# The three-in-a-tree problem

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**Abstract**

We show that there is a polynomial time algorithm that, given three vertices of a graph, tests whether there is an induced subgraph that is a tree, containing the three vertices. (Indeed, there is an explicit construction of the cases when there is no such tree.) As a consequence, we show that there is a polynomial time algorithm to test whether a graph contains a "theta" as an induced subgraph (this was an open question of interest) and an alternative way to test whether a graph contains a "pyramid" (a fundamental step in checking whether a graph is perfect).

# 1 Introduction

All graphs in this paper are finite and simple. If $G$ is a graph, its vertex- and edge-sets are denoted $V(G), E(G)$. If $X \subseteq V(G)$, the subgraph with vertex set $X$ and edge set all edges of $G$ with both ends in $X$ is denoted $G|X$, and called the subgraph *induced* on $X$.

There are many algorithmic questions of interest concerning the existence of an induced subgraph of some specific type containing some specified vertices, but almost all of them seem to be NP-complete, by virtue of the following result of Bienstock [1]:

**1.1** *The following problem is NP-complete:*
**Input:** *A graph $G$ and two edges $e, f$ of $G$.*
**Question:** *Is there a subset $X \subseteq V(G)$ such that $G|X$ is a cycle containing $e, f$?*

Bienstock's result leaves very little room between the trivial problems and the NP-complete problems, but in this paper we report on a problem that falls into the gap. We call the following the "three-in-a-tree" problem:

**Input:** A graph $G$, and three vertices $v_1, v_2, v_3$ of $G$.
**Question:** Does there exist $X \subseteq V(G)$ with $v_1, v_2, v_3 \in X$ such that $G|X$ is a tree?

For most graphs one would expect a "yes" answer, but there are interesting graphs for which the answer is "no"; for instance, if $e_1, e_2, e_3$ are edges of a graph $H$ each incident with a vertex of degree one, and $G$ is the line graph of $H$, then $e_1, e_2, e_3$ are vertices of $G$ and there is no induced tree in $G$ containing $e_1, e_2, e_3$. Nevertheless, we will show that the three-in-a-tree problem can be solved in time $O(|V(G)|^4)$. We shall give an explicit construction of all instances $(G, v_1, v_2, v_3)$ such that the desired tree does not exist, and the proof that all such instances must fall under this construction can be converted to an algorithm to check whether the desired tree exists or not.

# 2 Thetas, pyramids and prisms

We were led to the three-in-a-tree problem while working on the question of deciding if a graph contains a theta, so let us describe that. First we need some definitions. If $G, H$ are graphs, and $H$ is isomorphic to $G|X$ for some $X \subseteq V(G)$, we say that $G$ contains $H$ as an induced subgraph. A *path* is a graph $P$ whose vertex set and edge set can be labeled as $V(P) = \{v_1, \dots, v_k\}$ and $E(P) = \{e_1, \dots, e_{k-1}\}$ for some $k \geq 1$, such that $e_i$ is incident with $v_i, v_{i+1}$ for $1 \leq i \leq k-1$. A *cycle* is a graph $C$ with $V(C) = \{v_1, \dots, v_k\}$ and $E(C) = \{e_1, \dots, e_k\}$ for some $k \geq 3$, such that $e_i$ is incident with $v_i, v_{i+1}$ for $1 \leq i \leq k-1$, and $e_k$ is incident with $v_1, v_k$. The *length* of a path or cycle is the number of edges in it, and a path or cycle if *odd* or *even* if its length is odd or even respectively. A path or cycle *of* $G$ means a subgraph (not necessarily induced) of $G$ that is a path or cycle. A *hole* of $G$ means a cycle in $G$ that is an induced subgraph and has length at least four. A *triangle* is a set of three pairwise adjacent vertices.

Here are three types of graph that will be important to us:

- A *pyramid* is a graph consisting of a vertex $a$ and a triangle $\{b_1, b_2, b_3\}$, and three paths $P_1, P_2, P_3$, such that: $P_i$ is between $a$ and $b_i$ for $i = 1, 2, 3$; for $1 \leq i < j \leq 3$ $P_i, P_j$ are vertex-disjoint except for $a$ and the subgraph induced on $V(P_i) \cup V(P_j)$ is a cycle; and at most one of $P_1, P_2, P_3$ has only one edge.

- A *theta* is a graph consisting of two nonadjacent vertices $a, b$ and three paths $P_1, P_2, P_3$, each joining $a, b$ and otherwise vertex-disjoint, such that for $1 \leq i < j \leq 3$ the subgraph induced on $V(P_i) \cup V(P_j)$ is a cycle.
- A *prism* is a graph consisting of two vertex-disjoint triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, and three paths $P_1, P_2, P_3$, pairwise vertex-disjoint, such that for $1 \leq i < j \leq 3$ the subgraph induced on $V(P_i) \cup V(P_j)$ is a cycle.

Perhaps the main reason for interest in pyramids, thetas and prisms is that every graph containing a pyramid as an induced subgraph has an odd hole, and every graph containing a theta or prism has an even hole; and there seems to be some parallel between pyramids and "thetas or prisms". Yet although pyramids, thetas and prisms are superficially similar, there is a real difference in the difficulty of detecting their presence. We showed in [3] that

**2.1** *There is an algorithm with running time $O(|V(G)|^9)$, that, with input a graph $G$, tests whether $G$ contains a pyramid as an induced subgraph.*

Motivated by the parallel between pyramids and thetas-or-prisms, Chudnovsky and Kapadia [2, 4] proved the following:

**2.2** *There is a polynomial-time algorithm to test whether a graph $G$ contains either a theta or a prism as an induced subgraph.*

In contrast, Maffray and Trotignon[5] showed that

**2.3** *It is NP-complete to test whether a graph contains a prism as an induced subgraph.*

There are two useful applications of the three-in-a-tree algorithm here. First, until now the complexity of testing whether $G$ contains a theta has been open; but it can be solved in polynomial time as follows.

**2.4 Algorithm to test if $G$ contains a theta as an induced subgraph.**

There are two kinds of thetas, those with a cycle of length four and those with no such cycle. To test for thetas of the first kind, enumerate all induced cycles of length four, say $a_1$-$a_2$-$a_3$-$a_4$-$a_1$; then for each such choice delete $a_1, a_3$ and all their neighbours except $a_2, a_4$, and test whether there remains a path between $a_2, a_4$. It is easy to see that this works. To test for thetas of the second kind, enumerate all seven-tuples $(a, b_1, b_2, b_3, c_1, c_2, c_3)$ of distinct vertices such that $a$ is adjacent to $b_1, b_2, b_3$ and not to $c_1, c_2, c_3$, and for $1 \leq i, j \leq 3$, $b_i$ is adjacent to $c_j$ if and only if $i = j$. For each such seven-tuple, test if there is an induced tree in $G \setminus \{a, b_1, b_2, b_3\}$ that contains $c_1, c_2, c_3$. Again, it is easy to see that $G$ contains a theta in which all cycles have length at least five if and only if the induced tree exists for some choice of the seven-tuple. ■

Second, the algorithm of 2.1 depended heavily on some fortuitous properties of the smallest pyramid in a graph, and this was a little disturbing because testing for a pyramid was a crucial step in our algorithm to test whether a graph is perfect. The three-in-a-tree algorithm can be used to give another, less miraculous, way to test for pyramids, as follows.

**2.5 Algorithm to test if $G$ contains a pyramid as an induced subgraph.**

Enumerate all six-tuples $(a_1, a_2, a_3, b_1, b_2, b_3)$ of distinct vertices such that $\{b_1, b_2, b_3\}$ is a triangle, and $a_i$ is adjacent to $b_j$ if and only if $i = j$ (for $1 \leq i, j \leq 3$). For each such six-tuple, test whether there is an induced tree in $G \setminus \{b_1, b_2, b_3\}$ that contains all of $a_1, a_2, a_3$. It is easy to see that $G$ contains a pyramid if and only if for some six-tuple there is an induced tree as described. ■

# 3 Strip structures

Let us say that $Z \subseteq V(G)$ is *constricted* if $|Z \cap V(T)| \leq 2$ for every induced tree $T$ of $G$. We wish to study which three-vertex subsets of $V(G)$ are constricted; but we might as well study which sets are constricted in general, because that question is no more difficult, and does seem to be strictly more general. We will prove that if $|Z| \geq 2$, then $Z$ is constricted if and only if $G$ admits a certain decomposition with respect to $Z$, that we call an "extended strip decomposition". Our next goal is to define this.

Thus, let $G$ be a graph and $Z \subseteq V(G)$. Let $H$ also be a graph, let $W$ be the set of vertices of $H$ that have degree one in $H$, and let $\eta$ be a map satisfying the following conditions:

- for each edge $e \in V(G)$, $\eta(e) \subseteq V(G)$, and for each $v \in V(H)$ incident with $e$, $\eta(e, v) \subseteq \eta(e)$
- $\eta(e) \cap \eta(f) = \emptyset$ for all distinct $e, f \in E(H)$
- for all distinct $e, f \in E(H)$, let $x \in \eta(e)$ and $y \in \eta(f)$; then $x, y$ are adjacent in $G$ if and only if $e, f$ share an end-vertex $v$ in $H$, and $x \in \eta(e, v)$ and $y \in \eta(f, v)$
- $|Z| = |W|$, and for each $z \in Z$ there is a vertex $v \in W$ incident in $H$ with only one edge $e$, and $\eta(e, v) = \{z\}$.

Let us call such a map $\eta$ an *$H$-strip structure* in $(G, Z)$.

Let $\eta$ be an $H$-strip structure, and let us extend the domain of $\eta$ as follows. For each vertex $v \in V(H)$, let $\eta(v) \subseteq V(G)$, and for each triangle $D$ of $H$ let $\eta(D) \subseteq V(G)$, satisfying the following:

- all the sets $\eta(e)$ $(e \in E(H))$, $\eta(v)$ $(v \in V(H))$ and $\eta(D)$ (for all triangles $D$ of $H$) are pairwise disjoint, and their union is $V(G)$
- for each $v \in V(H)$, if $x \in \eta(v)$ and $y \in V(G) \setminus \eta(v)$ are adjacent in $G$ then $y \in \eta(e, v)$ for some $e \in E(H)$ incident with $H$ with $v$
- for each triangle $D$ of $H$, if $x \in \eta(D)$ and $y \in V(G) \setminus \eta(D)$ are adjacent in $G$ then $y \in \eta(e, u) \cap \eta(e, v)$ for some distinct $u, v \in D$, where $e$ is the edge $uv$ of $H$.

In this case we say that $\eta$ is an *extended $H$-strip decomposition* of $(G, Z)$. Our main theorem asserts the following:

**3.1** *Let $G$ be a graph and let $Z \subseteq V(G)$ with $|Z| \geq 2$. Then $Z$ is constricted if and only if for some graph $H$, $(G, Z)$ admits an extended $H$-strip decomposition.*

**Proof of the "if" half of 3.1.**

Suppose that $Z$ is not constricted, and yet $(G, Z)$ admits an extended $H$-strip-decomposition $\eta$. Choose an induced tree $T$ in $G$ with $|V(T) \cap Z| \geq 3$, with $V(T)$ minimal. It follows that every vertex of $T$ that has degree one in $T$ belongs to $Z$ (for otherwise it could be deleted from $T$); and since there is such a vertex, and it cannot be deleted from $T$, it follows that $|V(T) \cap Z| = 3$. Now either $T$ is an induced path with both end-vertices in $Z$ (in this case there is a unique vertex of $Z \cap V(T)$ that is an internal vertex of $P$, say $y$) or the three members of $Z \cap V(T)$ all have degree one in $T$, and $T$ has a unique vertex $y$ of degree three (and possibly some vertices of degree two). If $z \in Z \cap V(T) \setminus \{y\}$, a path $P$ of $T$ is said to be a *$z$-limb* if $y \notin V(P)$ and $z \in V(P)$ (and consequently $z$ is an end-vertex of $P$, and no other vertex of $P$ belongs to $Z$).

