, \alpha)$  as

$$(6) \quad \sum_{\mathbf{p} \in P(d)} \mathbf{C}_{\alpha}^{\mathbf{p}} \langle \mathbf{p} \rangle = 0 .$$

The coefficients are

$$\mathbf{C}_{\alpha}^{\mathbf{p}} = \frac{1}{|\text{Aut}(\mathbf{p})|} \prod_{i=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \sum_{\phi} \prod_{i=1}^m p_{\phi(i)}^{-\alpha_i} \prod_{j \in \text{Im}(\phi)^c} (-p_j)^{-1} ,$$

where the sum is over all injections

$$\phi : \{1, \dots, m\} \rightarrow \{1, \dots, \ell\}$$

and

$$\text{Im}(\phi)^c \subset \{1, \dots, \ell\}$$

is the complement of the image of  $\phi$ .

<sup>3</sup>The parallel equation on page 106 of [5] has a sign error in the normalization. Instead of  $(-1)^{g+1}I(g, d, \alpha)$  there, the normalization should be  $(-1)^{g+1+|\alpha|+\ell(\alpha)}I(g, d, \alpha)$ . The sign change makes no difference.

To prove Proposition 2, we will show the system (6) is of rank at least  $|P(d)| - |P(d, \delta + 1)|$ . The claim is empty unless  $0 \leq \delta \leq d - 2$ .

**4.2. Ordering.** For  $0 \leq \delta \leq d - 2$ , define the subset  $P_\delta(d) \subset P(d)$  by removing partitions of length at most  $\delta + 1$ ,

$$P_\delta(d) = P(d) \setminus P(d, \delta + 1) .$$

We order  $P_\delta(d)$  by the following rules

- longer partitions appear before shorter partitions,
- for partitions of the same length, we use the lexicographic ordering with larger parts<sup>4</sup> appearing before smaller parts.

For example, the ordered list of the 10 elements of  $P_0(6)$  is

$$(1^6), (2, 1^4), (3, 1^3), (2^2, 1^2), (4, 1^2), (3, 2, 1), (2^3), (5, 1), (4, 2), (3, 3) .$$

Given a partition  $\mathbf{p} \in P(d)$ , let  $\widehat{\mathbf{p}}$  be the partition obtained removing all parts equal to 1. For example,

$$\widehat{(1^6)} = \emptyset, \quad \widehat{(3, 2, 1)} = (3, 2) .$$

Let  $\mathbf{p}^-$  be the partition obtained by lowering all the parts of  $\mathbf{p}$  by 1,

$$(1^6)^- = \emptyset, \quad (3, 2, 1)^- = (2, 1) .$$

If  $\mathbf{p}$  has length  $\ell$ , then

$$\mathbf{p}^- \in P(d - \ell) .$$

To each partition  $\mathbf{p} \in P_\delta(d)$ , we associate data  $\alpha[\mathbf{p}]$  satisfying conditions (i)-(ii) with respect to  $\delta$  by the following rules. The special designation

$$\alpha[(1^d)] = (0)$$

is given. Otherwise

$$\alpha[\mathbf{p}] = \mathbf{p}^- .$$

We note condition (i) of Section 3.2.1,

$$|\alpha[\mathbf{p}]| \leq d - 2 - \delta ,$$

is satisfied in all cases.

Let  $\mathbf{M}_\delta(d)$  be the square matrix indexed by the ordered set  $P_\delta(d)$  with elements

$$M_\delta(d)[\mathbf{p}, \mathbf{q}] = \mathbf{C}_{\alpha[\mathbf{p}]}^{\mathbf{q}} .$$

---

<sup>4</sup>Remember the parts of  $\mathbf{p} = (p_1, \dots, p_\ell)$  are ordered by  $p_1 \geq \dots \geq p_\ell$ .

The rank of the system (6) is at least

$$|P_\delta(d)| = |P(d)| - |P(d, \delta + 1)|$$

by the following nonsingularity result proven in Sections 4.3 - 4.6 below.

**Proposition 4.** *For  $0 \leq \delta \leq d - 2$ , the matrix  $M_\delta(d)$  is nonsingular.*

Proposition 4 implies Proposition 2 and thus Theorem 1. Moreover, Proposition 4 provides a new approach to [5].