For each $v \in V(H)$, let $N(v)$ denote the union of all the sets $\eta(e, v)$, as $e$ ranges over all edges of $H$ incident with $v$.

(1) *For each $v \in V(H)$, there do not exist distinct $z_1, z_2 \in Z \cap V(T) \setminus \{y\}$ such that some $z_1$-limb contains a vertex in $N(v)$ and some $z_2$-limb contains a vertex in $N(v)$.*

For suppose this is false; then for $i = 1, 2$ we may choose a $z_i$-limb $P_i$, with end-vertices $z_i, y_i$ say, and $v \in V(H)$, such that $y_1, y_2 \in N(v)$. Choose $Q_1, Q_2, v$ so that the sum of the lengths of $Q_1, Q_2$ is minimum. It follows that for $i = 1, 2$, no vertex of $Q_i$ belongs to $N(v)$ except $y_i$. Since $y_1$ is not adjacent to $y_2$ (from the definition of $z_1$-, $z_2$-limb) it follows that for some $e \in E(H)$, $y_1, y_2 \in \eta(e, v)$. Let $e$ be incident with $v, u \in V(H)$ say. Since $|\eta(e, v)| > 1$, it follows that not both $z_1, z_2 \in \eta(e)$, so we may assume that $z_1 \notin \eta(e)$. Let $R_1$ be a maximal $z_1$-limb such that $R_1$ is a subpath of $Q_1$ and no vertex of $R_1$ belongs to $\eta(e)$. Let $R_1$ have ends $r_1, z_1$ say; and let $s_1$ be the neighbour of $r_1$ in $Q_1$ that does not belong to $R_1$. Thus $s_1 \in \eta(e)$. If $z_2 \notin \eta(e)$ define $R_2, r_2, s_2$ similarly. Since $r_1 \notin \eta(e)$, it follows that either $r_1 \in \eta(D)$ for some triangle $D$ of $H$, or $r_1 \in \eta(w)$ for some $w \in V(H)$, or $r_1 \in \eta(f)$ for some $f \in E(H) \setminus \{e\}$.

Suppose that the first holds, that is, $r_1 \in \eta(D)$ for some triangle $D$ of $H$. Since $s_1 \in \eta(e)$ and $r_1, s_1$ are adjacent, it follows that $u, v \in D$, and $s_1 \in \eta(e, u) \cap \eta(e, v)$. Let $D = \{u, v, w\}$ say. Since $z_1 \notin \eta(D)$, there is an edge $ab$ of $R$ such that $a \in \eta(D)$ and $b \notin \eta(D)$. Consequently $b$ belongs to one of

$$\eta(uv, u) \cap \eta(uv, v), \eta(vw, v) \cap \eta(vw, w), \eta(uw, u) \cap \eta(uw, w).$$

But from the choice of $R_1$, $b \notin \eta(e)$, and from the choice of $P_1$, $b \notin N(v)$. Hence $b \in \eta(uw, u) \cap \eta(uw, w)$; but then $s_1 \in \eta(uv, u)$ and $b \in \eta(uw, u)$, and so $s_1, b$ are adjacent, contrary to the minimality of $T$. Thus the first case cannot occur.

Suppose that the second holds, and $r_1 \in \eta(w)$ for some $w \in V(H)$. Since $r_1, s_1$ are adjacent, it follows that $s_1 \in N(w)$, and so $s_1 \in \eta(f, w)$ for some edge $f$ incident with $w$; and since $s_1 \in \eta(e)$, it follows that $f = e$, so $w$ is one of $u, v$. Moreover, since $z_1 \notin \eta(w)$, there is an edge $ab$ of $R_1$ such that $a \in \eta(w)$ and $b \notin \eta(w)$, and therefore $b \in N(w)$. Since $b \notin \eta(v)$ it follows that $w \neq v$, and so $w = u$, and hence $s_1 \in \eta(e, u)$. Moreover from the minimality of $T$, $b$ and $s_1$ are not adjacent. Since $s_1, b \in N(u)$, it follows that $b \in \eta(e, u)$, contrary to the choice of $R_1$. Thus the second case cannot hold.

We deduce that the third holds, and so $r_1 \in \eta(f)$ for some $f \in E(H) \setminus \{e\}$. Since $r_1, s_1$ are adjacent, it follows that there is a vertex $w$ of $H$ incident with $e, f$ such that $s_1 \in \eta(e, w)$ and $r_1 \in \eta(f, w)$. Since $r_1 \notin N(v)$, it follows that $w = u$, and so $s_1 \in \eta(e, u)$.

In particular, $\eta(e,u) \neq \{z_2\}$, and since $\eta(e,v) \neq \{z_2\}$, it follows that $z_2 \notin \eta(e)$, and so $R_2, r_2, s_2$ are defined. By the same argument with $z_1, z_2$ exchanged, it follows that $s_2 \in \eta(e,u)$. Since $r_1 \in \eta(f,u)$ for some $f \in E(H) \setminus \{e\}$, and $s_2 \in \eta(e,u)$, it follows that $r_1, s_2$ are adjacent, a contradiction. This proves (1).

Let $Z \cap V(T_i) = \{z_1, z_2, z_3\}$.

(2) *$y \in \eta(e)$ for some $e \in E(H)$.*

For either $y \in \eta(D)$ for some triangle $D$ of $H$, or $y \in \eta(v)$ for some $v \in V(H)$, or $y \in \eta(e)$ for some $e \in E(H)$. Suppose that the first holds. Then for $i = 1,2,3$, $z_i \notin \eta(D)$; let $R_i$ be a maximal $z_i$-limb containing no vertex of $\eta(D)$, with ends $z_i, r_i$ say, and $r_i$ has a neighbour in $\eta(D)$. Let $D = \{u,v,w\}$; then for $i = 1,2,3$, $r_i$ belongs to at least two of $N(u), N(v), N(w)$. Hence one of $N(u), N(v), N(w)$ meets both of $R_1, R_2$, contrary to (1). Next suppose the second holds, and $y \in \eta(v)$ for some $v \in V(H)$. Again, let $Z \cap V(T_i) = \{z_1, z_2, z_3\}$; for $i = 1,2,3$, since $z_i \notin \eta(v)$, there is a maximal $z_i$-limb $R_i$ with no vertex in $\eta(v)$, with ends $z_i, r_i$ say, and $r_i$ has a neighbour in $\eta(v)$. Hence $r_1, r_2, r_3$ all belong to $N(v)$, contrary to (1). Thus the third holds, that is, $y \in \eta(e)$ for some $e \in E(H)$. This proves (2).

For $i = 1,2,3$, if $z_i \notin \eta(e)$, let $R_i$ be a maximal $z_i$-limb containing no vertex of $\eta(e)$, with ends $z_i, r_i$ say, and let $r_i$ be adjacent in $T$ to $s_i \in \eta(e)$. Let $e$ be incident in $H$ with $u, v \in V(H)$.

(3) *For $i = 1,2,3$, if $z_i \notin \eta(e)$, then there exists $w \in \{u,v\}$ and an edge $f \in E(H) \setminus \{e\}$ incident with $w$ such that $r_i \in \eta(f,w)$ and $s_i \in \eta(e,w)$.*

For either $r_i \in \eta(D)$ for some triangle $D$ of $H$, or $r_i \in \eta(w)$ for some $w \in V(H)$, or $r_i \in \eta(f)$ for some $f \in E(H) \setminus \{e\}$. Suppose that the first holds, and $r_i \in \eta(D)$. Since $s_i \notin \eta(D)$, and $r_i, s_i$ are adjacent, and $s_i \in \eta(e)$, it follows that $u, v \in D$, and $s_i \in \eta(e,u) \cap \eta(e,v)$. Let $D = \{u,v,w\}$ say. Since $z_i \notin \eta(D)$, there is an edge $ab$ of $R_i$ with $a \in \eta(D)$ and $b \notin \eta(D)$. Consequently $b$ belongs to one of

$$\eta(uv,u) \cap \eta(uv,v), \eta(vw,v) \cap \eta(vw,w), \eta(uw,u) \cap \eta(uw,w).$$

Since $b, s_i$ are nonadjacent, it follows that $b \notin \eta(uw,u) \cup \eta(vw,v)$, and since $b \notin \eta(e)$, this is impossible. Thus the first case cannot occur.

Suppose that the second holds, and $r_i \in \eta(w)$ for some $w \in V(H)$. Since $r_i, s_i$ are adjacent, and $s_i \in \eta(e)$, it follows that $w \in \{u,v\}$ and $s_i \in \eta(e,w)$; and we may assume that $w = v$. Since $z_i \notin \eta(v)$, there is an edge $ab$ of $R_i$ with $a \in \eta(v)$ and $b \notin \eta(v)$. Hence $b \in N(v)$, and so $b \in \eta(f,v)$ for some edge $f \in E(H)$ incident with $v$. Since $s_i, b$ are nonadjacent, it follows that $f = e$, contrary to the definition of $R_i$. Thus the second case cannot hold. We deduce that the third holds, and $r_i \in \eta(f)$ for some $f \in E(H) \setminus \{e\}$. Since $r_i, s_i$ are adjacent, and $s_i \in \eta(e)$, there exists $w \in V(H)$ incident with both $e, f$, such that $r_i \in \eta(f,w)$ and $s_i \in \eta(e,w)$. Hence $w \in \{u,v\}$. This proves (3).

Now if none of $z_1, z_2, z_3$ belong to $\eta(e)$, then by (3), we may assume that $r_i \in \eta(f,u)$ and $s_i \in \eta(e,u)$ for $i = 1,2$, contrary to (1). Thus we may assume that $z_3 \in \eta(e)$, and therefore we may assume that $N(u) = \eta(e,u) = \{z_3\}$. If $z_1, z_2 \notin \eta(e)$, then for $i = 1,2$, $r_i \notin N(u)$, and so $r_i \in N(v)$ by (3), contrary to (1). Consequently one of $z_1, z_2 \in \eta(e)$, and so we may assume that

$N(v) = \eta(e,v) = \{z_2\}$. But then $z_1 \notin \eta(e)$, and yet $r_1 \notin N(u) \cup N(v)$, contrary to (3). This completes the proof. ■

## 4 A lemma

Before we start on the proof of the "only if" half of 3.1, we will prove several lemmas. If $G'$ is a subgraph of a graph $G$, and $C$ is a subgraph of $G \setminus V(G')$, and $v \in V(G')$ has a neighbour in $V(C)$, we say that $v$ is an *attachment* of $C$. A *separation* of a graph $K$ is a pair $(A,B)$ of subsets of $V(K)$ with union $V(K)$, such that no edge joins a vertex in $A \setminus B$ and a vertex in $B \setminus A$. We call $|A \cap B|$ the *order* of the separation. Let $W \subseteq V(K)$. We say that $(K,W)$ is a *frame* if

- every vertex in $W$ has degree one in $K$
- $|W| \geq 3$
- $K$ is connected
- for every separation $(A,B)$ of $K$ of order at most two with $W \subseteq B \neq V(K)$, we have that $|A \cap B| = 2$ and $K|A$ is a path between the two members of $A \cap B$.

If $(K,W)$ is a frame, we see that $W$ is the set of all vertices of $K$ that have degree one. A *branch* of $(K,W)$ is a path of $K$ with distinct ends, such that both its ends have degree in $K$ different from two, and all its internal vertices have degree two in $K$. Since $(K,W)$ is a frame, it follows that every branch is an induced subgraph of $K$, and every edge of $K$ belongs to a unique branch. If $v \in V(K)$, $\delta_K(v)$ or $\delta(v)$ denotes the set of all edges of $K$ incident with $v$.

Let $(K,W)$ be a frame and let $F \subseteq E(K)$. An *$F$-line* is a path $P$ in $K$ such that one end of $P$ belongs to $W$ and some edge of $P$ belongs to $F$. A *double $F$-line* is a path $P$ such that both ends of $P$ belong to $W$ and exactly one edge of $P$ belongs to $F$. We say that $F \subseteq E(K)$ is *focussed* if

- there do not exist three $F$-lines that are pairwise vertex-disjoint, and
- there do not exist an $F$-line and a double $F$-line that are vertex-disjoint.

We need to study which subsets $F$ are focussed. We shall prove the following:

**4.1** *Let $(K,W)$ be a frame and let $F \subseteq E(K)$ be focussed. Then either:*

1. *there exists $x \in V(K)$ with $F \subseteq \delta(x)$, or*
2. *$|F| = 3$ and the three edges of $F$ form a triangle, or*
3. *there exist $x, y \in V(K)$, not in the same branch of $K$, such that $F = \delta(x) \cup \delta(y)$, or*
4. *there is a branch $B$ of $(K,W)$ with $F \subseteq E(B)$, or*
5. *there is a branch $B$ of $(K,W)$ with ends $x, y$ such that $x \notin W$ and $F \setminus E(B) = \delta(x) \setminus E(B)$ and $F \cap E(B) \not\subseteq \delta(x)$, or*
6. *there is a branch $B$ of $(K,W)$ with ends $x, y$ such that $x, y \notin W$ and $F \setminus E(B) = (\delta(x) \cup \delta(y)) \setminus E(B)$.*

**Proof.** We proceed by induction on $|E(K)|$. We may assume (for a contradiction) that:

(1) *For every $X \subseteq V(K)$ with $|X| \leq 2$, there is an $F$-line with no vertex in $X$.*

For suppose that $X \subseteq V(K)$ with $|X| \leq 2$, and every $F$-line has a vertex in $X$. Choose such a set $X$ with $|X|$ minimum. Since all vertices in $W$ have degree one and their neighbours are not in $W$, we may also choose $X$ with $X \cap W = \emptyset$. If $|X| = 2$ and the two members of $X$ belong to the same branch, let $B_0$ be the path in this branch between the two members of $X$, and otherwise let $B_0$ be the subgraph with vertex set $X$ and no edges. Suppose that there exists $f \in F \setminus E(B_0)$ not incident with any member of $X$. There is no path between $f$ and $W$ in $K \setminus X$, from the property of $X$, and so there is a separation $(A, B)$ of $K$ with $A \cap B = X$ and $W \subseteq B$ such that both ends of $f$ belong to $A$. Since $f$ has no end in $X$, it follows that $B \neq V(K)$, and so $|X| = 2$ and $K|A$ is a path between the two members of $X$; but then $B_0$ exists and contains $f$, a contradiction. This proves that every member of $F$ either belongs to $E(B_0)$ or is incident with a member of $X$. If $|X| \leq 1$ then the first outcome holds, so we may assume that $X = \{x_1, x_2\}$ say. For $i = 1, 2$, let $V_i$ be the set of all neighbours of $x_i$ that are not in $V(B_0)$, let $Y_i$ be the set of all $v \in V_i$ such that the edge $yx_i$ belongs to $F$, and let $Z_i = V_i \setminus Y_i$. If one of $Y_1, Z_1$ is empty, and one of $Y_2, Z_2$ is empty, then one of the outcomes of the theorem holds; so we may assume that $Y_1, Z_1$ are both nonempty. Let $J = K \setminus V(B_0)$. Since $X \cap W = \emptyset$, it follows that $W \subseteq V(J)$. If $G$ is a graph and $X, Y \subseteq V(G)$, we denote by $\kappa(G, X, Y)$ the maximum $k$ such that there are $k$ vertex-disjoint paths in $G$ between $X$ and $Y$. Since $(K, W)$ is a frame, it follows that

- $\kappa(J, Y_1, W) \geq 1$, for otherwise there would be a separation $(A, B)$ of $K$ with $A \cap B = X$ and $V(B_0) \subseteq A$ and $Y_1 \subseteq A$, which is impossible since $(K, W)$ is a frame and no branch of $(K, W)$ includes $V(B_0) \cup Y_1$, since $Z_1 \neq \emptyset$
- $\kappa(J, Y_1, W) \geq 1$, similarly
- $\kappa(J, V_2, W) \geq 1$; indeed, $\kappa(J, V_2, W) = \kappa(K \setminus \{x_1\}, V_2, W)$, and therefore is at least two unless $|W_2| = 1$
- $\kappa(J, Y_1 \cup Z_1, W) \geq 2$, similarly
- $\kappa(J, Y_1 \cup V_2, W), \kappa(J, Z_1 \cup V_2, W) \geq 2$, since no branch of $(K, W)$ includes both $V(B_0)$ and one of $Y_1, Z_1$
- $\kappa(J, Y_1 \cup Z_1 \cup V_2, W) \geq 3$, since $\kappa(J, Y_1 \cup Z_1 \cup V_2, W) = \kappa(K, V_1 \cup V_2, W)$, and the latter is at least three since no branch of $K$ includes $x_1, x_2$ are all their neighbours.

From this and Menger's theorem (applied to the graph obtained from $J$ by adding three new vertices with neighbour sets $Y_1, Z_1, V_2$ respectively, and asking for three vertex-disjoint paths between the three new vertices and $W$), we deduce that there are three vertex-disjoint paths $P_1, P_2, P_3$ of $J$, from some $y_1 \in Y_1$ to $w_1 \in W$, from $z_1 \in Z_1$ to $w_2 \in W$, and from $v_2 \in V_2$ to $w_3 \in W$ respectively. The path $w_1$-$P_1$-$y_1$-$x_1$-$y_2$-$P_2$-$w_2$ (with the obvious notation) is a double $F$-line $Q$ say. Hence the path $x_2$-$v_2$-$P_3$-$w_3$, and so $v_2 \in Z_2$. For $y_2 \in Y_2$, the path $y_2$-$x_2$-$v_2$-$P_3$-$w_3$ is an $F$-line, and therefore is not disjoint from $Q$; and so $Y_2 \subseteq V(Q)$; and by a similar argument, every edge of $B_0$ in $F$ is incident with $x_1$. Thus if $Y_2 = \emptyset$ then the first outcome of the theorem holds, so we assume that

$Y_2 \neq \emptyset$. Let $y_2 \in Y_2$; then we have seen that $y_2 \in V(P_1) \cup V(P_2)$. If $y_2 \in V(P_2)$, then the path $w_3$-$P_3$-$v_2$-$x_2$-$y_2$-$P_2$-$w_2$ is a double $F$-line (here the notation $y_2$-$P_2$-$w_2$ at the end of this sequence of concatenations means that we take the subpath of $P_2$ between $y_2$ and $w_2$; we will use this and similar notation repeatedly without further explanation); and this double $F$-line is vertex-disjoint from the $F$-line $x_1$-$y_1$-$P_1$-$w_1$, a contradiction. Thus $y_2 \in V(P_1)$. If $y_2 \neq y_1$, then $w_3$-$P_3$-$v_2$-$x_2$-$y_2$-$P_1$-$w_1$ is a double $F$-line vertex-disjoint from the $F$-line $y_1$-$x_1$-$z_1$-$P_2$-$w_2$, a contradiction; and so $y_2 = y_1$. This proves that $Y_2 = \{y_1\}$. Since $Y_2, Z_2 \neq \emptyset$, this restores the symmetry between $x_1, x_2$, and so it follows that $Y_1 = \{y_1\}$, and every edge of $B_0$ in $F$ is incident with $x_2$ (as well as with $x_1$). If no edge of $B_0$ belongs to $F$, then every edge in $F$ is incident with $y_1$, contrary to the minimality of $X$; so $F \cap E(B_0) \neq \emptyset$, and therefore $B_0$ is a path of length one, and the second outcome holds. This proves (1).

Henceforth, therefore, we make the assumption of (1), and obtain a contradiction.

(2) *$K$ is not a tree.*

For suppose $K$ is a tree. It follows that for any set $\mathcal{C}$ of trees of $K$, either there are $k$ members of $\mathcal{C}$ pairwise vertex-disjoint, or there is a set $X \subseteq V(K)$ with $|X| < k$ meeting every member of $\mathcal{C}$. Since there do not exist three pairwise disjoint $F$-lines, we deduce (by taking $\mathcal{C}$ to be the set of all $F$-lines, and $k = 3$) that there exists $X \subseteq V(K)$ with $|X| \leq 2$, such that every $F$-line contains a member of $X$. But this contradicts (1), and so proves (2).

(3) *There is a branch $B$ of $(K, W)$ such that, if $K'$ denotes the graph obtained from $K$ by deleting the edges and internal vertices of $B$, then $(K', W)$ is a frame.*

For let $T$ be a minimal connected subgraph of $K$ with $W \subseteq V(T)$; then $T$ is a tree, and it is easy to see that $(T, W)$ is a frame. Since $K$ is not a tree, it follows that $T \neq K$, and so there is a frame $(K', W)$ with $K' \neq K$. Choose such a frame with as many branches as possible. Suppose that there exist $u, v \in V(K')$ that are are joined by a path $P$ of $K$ such that no edges or internal vertices of $P$ belong to $K'$, such that $u, v$ are not in the same branch of $(K', W)$. It follows that $(K' \cup P, W)$ is a frame, and from our choice of $(K', W)$, we deduce that $K' \cup P = K$. But then $P$ is a branch of $(K, W)$, and (2) holds taking $B = P$. We may therefore assume that for every two vertices $u, v$ that are joined by a path with no edges or internal vertices in $K'$, some branch of $(K', W)$ contains $u, v$. In particular every edge of $K$ that does not belong to $E(K')$ but has both ends in $V(K')$ is between two vertices in the same branch. We have seen that every two attachments of $C$ belong to the same branch. If for every such $C$ there is a branch of $(K', W)$ that contains every attachment of $C$, this contradicts that $(K, W)$ is a frame. Thus there is a component $C$ of $K \setminus V(K')$ such that no branch of $(K', W)$ contains all attachments of $C$. In particular $C$ has at least two attachments, and every two of them belong to a branch; let $B_1$ be a branch, with ends $x_2, x_3$, containing at least two attachments $v_1, v_2$ of $C$. Let $v_3$ be an attachment of $C$ that is not in $B_1$. Since some branch $B_2$ contains $v_1, v_3$, it follows that $v_3$ is one of $x_2, x_3$, say $v_3 = x_3$, and $B_2$ has ends $x_1, x_3$ say. Similarly there is a branch $B_3$ containing $v_1, v_2$; so $v_2 = x_2$, and $v_1$ is a common end of $B_2, B_3$. Since $v_1, v_2, v_3$ are attachments of $C$, we may choose a vertex $c$ of $C$ and three paths $P_1, P_2, P_3$ from $c$ to $v_1, v_2, v_3$ respectively, pairwise vertex-disjoint except for $c$, such that $V(P_i) \subseteq V(C) \cup \{v_i\}$. But then if $K''$ denotes the graph obtained from $K'$ by deleting the edges and internal vertices of $B_1$, and adding

$P_1 \cup P_2 \cup P_3$, then $(K'', W)$ is a frame, and $K'' \neq K$, since the edges of $P_1$ do not belong to $K''$, contrary to the choice of $K'$. This proves (3).

Henceforth, let $B, K'$ be as in (3). Let the ends of $B$ be $x_1, x_2$. Let $F' = F \cap E(K')$.