**4.3. Scaling.** Let  $X_\delta(d)$  be the square matrix indexed by the ordered set  $P_\delta(d)$  with elements

$$\begin{aligned} X_\delta(d)[(1)^d, \mathbf{q}] &= (-1)^{\ell(\mathbf{q})-1} d \\ X_\delta(d)[\mathbf{p} \neq (1)^d, \mathbf{q}] &= \sum_{\phi} (-1)^{\ell(\mathbf{q})-\ell(\widehat{\mathbf{p}})} \prod_{i=1}^{\ell(\widehat{\mathbf{p}})} q_{\phi(i)}^{-\widehat{p}_i+2}, \end{aligned}$$

where the sum is over all injections

$$\phi : \{1, \dots, \ell(\widehat{\mathbf{p}})\} \rightarrow \{1, \dots, \ell(\mathbf{q})\}.$$

For example,  $X_0(6)$  is

$$\begin{pmatrix} -6 & 6 & -6 & -6 & 6 & 6 & 6 & -6 & -6 & -6 \\ -6 & 5 & -4 & -4 & 3 & 3 & 3 & -2 & -2 & -2 \\ -6 & \frac{9}{2} & -\frac{10}{3} & -3 & \frac{9}{4} & \frac{11}{6} & \frac{3}{2} & -\frac{6}{5} & -\frac{3}{4} & -\frac{2}{3} \\ 30 & -20 & 12 & 12 & -6 & -6 & -6 & 2 & 2 & 2 \\ -6 & \frac{17}{4} & -\frac{28}{9} & -\frac{5}{2} & \frac{33}{16} & \frac{49}{36} & \frac{3}{4} & -\frac{26}{25} & -\frac{5}{16} & -\frac{2}{9} \\ 30 & -18 & 10 & 9 & -\frac{9}{2} & -\frac{11}{3} & -3 & \frac{6}{5} & \frac{3}{4} & \frac{2}{3} \\ -120 & 60 & -24 & -24 & 6 & 6 & 6 & 0 & 0 & 0 \\ -6 & \frac{33}{8} & -\frac{82}{27} & -\frac{9}{4} & \frac{129}{64} & \frac{251}{216} & \frac{3}{8} & -\frac{126}{125} & -\frac{9}{64} & -\frac{2}{27} \\ 30 & -17 & \frac{28}{3} & \frac{15}{2} & -\frac{33}{8} & -\frac{49}{18} & -\frac{3}{2} & \frac{26}{25} & \frac{5}{16} & \frac{2}{9} \\ 30 & -16 & 8 & \frac{13}{2} & -3 & -2 & -\frac{3}{2} & \frac{2}{5} & \frac{1}{4} & \frac{2}{9} \end{pmatrix}.$$

The matrix  $X_\delta(d)$  is obtained from  $M_\delta(d)$  by dividing each column corresponding to  $\mathbf{q}$  by

$$\frac{1}{|\text{Aut}(\mathbf{q})|} \prod_{i=1}^{\ell(\mathbf{q})} \frac{q_i^{q_i-1}}{q_i!}.$$

Hence,  $X_\delta(d)$  is nonsingular if and only if  $M_\delta(d)$  is nonsingular.

4.4. **Elimination.** Our strategy for proving Proposition 4 is to find an upper-triangular square matrix  $Y_0(d)$  for which the product

$$(7) \quad X_0(d) \cdot Y_0(d)$$

is lower-triangular with  $\pm 1$ 's on the diagonal. Since  $X_\delta(d)$  for

$$0 \leq \delta \leq d - 2$$

occurs as an upper left minor of  $X_0(d)$ , the lower-triangularity of the product (7) will establish Proposition 4 for the full range of  $\delta$  values.