(4) *There does not exist $v \in V(K')$ such that $F' \subseteq \delta_{K'}(v)$.*

For suppose that $v$ has this property. Let $Y$ be the set of neighbours $y$ of $v$ such that the edge $vy$ belongs to $F$. By (1) there is an $F$-line disjoint from $\{x_1, x_2\}$, and in particular, $v \notin V(B)$, and $Y \not\subseteq V(B)$. Since $(K, W)$ is a frame, there are three paths $P_1, P_2, P_3$ from $x_1, x_2, v$ respectively to $W$, pairwise vertex-disjoint. For $i = 1, 2, 3$ let $w_i$ be the end of $P_i$ in $W$. By (1), no vertex of $B$ meets every edge in $E(B) \cap F$ (since otherwise this vertex together with $v$ would meet every edge in $F$), and so there are two disjoint edges in $E(B) \cap F$, and therefore there are two vertex-disjoint $F$-lines $Q_1, Q_2$, both subpaths of $P_1 \cup B \cup P_2$. Hence $P_3$ is not an $F$-line, and for each $y \in Y \setminus V(B)$, the $F$-line $y$-$v$-$P_3$-$w_3$ is not disjoint from both $Q_1, Q_2$. In particular, $Y \subseteq V(P_1 \cup B \cup P_2)$. We have already seen that there exists $y \in Y \setminus V(B)$, and we may therefore assume that $y \in V(P_1) \setminus \{x_1\}$. The path $w_3$-$P_3$-$v$-$y$-$P_1$-$w_1$ is a double $F$-line, and it is disjoint from the $F$-line $B \cup P_2$, a contradiction. This proves (4).

(5) *It is not the case that $|F'| = 3$, and the three edges in $F'$ form a triangle.*

For suppose there is a triangle $\{v_1, v_2, v_3\}$ of $K'$ such that $F'$ consists of the three edges $v_1v_2, v_2v_3, v_3v_1$. By (1) (applied to $\{v_1, v_2\}$) there is an edge of $B$ in $F$. For any $M \subseteq \{v_1, v_2, v_3, x_1, x_2\}$ with $|M| = 3$, there is no branch of $(K, W)$ including all members of $M$, and so, since $(K, W)$ is a frame, there exist three vertex-disjoint paths in $K$ between $M$ and $W$. Consequently there are three vertex-disjoint paths $P_1, P_2, P_3$, such that at least one of them has first vertex in $\{x_1, x_2\}$, and at least two of them have first vertex in $\{v_1, v_2, v_3\}$, and they all have last vertex in $W$. Choose three such paths $P_1, P_2, P_3$ with minimal union, and let $P_i$ be between $u_i$ and $w_i \in W$ say. We may assume that $u_1 = x_1, u_2 = v_2$ and $u_3 = v_3$. Moreover, from the minimality of $P_1 \cup P_2 \cup P_3$, $v_1$ is not a vertex of $P_2 \cup P_3$, and $x_2$ is not a vertex of $P_1$. Hence $B$ is not a path of any of $P_1, P_2, P_3$. The path $w_2$-$P_2$-$v_2$-$v_3$-$P_3$-$w_3$ is a double $F$-line, disjoint from $P_1 \cup B \setminus \{x_2\}$; so the latter is not an $F$-line. But some edge of $B$ is in $F$, and so the edge of $B$ incident with $x_2$ is the unique edge of $B$ in $F$. Since the $F$-line $B \cup P_1$ is not disjoint from the double $F$-line $w_2$-$P_2$-$v_2$-$v_3$-$P_3$-$w_3$, it follows that $x_2$ belongs to one of $P_2, P_3$, say $P_2$. The double $F$-line $w_2$-$P_1$-$x_1$-$B$-$x_2$-$P_2$-$w_2$ is not vertex-disjoint from the $F$-line $v_2$-$v_3$-$P_3$-$w_3$, so $v_2 = x_2$. But then every edge in $F$ is incident with one of $v_2, v_3$, contrary to (1). This proves (5).

(6) *There do not exist two vertices $x_3, x_4 \in V(K')$ such that $F' = \delta_{K'}(x_3) \cup \delta_{K'}(x_4)$.*

For suppose such $x_3, x_4$ exist. By (1) (applied to $\{x_3, x_4\}$) there is an edge $f$ of $B$ in $F$ and incident with neither of $x_3, x_4$. Also by (1) $\{x_1, x_2\} \neq \{x_3, x_4\}$, so there are three pairwise vertex-disjoint paths $P_2, P_3, P_4$ from $\{x_1, x_2, x_3, x_4\}$ to $W$, where $P_i$ is from $u_i \in \{x_1, x_2, x_3, x_4\}$ to $w_i \in W$, and $u_3 = x_3, u_4 = x_4$, and $u_2 \in \{x_1, x_2\}$. Choose such paths with $P_2$ minimal; then we may assume that $u_2 = x_2$, and $x_1 \notin V(P_2)$. Hence $B$ is not a path of any of $P_2, P_3, P_4$. Moreover, $P_3, P_4$ are disjoint $F$-lines, and $P_2 \cup B \setminus \{x_1\}$ is disjoint from both of them; so the latter is not an $F$-line. Hence $f$ is incident with $x_1$, and therefore $x_1 \neq x_3, x_4$, and so $x_1, \ldots, x_4$ are all distinct. Since $P_2 \cup B$ is an

$F$-line, it meets one of $P_3, P_4$, and so $x_1 \in V(P_3) \cup V(P_4)$, and we may assume that $x_1 \in V(P_3)$ say. But then the double $F$-line $w_2$-$P_2$-$x_2$-$B$-$x_1$-$P_3$-$w_3$ is disjoint from the $F$-line $P_4$, a contradiction. This proves (6).

(7) *There is no branch $B'$ of $(K', W)$ such that $F' \subseteq E(B')$.*

For suppose that $B'$ is such a branch, with ends $x_3, x_4$. First suppose that one of $x_1, x_2$, say $x_1$, belongs to $V(B')$. Since $(K, W)$ is a frame, there are three paths $P_i$ of $K$ from $x_i$ to $w_i \in W$ for $i = 2, 3, 4$, pairwise vertex-disjoint. Consequently they are all paths of $K \setminus \{x_1\}$. For $i = 3, 4$, let $Q_i$ be the subpath of $B'$ between $x_1$ and $x_i$, and let $Q_2 = B$. Now $F \subseteq E(Q_2) \cup E(Q_3) \cup E(Q_4)$. By (1), not every edge in $F$ is incident with $x_1$, so we may assume that $P_i \cup Q_i \setminus \{x_1\}$ is an $F$-line for some $i \in \{2, 3, 4\}$. Let $\{i, j, k\} = \{2, 3, 4\}$. Consequently there do not exist two vertex-disjoint $F$-lines in $P_j \cup Q_j \cup Q_k \cup P_k$, and so there are at most two edges in $F \cap E(Q_j \cup Q_k)$, and if there are two then they have a common end. By (1) (applied to $\{x_1, x_i\}$) at least one edge of $Q_j \cup Q_k$ is in $F$, and if there is only one then $P_j \cup Q_j \cup Q_k \cup P_k$ is a double $F$-line, disjoint from the $F$-line $P_i \cup Q_i \setminus \{x_1\}$, a contradiction. Hence exactly two edges of $Q_j \cup Q_k$ belong to $F$, and they have a common end $y_j \in V(Q_j)$ say. By (1) applied to $\{x_1, x_j\}$ it follows that $y_j \neq x_1$, so $y_j$ is an internal vertex of $Q_j$, and in particular $F \cap E(Q_k) = \emptyset$, and $F \cap E(Q_j) = \delta(y_j)$. Then $P_j \cup Q_j \setminus \{x_1\}$ is an $F$-line, and so by the same argument there is an internal vertex $y_i$ of $Q_i$ such that $F \cap E(Q_i) = \delta(y_i)$. But this contradicts (1) (applied to $\{y_i, y_j\}$). This completes the proof of (7) in the case when one of $x_1, x_2$ belongs to $B'$.

Thus we may assume that $B, B'$ are vertex-disjoint. There is symmetry between $B$ and $B'$ (for we will not use any more that $(K', W)$ is a frame). By two applications of (1), both of $B, B'$ contains an edge in $F$. There are three vertex-disjoint paths between $\{x_1, \ldots, x_4\}$ and $W$, and we may assume that none of them has an internal vertex in $\{x_1, \ldots, x_4\}$; and from the symmetry we may assume that these paths are $P_1, P_2, P_3$, where $P_i$ is between $x_i$ and $w_i \in W$. Now $P_3 \cup B'$ is an $F$-line, and so $F_1 \cup B \cup F_2$ is not a double $F$-line and does not include two disjoint $F$-lines; so there is an internal vertex $y$ of $B$ such that $F \cap E(B) = \delta(y)$. There are three disjoint paths $Q_2, Q_3, Q_4$ from $\{x_1, \ldots, x_4\}$ to $W$, such that for $i = 3, 4$, $Q_i$ has first vertex $x_i$; choose them with $Q_2$ minimal, then we may assume that $Q_2$ has first vertex $x_2$ and $x_1 \notin V(Q_2)$ (possibly $x_1 \in V(Q_3 \cup Q_4)$). The path $y$-$B$-$x_2$-$Q_2$-$w'_2$ (where $Q_2$ is from $x_2$ to $w'_2 \in W$) is an $F$-line, disjoint from the path $Q_3 \cup B' \cup Q_4$; so the latter is not a double $F$-line, and does not include two disjoint $F$-lines. Hence there is an internal vertex $y'$ of $B'$ such that $F \cap E(B') = \delta(y')$; but this contradicts (1) (applied to $\{y, y'\}$). This proves (7).

(8) *There is no branch $B'$ of $(K', W)$ with ends $x_3, x_4$, such that $F' \setminus E(B') = \delta_{K'}(x_3)$.*

For suppose $B'$ is such a branch. Again there are two cases depending whether $B, B'$ are vertex-disjoint or not. First suppose that $x_1 \in V(B')$ say, and as in (7) we may choose three paths $P_i$ of $K$ from $x_i$ to $w_i \in W$ for $i = 2, 3, 4$, pairwise vertex-disjoint. For $i = 3, 4$, let $Q_i$ be the subpath of $B'$ between $x_1$ and $x_i$, and let $Q_2 = B$. Suppose that $x_1 = x_3$. By (1) (applied to $\{x_2, x_3\}$), there is an edge of $B' \setminus \{x_3\}$ in $F$, and similarly an edge of $B \setminus \{x_3\}$ in $F$. But then $P_2 \cup Q_2 \setminus \{x_3\}, P_3, P_4 \cup Q_4 \setminus \{x_3\}$ are three disjoint $F$-lines, a contradiction. Thus $x_1 \neq x_3$. By (1) (applied to $\{x_1, x_3\}$), at least one edge of $Q_2 \cup Q_4$ is in $F$ and not incident with $x_1$. Since $P_1$ is an $F$-line, the path $P_2 \cup Q_2 \cup Q_4 \cup P_4$ is not a double $F$-line, and does not include two disjoint $F$-lines; so exactly two edges of $Q_2 \cup Q_4$ belong to $F$ and they have a common end $y \neq x_1$. From the symmetry we may assume that $y$ belongs to the interior of $Q_2$ say, and so $F \cap E(Q_4) = \emptyset$. By (1) (applied to $\{x_3, y\}$) there is an edge of $Q_3 \setminus \{x_3\}$ in $F$; but then $P_2 \cup Q_2 \setminus \{x_1\}, P_3, P_4 \cup Q_4 \cup Q_3 \setminus \{x_3\}$ are three vertex-disjoint $F$-lines, a contradiction. This proves (8) in the case that $B, B'$ are not disjoint.

We may therefore assume that $B, B'$ are disjoint. By (1) (applied to $\{x_3, x_4\}$) at least one edge of $B$ is in $F$. By (5) at least one edge of $B' \setminus \{x_3\}$ belongs to $F$. There are three vertex-disjoint paths $P_2, P_3, P_4$ from $\{x_1, \dots, x_4\}$ to $W$, such that for $i = 3, 4$, $P_i$ is from $x_i$ to $w_i \in W$ say. We may assume that $P_2$ is from $x_2$ to $w_2$, and $x_1 \notin V(P_2)$ (possibly $x_1 \in V(P_3 \cup P_4)$). Thus $P_3, P_4 \cup B' \setminus \{x_3\}$ are disjoint $F$-lines, and so $P_2 \cup B \setminus \{x_1\}$ is not an $F$-line; and therefore the edge of $B$ incident with $x_1$ is the unique edge of $B$ in $F$. Since $B \cup P_2$ is an $F$-line, it follows that $x_1$ belongs to one of $P_3, P_4$. If $x_1 \in V(P_3)$, then $w_2$-$P_2$-$x_2$-$B$-$x_1$-$P_3$-$w_3$ is a double $F$-line, and $B' \cup P_4$ is an $F$-line, and they are disjoint, a contradiction. If $x_1 \in V(P_4)$, then $w_2$-$P_2$-$x_2$-$B$-$x_1$-$P_4$-$w_4$ is a double $F$-line, and $P_3$ is an $X$-line, and they are disjoint, a contradiction. This proves (8).

(9) *There is no branch $B'$ of $(K', W)$ with ends $x_3, x_4$, such that $F' \setminus E(B') = \delta_{K'}(x_3) \cup \delta_{K'}(x_4)$.*

For suppose that $B'$ is such a branch. Again there are two cases depending whether $B, B'$ are vertex-disjoint or not. First suppose that $x_1 \in V(B')$ say, and as in (7) we may choose three paths $P_i$ of $K$ from $x_i$ to $w_i \in W$ for $i = 2, 3, 4$, pairwise vertex-disjoint. For $i = 3, 4$, let $Q_i$ be the subpath of $B'$ between $x_1$ and $x_i$, and let $Q_2 = B$. Thus $P_3, P_4$ are $F$-lines. Suppose that $x_1 = x_3$. Then from (1) (applied to $\{x_3, x_4\}$) there is an edge of $Q_2 \setminus \{x_3\}$ in $F$, and so $P_2 \cup Q_2 \setminus \{x_3\}$ is an $F$-line disjoint from $P_3, P_4$, a contradiction. Thus $x_1 \neq x_3$, and similarly $x_1 \neq x_4$. By (1) (applied to $\{x_3, x_4\}$) there is an edge of $F$ in $Q_2 \cup Q_3 \cup Q_4$ not incident with either of $x_3, x_4$; but hence there is an $F$-line in $P_2 \cup Q_2 \cup Q_3 \cup Q_4 \setminus \{x_3, x_4\}$, and it is disjoint from $P_3, P_4$, a contradiction. This proves (9) in the case that $B, B'$ are not disjoint.

Thus we may assume that $B, B'$ are disjoint. By (1) (applied to $\{x_3, x_4\}$) at least one edge of $B$ is in $F$. There are three vertex-disjoint paths $P_2, P_3, P_4$ from $\{x_1, \dots, x_4\}$ to $W$, such that for $i = 3, 4$, $P_i$ is from $x_i$ to $w_i \in W$ say. We may assume that $P_2$ is from $x_2$ to $w_2$, and $x_1 \notin V(P_2)$ (possibly $x_1 \in V(P_3 \cup P_4)$). Thus $P_3, P_4$ are disjoint $F$-lines, and so $P_2 \cup B \setminus \{x_1\}$ is not an $F$-line; and therefore the edge of $B$ incident with $x_1$ is the unique edge of $B$ in $F$. Since $B \cup P_2$ is an $F$-line, it follows that $x_1$ belongs to one of $P_3, P_4$, and we may assume it belongs to $P_3$ from the symmetry. Then $w_2$-$P_2$-$x_2$-$B$-$x_1$-$P_3$-$w_3$ is a double $F$-line, and $P_4$ is an $F$-line, and they are disjoint, a contradiction. This proves (9).

But (4)–(9) are contrary to the inductive hypothesis applied to the frame $(K', W)$. This proves that our assumption of (1) was false, and so proves 4.1. ■

## 5 The main proof

In this section we prove the "only if" half of 3.1. We need to show that if $G$ is a graph and $Z \subseteq V(G)$ with $|Z| \geq 2$ is constricted, then $(G, Z)$ admits an extended $H$-strip decomposition for some graph $H$. For each component $C$ of $G$, $Z \cap V(C)$ is constricted in $C$, and if the result holds for $(C, Z \cap V(C))$ for all components $C$ with $|Z \cap V(C)| \geq 2$, then it holds for $(G, Z)$. Thus we may assume that $G$ is connected. Moreover, the result is trivial if $|Z| = 2$, so we may assume that $|Z| \geq 3$. Therefore, throughout this section we assume that $G$ is a connected graph, $Z \subseteq V(G)$ with $|Z| \geq 3$, and $Z$ is constricted in $G$. We shall prove a series of lemmas about the pair $(G, Z)$.

Let $(K, W)$ be a frame. We say it is a *frame for* $(G, Z)$ if $E(K) \subseteq V(G)$, and

- for all distinct $e, f \in E(K)$, $e, f$ have a common end in $K$ if and only if $e, f \in V(G)$ are adjacent in $G$
- $Z$ is the set of edges of $K$ incident with a vertex in $W$.

We begin with:

**5.1** *There is a frame for* $(G, Z)$.

**Proof.** Since $G$ is connected, we may choose $X \subseteq V(G)$ with $Z \subseteq X$, minimal such that $G|X$ is connected.

(1) *For each* $v \in X \setminus Z$, $G|(X \setminus \{v\})$ *has exactly two components, and they both contain at least one vertex of* $Z$.

For $G|(X \setminus \{v\})$ is not connected, from the minimality of $X$. Let its components be $C_1, \ldots, C_k$ say where $k \geq 2$. If $C_i \cap Z = \emptyset$, let $X' = X \setminus V(C_i)$; then $G|X'$ is connected and $Z \subseteq X'$, contrary to the minimality of $X$. Thus each $C_i$ contains at least one vertex of $Z$. Suppose that $k \geq 3$, and choose $z_i \in V(C_i) \cap Z$ for $i = 1, 2, 3$. Since $G|X$ is connected, there are paths $P_1, P_2, P_3$ of $G|X$ between $v$ and $z_1, z_2, z_3$ respectively, with $V(P_i) \subseteq C_i \cup \{v\}$, and if we choose $P_1, P_2, P_3$ with minimal union then their union is an induced tree of $G$, containing three members of $Z$, contradicting that $Z$ is constricted in $G$. This proves (1).

For each $v \in X \setminus Z$, let $A_v, B_v$ be the vertex sets of the two components of $G|(X \setminus \{v\})$.

(2) *For each* $v \in X \setminus Z$, *the set of neighbours of* $v$ *in* $A_v$ *is a clique, and so is the the set of neighbours of* $v$ *in* $B_v$.

For suppose that $u_1, u_2 \in A_v$ are nonadjacent, and are both adjacent to $v$. Choose $z_3 \in B_v$, and let $P_3$ be an induced path between $v$ and $z_3$ with vertex set in $B_v \cup \{v\}$. From the minimality of $X$, for $i = 1, 2$ there exists $z_i \in Z$ such that every path of $G|X$ between $v$ and $z_i$ contains $u_i$; let $P_i$ be some such path, induced. Consequently $u_1 \notin V(P_2)$, since $P_2$ is induced and $u_2 \in V(P_2)$, and similarly $u_2 \notin V(P_1)$. Hence $z_1 \neq z_2$, and $V(P_1), V(P_2) \subseteq A_v \cup \{v\}$. Since every path of $G|X$ between $v$ and $z_1$ contains $u_1$, it follows that $V(P_1) \setminus \{v, u_1\}$ is disjoint from $V(P_2) \setminus \{v\}$, and there is no edge bteween these two sets. Similarly there is no edge between $V(P_1) \setminus \{v\}$ and $V(P_2) \setminus \{vi, u_2\}$; and therefore there is no edge between $V(P_1) \setminus \{v\}$ and $V(P_2) \setminus \{v\}$, since $u_1, u_2$ are nonadjacent. Hence $p_1 \cup P_2 \cup P_3$ is an induced tree in $G$, contradicting that $Z$ is constricted. This proves (2).

(3) *For each* $v \in Z$, *the set of neighbours of* $v$ *in* $X \setminus \{z\}$ *is a clique.*

The proof is similar to that of (2). Suppose that $u_1, u_2 \in X \setminus \{v\}$ are nonadjacent, and both adjacent to $v$. From the minimality of $X$, there exist $z_i \in Z$ such that every path of $G|X$ between $v$

and $z_i$ contains $u_i$; let $P_i$ be such a path, induced, for $i = 1, 2$. Then $z_1, z_2 \neq v$, and as in (2), there are no edges between $V(P_1) \setminus \{v\}$ and $V(P_2) \setminus \{v\}$. But then $P_1 \cup P_2$ is an induced tree containing $v, z_1, z_2$, a contradiction. This proves (3).

From (2) and (3) it follows that $G|X$ is the line graph of a tree $K$; thus $E(K) = X$, and for $x, y \in X$, $x, y$ are adjacent in $G$ if and only if some vertex of $K$ is incident with them both. Let $W$ be the set of vertices of $K$ that have degree one in $K$. By (3), every $z \in Z$ is incident in $K$ with a member of $W$. Moreover, if $x \in E(K)$ is incident with a member of $W$, and $x \notin Z$, then one of $A_x, B_x$ is empty, which is impossible; so $Z$ is equal to the set of edges of $K$ incident with members of $W$. But then $(K, W)$ is a frame for $(G, Z)$. This proves 5.1. ■

Let $\eta$ be an $H$-strip structure $\eta$ in $(G, Z)$. If $e \in E(H)$ with ends $u, v$, a *$e$-rung* of $\eta$ means an induced path $G|\eta(e)$ with vertices $p_1, \dots, p_k$ in order, where for $1 \leq i \leq k$, $p_i \in \eta(e, u)$ if and only if $i = 1$, and $p_i \in \eta(e, v)$ if and only if $i = k$. (Possibly $k = 1$.) An $H$-strip structure $\eta$ in $(G, Z)$ is said to be *connected* if for every $e \in E(H)$, $\eta(e)$ is nonempty, and $\eta(e)$ is the union of the vertex sets of the $e$-rungs of $\eta$.

**5.2** *There is a graph $H$ with the following properties, where $W$ denotes the set of vertices of $H$ of degree one:*

- *$(H, W)$ is a frame*
- *no vertex of $H$ has degree two*
- *there is a connected $H$-strip structure in $(G, Z)$*
- *subject to these three conditions, $|E(H)|$ is maximum.*

**Proof.** By 5.1, there is a frame $(K, W)$ for $(G, Z)$. Let $W_2$ be the set of vertices of $K$ that have degree two, and let $W_3$ be the set that have degree at least three; thus $W, W_2, W_3$ are pairwise disjoint and have union $V(K)$. Let $H$ be the graph with vertex set $W \cup W_3$, in which vertices $u, v$ are adjacent if there is a branch of $K$ with ends $u, v$. Hence for each edge $e \in E(H)$ there is a branch $B_e$ of $K$ with the same ends as $e$. Thus $(H, W)$ is a frame (though no longer a frame for $(G, Z)$, in general), and no vertex of $H$ has degree two. Define $\eta$ as follows:

- for each $e \in E(H)$, $\eta(e) = E(B_e) \subseteq V(G)$
- for each $e \in E(H)$ incident with $v \in V(H)$, $\eta(e, v) = \{f\}$ where $f \in E(B_e) \subseteq V(G)$ is the edge of $B_e$ incident with $v$.