We define  $Y_0(d)$  to be the square matrix indexed by the ordered set  $P_0(d)$  given by the following rules. The upper left corner is

$$Y_0(d)[(1^d), (1^d)] = \frac{1}{d}$$

If at least one of  $\{\mathbf{p}, \mathbf{q}\}$  is not equal to  $(1^d)$ , then the matrix elements are

$$Y_0(d)[\mathbf{p}, \mathbf{q}] = \frac{1}{|\text{Aut}(\mathbf{p})|} \frac{1}{|\text{Aut}(\widehat{\mathbf{q}})|} \sum_{\theta} \prod_{i=1}^{\ell(\mathbf{q})} \binom{q_i}{p_{i[1]}, \dots, p_{i[\ell_i]}} q_i^{\ell_i - 2} \prod_{j=1}^{\ell_i} p_{ij}^{p_{ij} - 1},$$

where the sum is over all functions

$$\theta : \{1, \dots, \ell(\mathbf{p})\} \rightarrow \{1, \dots, \ell(\mathbf{q})\}$$

with

$$\theta^{-1}(i) = \{i[1], \dots, i[\ell_i]\}$$

satisfying

$$q_i = \sum_{j=1}^{\ell_i} p_{i[j]}.$$

For example,  $Y_0(6)$  is

$$\begin{pmatrix} \frac{1}{6} & 1 & 3 & \frac{1}{2} & 16 & 3 & \frac{1}{6} & 125 & 16 & \frac{9}{2} \\ 0 & 1 & 6 & 1 & 48 & 9 & \frac{1}{2} & 500 & 64 & 18 \\ 0 & 0 & 3 & 0 & 36 & 3 & 0 & 450 & 36 & 9 \\ 0 & 0 & 0 & \frac{1}{2} & 12 & 6 & \frac{1}{2} & 300 & 60 & 18 \\ 0 & 0 & 0 & 0 & 16 & 0 & 0 & 320 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 180 & 36 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 125 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \end{pmatrix}.$$

By the conditions on  $\theta$  in the definition,  $Y_0(d)$  is easily seen to be upper-triangular.

**4.5. Generating functions.** Let  $\mathbb{Q}[t]$  denote the polynomial ring in infinitely many variables

$$t = \{t_1, t_2, t_3, \dots\}.$$

Define a  $\mathbb{Q}$ -linear function

$$\langle \rangle : \mathbb{Q}[t] \rightarrow \mathbb{Q}$$

by the equations  $\langle 1 \rangle = 1$  and

$$\langle t_{d_1} t_{d_2} \cdots t_{d_k} \rangle = (d_1 + d_2 + \dots + d_k)^{k-3}.$$

We may extend  $\langle \rangle$  uniquely to define a  $x$ -linear function:

$$\langle \rangle : \mathbb{Q}[t][[x]] \rightarrow \mathbb{Q}[[x]].$$

For each non-negative integer  $i$ , let

$$Z_i(t, x) = \sum_{j>0} x^j t_j \frac{j^{j-i}}{j!} \in \mathbb{Q}[t][[x]].$$

Applying the bracket, we define

$$F_{\alpha_1, \dots, \alpha_m} = \langle \exp(-Z_1) \cdot Z_{\alpha_1} \cdots Z_{\alpha_m} \rangle \in \mathbb{Q}[[x]].$$

**Lemma 3.** *Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  be a non-empty sequence of non-negative integers satisfying  $\alpha_i > 0$  for  $i > 1$ . The series*

$$F_{\alpha_1, \dots, \alpha_m} \in \mathbb{Q}[[x]]$$

*is a polynomial of degree at most  $1 + \sum_{i=1}^m \alpha_i$  in  $x$ .*

**Lemma 4.** *Let  $\alpha_1 \geq 0$ . Then,*

$$F_{\alpha_1} = \frac{(-1)^{\alpha_1}}{(1 + \alpha_1)(1 + \alpha_1)!} x^{1+\alpha_1} + \dots$$

where the dots stand for lower order terms.

Lemma 3 can be proven by various methods. A proof via localization on moduli space is given in [5] in Section 1.7.  $\square$

Lemma 4 is more interesting. The integral

$$(8) \quad J_{1+\alpha_1} = \int_{\overline{M}_{0,1}(\mathbb{P}^1, 1+\alpha_1)} \rho_1 \psi_1^{\alpha_1} c_{top}(\mathbb{R})$$

can be evaluated by exactly following<sup>5</sup> the localization analysis of Section 3.3. We find

$$J_{1+\alpha_1} = (-1)^{\alpha_1} \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} p_1^{-\alpha_1} \prod_{i=2}^{\ell} (-p_i)^{-1} \prod_{i=1}^{\ell} \frac{p_i^{p_i}}{p_i!} (1 + \alpha_1)^{\ell-3}$$

where the sum is over all 1-pointed comb graphs (5) of total degree  $1 + \alpha_1$ . We conclude  $J_{1+\alpha_1}$  equals, up to the factor of  $(-1)^{\alpha_1}$ , the leading  $x^{1+\alpha_1}$  coefficient of  $\langle \exp(-Z_1) \cdot Z_{\alpha_1} \rangle$ .

To calculate the integral (8), we use well-known equations in Gromov-Witten theory. Certainly

$$(9) \quad J_1 = 1 .$$

By two applications of the divisor equation,

$$k^2 J_k = \int_{\overline{M}_{0,3}(\mathbb{P}^1, k)} \rho_1 \psi_1^{k-1} \rho_2 \rho_3 c_{top}(\mathbb{R})$$

By the topological recursion relation [2] applied to the right side,

$$k^2 J_k = \int_{\overline{M}_{0,2}(\mathbb{P}^1, k-1)} \rho_1 \psi_1^{k-2} \rho_2 c_{top}(\mathbb{R}) \cdot \int_{\overline{M}_{0,3}(\mathbb{P}^1, 1)} \rho_1 \rho_2 \rho_3 c_{top}(\mathbb{R}) .$$

We obtain the recursion

$$\begin{aligned} k^2 J_k &= (k-1) J_{k-1} J_1 \\ &= (k-1) J_{k-1} \end{aligned}$$

which we can easily solve

$$J_k = \frac{1}{k \cdot k!}$$

---

<sup>5</sup>The equivariant lifts are taken just as in Section 3.3.2.

starting with the initial condition (9).  $\square$

The case where the  $\alpha$  data is empty will arise naturally. We define

$$F_\emptyset = \langle \exp(-Z_1) \rangle.$$

The following result is derived from Lemma 3 by the relation

$$x \frac{d}{dx} F_\emptyset = -F_\emptyset.$$

**Lemma 5.**  $F_\emptyset = 1 - x$ .

**4.6. Product.** We will now prove the basic identity

$$(10) \quad X_0(d) \cdot Y_0(d) = L_0(d)$$

where  $L_0(d)$  is lower triangular with diagonal entries all  $\pm 1$ .

We first address the special upper left corner. The product on the left side of (10) is

$$L_0(d)[(1^d), (1^d)] = (-1)^{d-1} d \cdot \frac{1}{d} = (-1)^{d-1},$$

a diagonal entry of the specified form.

Next assume  $\mathbf{p} \neq (1^d)$ . Then, the matrix elements are

$$(11) \quad L_0(d)[\mathbf{p}, \mathbf{q}] = \frac{1}{|\text{Aut}(\widehat{\mathbf{q}})|} \sum_{\gamma} \prod_{i=1}^{\ell(\mathbf{q})} \text{Coeff}(F_{\gamma^{-1}(i)}, x^{q_i}) q_i q_i!,$$

where the sum is over all functions

$$\gamma : \{1, \dots, \ell(\widehat{\mathbf{p}})\} \rightarrow \{1, \dots, \ell(\mathbf{q})\}.$$

In case  $\gamma^{-1}(i) = \{i[1], \dots, i[\ell_i]\}$  is nonempty, we define

$$F_{\gamma^{-1}(i)} = F_{\widehat{p}_{i[1]-1}, \dots, \widehat{p}_{i[\ell_i]-1}}.$$

If  $\gamma^{-1}(i) = \emptyset$ , then

$$F_\emptyset = \langle \exp(-Z_1) \rangle = 1 - x.$$

Equation (11) is obtained from a simple unravelling of the definitions.

If  $q_i > 1$ ,  $\text{Coeff}(F_{\gamma^{-1}(i)}, x^{q_i})$  vanishes unless  $\gamma^{-1}(i)$  is nonempty by Lemma 5 and unless

$$(12) \quad q_i \leq 1 - \ell_i + \sum_{j=1}^{\ell_i} \widehat{p}_{i[j]}$$

by Lemma 3. Inequality (12) for all parts  $q_i > 1$  implies

$$\ell(\mathbf{q}) \geq \ell(\mathbf{p}).$$

Moreover, if equality of length holds, then inequality (12) implies either  $\mathbf{q}$  precedes  $\mathbf{p}$  in the ordering of  $P_0(d)$  or  $\mathbf{q} = \mathbf{p}$ .