It follows that $\eta$ is a connected $H$-strip structure in $(G, Z)$. Thus the first three conditions of the theorem are satisfied. Since the sets $\eta(e)$ ($e \in E(H)$) are nonempty (since $\eta$ is connected) and pairwise disjoint, it follows that $|E(H)| \leq |V(G)|$ for every choice of $H$ satisfying the first three conditions above, and therefore the fourth can also be satisfied. This proves 5.2 ■

Henceforth in the section, $H, W$ will be as in 5.2. Moreover, $\eta$ will be a connected $H$-strip structure in $(G, Z)$, chosen with $\cup\eta$ maximal, where $\cup\eta$ denotes the union of all the sets $\eta(e)$ ($e \in E(H)$). For each $e = uv \in E(H)$, define $M(e) = \eta(e, u) \cap \eta(e, v)$. For each $v \in V(H)$, define $N(v) = \bigcup_{e \in \delta_H(v)} \eta(e, v)$, and for every triangle $D = \{v_1, v_2, v_3\}$ of $H$, define $N(D) = M(v_1v_2) \cup M(v_2v_3) \cup M(v_3v_1)$.

**5.3** *Let $p \in V(G) \setminus \cup\eta$, and let $Y$ denote the set of all neighbours of $p$ in $\cup\eta$. Then either*

- *there is an edge $e$ of $H$ such that $Y \subseteq \eta(e)$, or*
- *there is a vertex $v$ of $H$ such that $Y \subseteq N(v)$, or*
- *there is a triangle $D$ of $H$ such that $Y \subseteq N(D)$.*

**Proof.**

(1) *For each $e \in E(H)$, let $R_e$ be an $e$-rung. Let $R$ be the union of all the sets $V(R_e)$ ($e \in E(H)$). Then one of the following holds:*

- *there exists $e \in E(H)$ with $Y \cap R \subseteq V(R_e)$, or*
- *there exists $v \in V(H)$ such that $Y \cap R \subseteq N(v)$, or*
- *there is a triangle $\{u, v, w\}$ of $H$ such that $R_{uv}, R_{vw}, R_{wu}$ all have length zero, and $Y \cap R = V(R_{uv}) \cup V(R_{vw}) \cup V(R_{wu})$, or*
- *there exists $e = uv \in E(H)$ such that $u \notin W$ and $Y \cap (R \setminus V(R_e)) = N(u) \cap (R \setminus V(R_e))$ and $Y \cap V(R_e) \not\subseteq N(u)$, or*
- *there exists $e = uv \in E(H)$ such that $u, v \notin W$ and $Y \cap (R \setminus V(R_e)) = (N(u) \cup N(v)) \cap (R \setminus V(R_e))$.*

For let $K$ be obtained from $H$ by replacing each edge $e \in E(H)$ by a path with edges the vertices of $R_e$ in order, in the natural way, so that $G|R$ is the line graph of $K$. Thus $(K, W)$ is a frame for $(G, Z)$. Let $F = R \cap Y$; then $F \subseteq E(K)$. Moreover, $F$ is focussed, since $Z$ is constricted in $G$. Hence one of the six outcomes of 4.1 holds. If 4.1.3 holds and $x, y$ are as in 4.1.3, then there is a frame $(K', W)$ for $G$, where $K'$ is obtained from $K$ by adding $p$ to $K$ as a new edge incident with $x, y$, contrary to the maximality of $|E(H)|$. If 4.1.1 holds, then the second outcome of (1) holds. Similarly if one of 4.1.2, 4.1.4, 4.1.5, 4.1.6 holds then respectively the third, first, fourth and fifth outcome of (1) holds. This proves (1).

(2) *If there is an edge $e = v_1v_2$ of $H$ such that $Y \subseteq \eta(e) \cup N(v_1) \cup N(v_2)$ then the theorem holds.*

Suppose there is such an edge $e = v_1v_2$. For each $f \in E(H)$ choose an $f$-rung $R_f$. For $i = 1, 2$, let $E_i$ be the set of all edges of $H$ that are incident with $v_i$ and different from $e$; thus $|E_i| \neq 1$. Let $A_i$ be the set of all $f \in E_i$ such that $Y$ contains the end of $R_f$ in $N(v_i)$, and let $B_i = E_i \setminus A_i$.

Suppose first that $Y \cap N(v_2) \subseteq \eta(e)$. If also $Y \cap N(v_1) \subseteq \eta(e)$ then $Y \subseteq \eta(e)$ and the theorem holds, so we may assume that $Y \cap N(v_1) \not\subseteq \eta(e)$. Moreover, we may assume that $Y \cap \eta(e) \not\subseteq N(v_1)$, for otherwise $Y \subseteq N(v_1)$ and the theorem holds. Hence we may choose the rungs $R_f$ ($f \in E(H)$) such that there exists $a_1 \in A_1$ and $Y \cap V(R_e) \not\subseteq N(v_1)$. By (1), it follows that $B_1 = \emptyset$. Since this holds for all choices of $R_f$ ($f \neq a_1, e$), we deduce that $Y \cap \eta(f) = \eta(f, v_1)$ for all $f \in E(H) \setminus \{a_1, e\}$ incident with $v_1$. Since $|E_1| \neq 1$, it follows by exchanging the roles of $a_1$ and some other member of $E_1$ that $Y \cap \eta(a_1) = \eta(a_1, v_1)$, and so $N(v_1) \setminus \eta(e) = Y \setminus \eta(e)$. But then $a$ can be added to $\eta(e)$ and to $\eta(e, v_1)$, contrary to the maximality of $\cup\eta$. Thus we may assume that $Y \cap N(v_2) \not\subseteq \eta(e)$, and similarly $Y \cap N(v_1) \not\subseteq \eta(e)$. Hence we may choose the $R_f$ ($f \in E(H)$) such that $A_1, A_2$ are both nonempty.

Suppose that there is a choice of the $R_f$ ($f \in E(H)$) such that for some $a_1 \in A_1$ and $a_2 \in A_2$, either $a_1, a_2$ are disjoint edges of $H$, or not both $R_{a_1}, R_{a_2}$ have length zero. By (1), $B_1, B_2$ are both empty. Since this holds for all choices of $R_f$ ($f \neq a_1, a_2$), we deduce that $Y \cap \eta(f) = \eta(f, v_i)$ for $i = 1, 2$ and for all $f \in E(G) \setminus \{e, a_1, a_2\}$ incident with $v_i$. Now there exist $a_1' \in A_1$ and $a_2' \in a_2$ such that $a_1' \neq a_1$ and $a_1', a_2$ are disjoint edges of $H$, since $|E_1|, |E_2| \geq 2$ (possibly $a_2' = a_2$). We have seen that we can choose $R_{a_1'}, R_{a_2'}$ such that they both meet $Y$, and so by the same argument with $a_1, a_1'$ exchanged, it follows that $Y \cap \eta(a_1) = \eta(a_1, v_1)$, and so $N(v_i) \setminus \eta(e) \subseteq Y$ for $i = 1$, and also for $i = 2$ by the symmetry. But then $a$ can be added to $\eta(e), \eta(e, v_1)$ and $\eta(e, v_2)$, contrary to the maximality of $\cup\eta$.

Hence for every choice of the $R_f$ ($f \in E(H)$) with $A_1, A_2$ both nonempty, we have that $|A_1| = |A_2| = 1$, say $A_i = \{a_i\}$ for $i = 1, 2$, and $R_{a_1}, R_{a_2}$ both have length zero, and $a_1, a_2$ have a common end $w$ in $H$. Take some such choice of the $R_f$ ($f \in E(H)$). For $f \in B_1$, we deduce that we cannot replace $R_f$ with some other $f$-rung that meets $Y$, and therefore $Y \cap (N(v_1) \setminus \eta(e)) \subseteq \eta(a_1, v_1)$. Moreover, there is no $a_1$-rung that meets $Y$ with length greater than zero, and so $Y \cap \eta(a_1) \subseteq \eta(a_1, v_1) \cap \eta(a_1, w)$. Similarly $Y \cap (N(v_2) \setminus \eta(e)) \subseteq \eta(a_2, v_2)$, and $Y \cap \eta(a_2) \subseteq \eta(a_2, v_2) \cap \eta(a_2, w)$. We may assume that $Y \not\subseteq N(w)$, and so $Y \cap \eta(e) \neq \emptyset$. Choose $R_e$ with $Y \cap V(R_e)$ nonempty. By (1), every such choice of $R_e$ has length zero. But then $Y \subseteq N(D)$ where $D$ is the triangle $\{v_1, v_2, w\}$, and the theorem holds. This proves (2).

(3) *For each $e \in E(H)$, let $R_e$ be an $e$-rung. If either the fourth or fifth outcome of (1) holds, then the theorem holds.*

For let $R$ be the union of the sets $V(R_e)$ ($e \in E(H)$). Suppose first that there exists $e = uv \in E(H)$ such that $u \notin W$ and $Y \cap (R \setminus V(R_e)) = N(u) \cap (R \setminus V(R_e))$ and $Y \cap V(R_e) \not\subseteq N(u)$. Then (1) implies that $\eta(f) \cap Y = \emptyset$ for all $f \in E(H)$ not incident with $u$ (because otherwise we could make another choice of $R_f$ so that (1) was violated); and $\eta(f) \cap Y \subseteq N(u)$ for each $f \neq e$ incident with $u$, for the same reason. But then $Y \subseteq \eta(e) \cup N(u) \cup N(v)$, and so the theorem holds by (2).

Next suppose that there exists $e = uv \in E(H)$ such that $u, v \notin W$ and $Y \cap (R \setminus V(R_e)) = (N(u) \cup N(v)) \cap (R \setminus V(R_e))$. Again by (1), $Y \cap \eta(f) = \emptyset$ for each edge $f$ of $H$ not incident with $u, v$, and $Y \cap \eta(f) \subseteq \eta(f, u)$ for every $f \neq e$ incident with $u$, and a similar result holds with $u, v$ exchanged. But then $Y \subseteq \eta(e) \cup N(u) \cup N(v)$ and the theorem holds by (2). This proves (3).

Suppose that there exists $e = v_1v_2 \in E(H)$ such that $Y \cap \eta(e) \not\subseteq \eta(e, v_1) \cup \eta(e, v_2)$. Choose an $e$-rung $R_e$ such that some internal vertex of $R_e$ belongs to $Y$. By (1) and (3), $Y \cap \eta(f) = \emptyset$ for all $f \in E(H) \setminus \{e\}$, and so the theorem holds. Hence every vertex in $Y$ belongs to at least one of the sets $N(v)$ ($v \in V(H)$). Suppose that some $y \in Y$ belongs to exactly one such set; say $y \in \eta(e, v) \setminus M(e)$, where $e = uv$. By (1), $Y \setminus \eta(e) \subseteq N(v)$, and so the theorem holds by (2). Thus we may assume that every vertex in $Y$ belongs to $M(e)$ for some $e \in E(H)$. Let $F$ be the set of all $f \in E(H)$ with $Y \cap M(f) \neq \emptyset$. If there exists $e, f \in F$ with no common end in $H$, then the theorem holds by (1) and (3); if there is some vertex $v$ of $H$ incident with every edge in $F$, then $Y \subseteq N(v)$ and the theorem holds; and if neither of these hold, then $|F| = 3$, and the three members of $F$ are the edges of a triangle $D$ of $H$, and $Y \subseteq N(D)$ and the theorem holds. This proves 5.3. ■

**5.4** *Let $X \subseteq V(G) \setminus \cup\eta$ such that $G|X$ is connected, and let $Y$ be the set of all attachments of $G|X$ in $\cup\eta$. Then either*

- *there is an edge $e$ of $H$ such that $Y \subseteq \eta(e)$, or*
- *there is a vertex $v$ of $H$ such that $Y \subseteq N(v)$, or*
- *there is a triangle $D$ of $H$ such that $Y \subseteq N(D)$.*

**Proof.** Suppose that this is false for some $X$, and choose $X$ minimal such that 5.4 is false for $X$.

(1) *There exist $y_1, y_2$ in $Y$ such that $\{y_1, y_2\} \not\subseteq \eta(e)$ for each $e \in E(H)$, and $\{y_1, y_2\} \not\subseteq N(v)$ for each $v \in V(H)$, and $\{y_1, y_2\} \not\subseteq N(D)$ for each triangle $D$ of $H$.*

For suppose first that for some $e = v_1v_2 \in E(H)$, there exists $y_1 \in Y \cap \eta(e)$ with $y_1 \notin \eta(e, v_1) \cup \eta(e, v_2)$. Now $Y \not\subseteq \eta(e)$; choose $y_2 \in Y \setminus \eta(e)$, and then $y_1, y_2$ satisfy (1). We may therefore assume that $Y \subseteq \cup_{v \in V(H)} N(v)$.

Next suppose that some $y_1 \in Y$ belongs to exactly one of the sets $N(v)$ $(v \in V(H))$; say $y_1 \in \eta(e, v_1) \setminus M(e)$, where $e = v_1v_2 \in E(H)$. Since $Y \not\subseteq N(v_1)$, there exists $y_2 \in Y \setminus N(v_2)$. If also $y_2 \notin \eta(e)$, then the pair $y_1, y_2$ satisfies (1), so we may assume that $y_2 \in \eta(e) \setminus N(v_1)$. We already assumed that every member of $Y$ belongs to one of the sets $N(v)$ $(v \in V(H))$, and so $y_2 \in \eta(e, v_2) \setminus M(e)$. This restores the symmetry between $v_1$ and $v_2$. Since $Y \not\subseteq \eta(e)$, there exists $y_3 \in Y$ with $y_3 \notin \eta(e)$. Since we may assume that the pair $y_1, y_3$ does not satisfy (1), it follows that $y_3 \in N(v_1)$, and similarly $y_3 \in N(v_2)$. Let $f \in E(H)$ with $y_3 \in \eta(f)$; then $f$ is incident with $v_1$ since $y_3 \in N(v)$, and similarly $f$ is incident with $v_2$, and so $f = e$, a contradiction since $y_3 \notin \eta(e)$. We may therefore assume that every $y \in Y$ belongs to $M(e)$ for some $e \in E(H)$.

Let $F$ be the set of all edges $f \in E(H)$ such that $Y \cap M(f) \neq \emptyset$. Suppose that there exist $e, f \in F$ with no common end in $H$. Choose $y_1 \in Y \cap M(e)$ and $y_2 \in Y \cap M(f)$; then $y_1, y_2$ satisfy (1). Thus we may assume that every two edges in $F$ share an end. Consequently either there is a vertex $v \in V(H)$ incident with every member of $F$, or $|F| = 3$ and the three edges in $F$ form a triangle $D$ of $H$. In the first case $Y \subseteq N(v)$, and in the second $Y \subseteq N(D)$, in either case a contradiction. This proves (1).

For each $p \in X$, let $Y(p)$ denote the set of all $v \in \cup\eta$ adjacent to $p$; and for $P \subseteq X$, let $Y(P) = \cup_{p \in P} Y(p)$. Thus $Y = Y(X)$. Let $y_1, y_2$ be as in (1). Then $y_1, y_2$ are nonadjacent. Since $G|X$ is connected, there is an induced path of $G$ with vertices $y_1$-$p_1$-$p_2$-$\cdots$-$p_k$-$y_2$ in order. By 5.3 it follows that $k > 1$. From the minimality of $X$, $X = \{p_1, \ldots, p_k\}$. Let $P_1 = X \setminus \{p_k\}$, and $P_2 = X \setminus \{p_1\}$. Then for $i = 1, 2$, the minimality of $X$ implies that either

- there is an edge $e$ of $H$ such that $Y(P_i) \subseteq \eta(e)$, or
- there is a vertex $v$ of $H$ such that $Y(P_i) \subseteq N(v)$, or
- there is a triangle $D$ of $H$ such that $Y(P_i) \subseteq N(D)$.

Thus there are three possibilities for $Y(P_1)$ and three for $Y(P_2)$, and we need to check these nine possibilities individually. For each $e \in E(H)$ choose an $e$-rung $R_e$ (in some cases we shall need to choose the $e$-rungs subject to some further conditions); let $R$ be the union of the sets $V(R_e)$ ($e \in E(H)$), and let $L = G|R$. Let $K$ be the graph obtained from $H$ by replacing every edge $e$ of $H$ by a path whose edges are the vertices of $R_e$ in the corresponding order. Then $L$ is the line graph of $K$ and $(K, W)$ is a frame for $(G, Z)$. (Thus $K, L, R$ all depend on the choice of the rungs $R_e$.)

(2) *There do not exist* $v_1, v_2 \in V(H)$ *such that* $Y(P_1) \subseteq N(v_1)$ *and* $Y(P_2) \subseteq N(v_2)$.

For suppose that such $v_1, v_2$ exist. Since $Y = Y(P_1) \cup Y(P_2)$, it follows that $v_1 \neq v_2$. Now $v_1, v_2$ may or may not be adjacent in $H$. If they are adjacent, let $f = v_1v_2 \in E(H)$, and otherwise $f$ is undefined. For $1 < i < k$,

$$Y(p_i) \subseteq Y(P_1) \cap Y(P_2) \subseteq N(v_1) \cap N(v_2) = M(f),$$

(where $M(f) = \emptyset$ if $f$ is not defined) and so $Y \subseteq Y(p_1) \cup Y(p_k) \cup M(f)$. Suppose first that for $i = 1, 2$, either $N(v_i) \setminus \eta(f, v_i) \subseteq Y(p_i)$ or $(N(v_i) \setminus \eta(f, v_i)) \cap Y(p_i) = \emptyset$ (where $\eta(f, v_i) = \emptyset$ if $f$ is undefined). If $(N(v_i) \setminus \eta(f, v_i)) \cap Y(p_i) = \emptyset$ for $i = 1, 2$, then $Y \subseteq \eta(f)$ (where $\eta(f) = \emptyset$ if $f$ is undefined), a contradiction, so we may assume that $N(v_1) \setminus \eta(f, v_1) \subseteq Y(p_1)$ say. If $N(v_2) \setminus \eta(f, v_2) \cap Y(p_k) = \emptyset$, then $f$ is defined since $Y(p_k) \neq \emptyset$, and we can add $p_1, \ldots, p_k$ to $\eta(f)$, and add $p_1$ to $\eta(f, v_1)$, contrary to the maximality of $\cup\eta$; if $N(v_2) \setminus \eta(f, v_2) \subseteq Y(p_k)$ and $f$ is defined, we can add $p_1, \ldots, p_k$ to $\eta(f)$, add $p_1$ to $\eta(f, v_1)$, and add $p_k$ to $\eta(f, v_2)$, again contrary to the maximality of $\cup\eta$; so we may assume that $N(v_2) \setminus \eta(f, v_2) \subseteq Y(p_k)$ and $f$ is undefined. Thus $Y(p_1) = N(v_1)$ and $Y(p_k) = N(v_2)$. Let $K'$ be obtained from $K$ by adding a path between $v_1, v_2$ with edges $p_1, \ldots, p_k$ in order; then $(K', W)$ is a frame for $(G, Z)$, contrary to the maximality of $|E(H)|$.

Thus we may assume that for some $i \in \{1, 2\}$, $N(v_i) \setminus \eta(f, v_i) \not\subseteq Y(p_i)$ and $(N(v_i) \setminus \eta(f, v_i)) \cap Y(p_i) \neq \emptyset$. For $i = 1, 2$, let $E_i$ is the set of edges of $K$ incident with $v_i$ if $f$ is undefined, and let $E_i$ is the set of edges of $K$ incident with $v_i$ not in the branch of $K$ between $v_1, v_2$ if $f$ is defined. Therefore we may choose the $R_e$ ($e \in E(H)$) such that at least three of the sets $A_1, B_1, A_2, B_2$ are nonempty, where for $i = 1, 2$, $A_i = E_i \cap Y(P_i)$, and $B_i = E_i \setminus A_i$. We may also assume that for $i = 1, 2$, either $y_i \in E(K)$, or $|B_i| = 1$ and one of $A_j, B_j$ is empty, where $\{i, j\} = \{1, 2\}$. (To see this, observe that if say $y_1 \notin E(K)$, let $y_1 \in \eta(e)$ say, and choose an $e$-rung $R'_e$ containing $y_1$; then if either $|B_1| \neq 1$, or $A_2, B_2$ are both nonempty, we may replace $R_e$ by $R'_e$, and still satisfy all the other requirements.) From the symmetry we may assume that $A_1, B_1 \neq \emptyset$, and hence $y_2 \in E(K)$. Since $(H, W)$ is a frame, there are three paths $Q_1, Q_2, Q_3$ of $H$ such that $Q_1$ is between $v_1$ and $w_1 \in W$, and $Q_2$ is between $v_1$ and $w_2 \in W$, and $Q_3$ is between $v_2$ and $w_3 \in W$, and $V(Q_1 \cap Q_2) = \{v_1\}$, and $Q_3$ is vertex-disjoint from both $Q_1, Q_2$, and the edge $a_1 = u_1v_1$ of $Q_1$ incident with $v_1$ belongs to $A_1$, and the edge of $Q_2$ incident with $v_1$ belongs to $B_1$. If the edge of $Q_3$ incident with $v_2$ belongs to $A_2$, then

$$G|(X \cup E(Q_1) \cup E(Q_2) \cup E(Q_3))$$

is an induced tree containing three vertices of $Z$, a contradiction. Thus the first edge of $Q_3$ is in $B_2$. If $y_2 \notin A_2$, then $f$ is defined and $y_2$ belongs to the branch of $K$ between $v_1, v_2$, and this branch has length at least two (since $y_2 \notin N(v_1)$); but then

$$G|(X \cup E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup \{y_2\})$$

is an induced tree containing three vertices of $Z$, a contradiction. Thus $y_2 \in A_2$. Let $a_2 \in A_2$, with ends $v_2, u_2$ say. Then $u_2 \notin V(Q_3)$. If $u_2 \notin V(Q_1) \cup V(Q_2)$, then

$$G|(X \cup E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup \{a_2\})$$

is an induced tree containing three vertices of $Z$, a contradiction. Thus $u_2 \in V(Q_1) \cup V(Q_2)$. If $u_2 \in V(Q_1) \setminus \{u_1, v_1\}$, let $Q_1'$ be the path of $Q_1$ between $u_2$ and $w_1$; then

$$G|(X \cup E(Q_1') \cup E(Q_2) \cup E(Q_3) \cup \{a_1, a_2\})$$

is an induced tree containing three vertices of $Z$, a contradiction. If $u_2 \in V(Q_2) \setminus \{v_1\}$, let $Q_2'$ be the path of $Q_2$ between $u_2$ and $w_2$; then

$$G|(X \cup E(Q_1) \cup E(Q_2') \cup E(Q_3) \cup \{a_1, a_2\})$$

is an induced tree containing three vertices of $Z$, a contradiction. Thus $u_2 \in \{u_1, v_1\}$. But $u_2 \neq v_1$ from the definition of $A_2$, and so $u_2 = u_1$. Hence $|A_2| = 1$, and so $A_2 = \{y_2\}$, and $B_2 \neq \emptyset$. This restores the symmetry between $v_1, v_2$. From the same argument with $v_1, v_2$ exchanged, applied to the paths $Q_1', Q_3, Q_2$ (where $Q_1'$ is the path $v_2$-$u_2$-$Q_1$-$w_1$), it follows that $A_1 = \{y_1\}$, contradicting that $y_1, y_2$ are nonadjacent in $G$. This proves (2).