We conclude the matrix  $L_0(d)$  is lower-triangular when the first coordinate  $\mathbf{p}$  is not  $(1^d)$ . The diagonal elements for  $\mathbf{p} \neq (1^d)$  are

$$L_0(d)[\mathbf{p}, \mathbf{p}] = \prod_{i=1}^{\ell(\widehat{\mathbf{p}})} (-1)^{\widehat{p}_i - 1} \cdot (-1)^{\ell(\mathbf{p}) - \ell(\widehat{\mathbf{p}})}$$

by Lemmas 4 and 5.

To complete the proof of the lower-triangularity of  $L_0(d)$ , we must show the vanishing of  $L_0(d)[(1^d), \mathbf{q} \neq (1^d)]$ . The matrix elements are

$$L_0(d)[(1^d), \mathbf{q} \neq (1^d)] = \frac{1}{|\text{Aut}(\widehat{\mathbf{q}})|} \sum_{\tilde{\gamma}} \prod_{i=1}^{\ell(\mathbf{q})} \text{Coeff}(\tilde{F}_{\tilde{\gamma}^{-1}(i)}, x^{q_i}) q_i q_i! ,$$

where the sum is over all functions

$$\tilde{\gamma} : \{1\} \rightarrow \{1, \dots, \ell(\mathbf{q})\} .$$

In case  $\tilde{\gamma}^{-1}(i) = \{1\}$  is nonempty, we define

$$\tilde{F}_{\tilde{\gamma}^{-1}(i)} = F_0 .$$

If  $\tilde{\gamma}^{-1}(i) = \emptyset$ , then

$$\tilde{F}_\emptyset = \langle \exp(-Z_1) \rangle = 1 - x .$$

Let  $q_1 > 1$  be the largest part of  $\mathbf{q}$ . Then

$$\text{Coeff}(\tilde{F}_{\tilde{\gamma}^{-1}(1)}, x^{q_1}) = 0$$

by Lemmas 3 and 5. Hence,

$$L_0(d)[(1^d), \mathbf{q} \neq (1^d)] = 0,$$

and the lower-triangularity of  $L_0(d)$  is fully proven.

The proof of Proposition 4 is complete. Following the implications back, the proof of Theorem 1 is also complete.  $\square$

Since we know explicitly the diagonal elements of the triangular matrices  $Y_0(d)$  and  $L_0(d)$ , the product

$$X_0(d) \cdot Y_0(d) = L_0(d)$$

yields a simple formula for the determinant,

$$\det(X_{0,d}) = (-1)^{d-1} \prod_{\mathbf{p} \in P_0(d) \setminus \{(1^d)\}} \left( \frac{|\text{Aut}(\widehat{\mathbf{p}})|}{\prod_{i=1}^{\ell(\mathbf{p})} p_i^{p_i-2}} (-1)^{\ell(\mathbf{p})} \prod_{i=1}^{\ell(\widehat{\mathbf{p}})} (-1)^{\widehat{p}_i} \right) .$$

## 5. GORENSTEIN CONJECTURE

5.1. **Proof of Theorem 2.** If  $n > 0$ , the pairing

$$\kappa^d(M_{g,n}^c) \times R^{2g-3+n-d}(M_{g,n}^c) \rightarrow \mathbb{Q}$$

is shown to have rank at least  $|P(d, 2g - 2 + n - d)|$  in [9]. Since

$$\dim_{\mathbb{Q}} \kappa^d(M_{g,n}^c) = |P(d, 2g - 2 + n - d)|$$

by Theorem 1 and [9], Theorem 2 follows.  $\square$

5.2. **Further directions.** Perhaps the universality of Theorem 1 extends to larger subrings of  $R^*(M_{g,n}^c)$ . A natural place to start is the ring

$$S^*(M_{g,n}^c) \subset R^*(M_{g,n}^c)$$

generated by all the  $\kappa$  and  $\psi$  classes.

**Question 2.** *Is  $S^*(M_{g,n}^c)$  canonically a subring of  $S^*(M_{0,2g+n}^c)$  ?*

At least the condition  $n > 0$  must be imposed in Question 2. How to include the strata classes in a universality statement is not clear.

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