(3) *There do not exist $v_1 \in V(H)$ and $e \in E(H)$ such that $Y(P_1) \subseteq N(v)$ and $Y(P_2) \subseteq \eta(e)$.*

For suppose that such $v_1, e$ exist. Then $e$ may or may not be incident with $v_1$. If $e$ is not incident with $v_1$, then $Y = Y(p_1) \cup Y(p_2)$, while if $e$ is incident with $v_1$ then $Y \subseteq Y_1 \cup Y_2 \cup \eta(e, v_1)$. Suppose first that $N(v_1) \subseteq Y(P_1) \cup \eta(e)$. Since $Y(P_2) \not\subseteq N(v_1)$, we may choose the $e$-rung $R_e$ such that at least one vertex of $V(R_e) \cap Y(P_2)$ does not belong to $N(v_1)$. If $e$ is incident with $v_1$, then we can add $p_1, \ldots, p_k$ to $\eta(e)$ and add $p_1$ to $\eta(e, v_1)$, contrary to the maximality of $\cup\eta$. Thus $e$ is not incident with $v_1$, and so $Y(p_1) = N(v_1)$. Let $e$ be incident with $v_2, v_3$ in $H$. There are three vertex-disjoint paths $Q_1, Q_2, Q_3$ of $K$ such that for $i = 1, 2, 3$, $Q_i$ is between $v_i$ and some $w_i \in W$. If exactly one vertex of $R_e$ is in $Y(p_2)$, then

$$G|(E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup V(R_e) \cup \{p_1, \ldots, p_k\})$$

is an induced tree of $G$ containing three members of $Z$, a contradiction. If there are two nonadjacent vertices of $R_e$ that are both in $Y(p_2)$, let $S_2, S_3$ be minimal subpaths of $R_e$ that meet both $Y(p_2)$ and $N(v_2), N(v_3)$ respectively; then there is no edge between $V(S_2)$ and $V(S_3)$, and

$$G|(E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup V(S_2) \cup V(S_3) \cup \{p_1, \ldots, p_k\})$$

is an induced tree of $G$ containing three members of $Z$, a contradiction. Thus there are exactly two vertices in $R_e$ that belong to $Y(p_2)$, say $x, y$, and they are adjacent in $G$. There is a branch of $K$ with edge set the vertex set of $R_e$; let $t$ be the vertex of this branch that is incident with $x, y$ in $K$. Let $K'$ be obtained from $K$ by adding a path between $v_1, t$ with edges $p_1, \ldots, p_k$ in order; then $(K', W)$ is a frame for $(G, Z)$, contrary to the maximality of $|E(H)|$. This proves that $N(v_1) \not\subseteq Y(P_1) \cup \eta(e)$.

Hence we may choose the $R_f$ ($f \in E(H)$) such that both $A_1, B_1$ are nonempty, where $E_1$ denotes the set of edges of $K$ incident with $v_1$ and not in $V(R_e)$, and $A_1 = E_1 \cap Y(p_1)$, and $B_1 = E_1 \setminus Y(p_1)$. Let $e$ be incident with $v_2, v_3 \in H$. Since $(K, W)$ is a frame, there are three paths $Q_1, Q_2, Q_3$ of $K$, such that $Q_1$ is from $v_1$ to some $w_1 \in W$, $Q_2$ is from $v_1$ to some $w_2 \in W$, $Q_3$ is from one of $v_2, v_3$ to some $w_3 \in W$, $V(Q_1) \cap V(Q_2) = \{v_1\}$, $Q_3$ is vertex-disjoint from both $Q_1$ and $Q_2$, the edge of $Q_1$ incident with $v_1$ belongs to $A_1$, and the edge of $Q_2$ incident with $v_1$ belongs to $B_1$. Moreover, we may assume that only one of $v_2, v_3$ belongs to $V(Q_3)$, say $v_3$. It follows that $v_1 \neq v_3$. Since none of $Q_1, Q_2, Q_3$ contain both $v_2, v_3$, we may alter our choice of $R_e$ without affecting the existence of $Q_1, Q_2, Q_3$; and since $Y(P_2) \not\subseteq N(v_2)$ by (2), we may choose $R_e$ such that some vertex of $R_e$ belongs to $Y(P_2) \setminus N(v_2)$. Let $S$ be a minimal subpath of $R_e$ that meets both $Y(P_2)$ and $N(v_3)$. Then

$$G|(E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup V(S) \cup \{p_1, \ldots, p_k\})$$

is an induced tree of $G$ containing three vertices of $Z$, a contradiction. This proves (3).

(4) *There do not exist edges $e_1, e_2$ of $H$ such that $Y(P_i) \subseteq \eta(e_i)$ for $i = 1, 2$.*

For suppose that such edges exist. Then $e_1 \neq e_2$; let $e_i$ have ends $u_i, v_i$ for $i = 1, 2$. Since $\eta(e_1) \cap \eta(e_2) = \emptyset$, it follows that $Y = Y(p_1) \cup Y(p_2)$. We may assume that $v_1 \neq u_2, v_2$ and $v_2 \neq u_1, v_1$; that is, $u_1, u_2, v_1, v_2$ are all distinct except that possibly $u_1 = u_2$. For $i = 1, 2$, choose $R_{e_i}$ such that some vertex of $R_{e_i}$ belongs to $Y(P_i)$, and in addition, if $u_1 = u_2$, choose $R_{e_i}$ such that some vertex of $R_{e_i}$ belongs to $Y(P_i)$ and not to $N(u_i)$ (this is possible since $Y(P_i) \not\subseteq V(u_i)$ by (3)). Suppose first that for $i = 1, 2$, exactly two vertices $x_i, y_i$ of $R_{e_i}$ belong to $Y(P_i)$ and they are adjacent. For $i = 1, 2$, let $t_i$ be the vertex incident with $x_i, y_i$ of the branch of $K$ with edge set $V(R_i)$. Let $K'$ be obtained from $K$ by adding a new path between $t_1, t_2$ with edges $p_1, \ldots, p_k$ in order; then $(K', W)$ is a frame for $(G, Z)$, contrary to the maximality of $|E(H)|$. We may therefore assume that either exactly one vertex of $R_{e_1}$ belongs to $Y(p_1)$, or two nonadjacent vertices of $R_{e_1}$ belong to $Y(p_1)$. Let $Q_1, Q_2, Q_3$ be vertex-disjoint paths of $K$ between $\{u_1, v_1, v_2\}$ and $W$, where $Q_1$ is between $u_1$ and some $w_1 \in W$, and $Q_2$ is between $v_1$ and some $w_2$ in $W$, and $Q_3$ is between $v_2$ and some $w_3 \in W$. (Possibly $u_2$ belongs to one of these paths.) Let $T$ be the branch of $K$ with edge set $V(R_{e_2})$; then some edge of $T$ belongs to $Y(p_2)$ and is not incident with $u_2$, from the choice of $R_{e_2}$. Hence there is a path $S_3$ of $K$, a subgraph of $T \cup Q_3$, with first vertex $t \in V(T)$ say and last vertex $w_3$, such that the first edge and not other edge of $S$ belongs to $Y(p_2)$, and $S, Q_1, Q_2$ are pairwise vertex-disjoint. If only one vertex of $R_{e_1}$ is in $Y(p_1)$, then

$$G|(E(Q_1) \cup E(Q_2) \cup E(S_3) \cup \{p_1, \ldots, p_k\})$$

is an induced tree of $G$ containing three members of $Z$, a contradiction. Thus there are two nonadjacent vertices in $V(R_{e_1}) \cap Y(p_1)$, and so there are vertex-disjoint subpaths $S_1, S_2$ of $Q_1 \cup Q_2$, such that for $i = 1, 2$, $S_i$ has first vertex $t_i$ say, first edge and no other edge in $Y(p_1)$, and last vertex $w_i$. But then

$$G|(E(S_1) \cup E(S_2) \cup E(S_3) \cup \{p_1, \ldots, p_k\})$$

is an induced tree of $G$ containing three members of $Z$, a contradiction. This proves (4).

From (2),(3),(4), we may assume that $Y(P_2) \subseteq N(D)$ for some triangle $D = \{u_1, u_2, u_3\}$ of $H$, and that $M(u_1u_2), M(u_2u_3), M(u_1u_3)$ all contain at least one member of $Y(P_2)$. Let $e_1, e_2, e_3$ be the edges $u_2u_3, u_3u_1, u_1u_2$ of $H$ respectively. Choose $R_{e_1}$ of length zero such that its vertex ($r_1$ say) is in $Y(P_2)$, and choose $R_{e_2}, R_{e_3}$ similarly. Thus $r_1$ is the edge of $K$ joining $u_2, u_3$. For $i = 1, 2, 3$, let $Q_i$ be a path of $K$ between $u_i$ and some $w_i \in W$, such that $Q_1, Q_2, Q_3$ are pairwise vertex-disjoint.

(5) $Y(P_1) \cap E(K) \subseteq \{r_1, r_2, r_3\}$.

For let $e \in Y(P_1) \cap E(K)$, and suppose first that at most one of $Q_1, Q_2, Q_3$ contains an end of $e$. Since $K$ is connected, we may choose a path $S$ of $K$ with first edge in $Y(p_1)$ that meets one of $Q_1, Q_2, Q_3$, and by choosing $S$ minimal, we may assume that $S$ meets $Q_3$ and not $Q_1, Q_2$, and only its first edge is in $Y(p_1)$. In particular, at most one of $u_1, u_2, u_3$ belongs to $S$, and so no edge of $S$ is in $N(D)$; and therefore no edge of $S$ except the first is adjacent in $G$ to any of $p_1, \dots, p_k$. Let $S'$ be a path of $K$, a subgraph of $S \cup Q_3$, between the first vertex of $S$ and $w_3$. Then

$$G|(E(Q_1) \cup E(Q_2) \cup E(S') \cup \{p_1, \dots, p_k\})$$

is an induced tree in $G$ containing three members of $Z$, a contradiction. This proves that two of $Q_1, Q_2, Q_3$ contain ends of $e$. Let $e = v_1v_2$ where $v_1 \in V(Q_1)$ and $v_2 \in V(Q_2)$ say. Suppose that $v_2 \neq u_2$. For $i = 1, 2$, let $S_i$ be a subpath of $Q_i$ between $v_i$ and $w_i$. Then

$$G|(E(S_1) \cup E(S_2) \cup E(Q_3) \cup \{e, r_1\} \cup \{p_1, \dots, p_k\})$$

is an induced tree in $G$ containing three members of $Z$, a contradiction. Thus $v_2 = u_2$, and similarly $v_1 = u_1$ and so $e = r_3$. This proves (5).

From (5), we deduce that $Y(P_1) \subseteq \eta(e_1) \cup \eta(e_2) \cup \eta(e_3)$ (for otherwise we could make a choice of the $R_f$ for $f \neq u_1u_2, u_1u_3, u_2u_3$ that would violate (5)). Since $Y(P_1) \not\subseteq N(D)$, we may assume that there is some $e_3$-rung $R'_{e_3}$ such that some vertex of $R'_{e_3}$ belongs to $Y(P_1)$ and not to $\eta(e_3, u_2)$ (and so $R'_{e_3}$ has length at least one). Let $S$ be a minimal subpath of $R'_{e_3}$ that meets both $Y(P_1)$ and $\eta(e_3, u_1)$. Then

$$G|(E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup V(S) \cup \{r_1\} \cup \{p_1, \dots, p_k\})$$

is an induced tree in $G$ containing three members of $Z$, a contradiction. This proves 5.4. ■

**Proof of 3.1.** We have already seen the proof of the "if" half. To prove the "only if" half, as we saw earlier in this section, we may assume that $Z$ is constricted in $G$, and $|Z| \geq 3$, and $G$ is connected. We choose $H, W$ as in 5.2. Choose $\eta$ as before; that is, $\eta$ is a connected $H$-strip structure in $(G, Z)$, chosen with $\cup\eta$ maximal. Let $\mathcal{C}$ be the set of all vertex sets of components of $G \setminus \cup\eta$. For $C \in \mathcal{C}$ we define its *home* as follows. Let $Y$ be the set of attachments of $G|C$ in $G| \cup \eta$. We say that $e \in E(H)$ is the home of $C$ if $Y \subseteq \eta(e)$; $v \in V(H)$ is the home of $C$ if $Y \subseteq N(v)$ and no edge of $H$ is the home of $C$; and a triangle $D$ of $H$ is the home of $C$ if $Y \subseteq N(D)$ and there is no vertex or edge of$H$ that is the home of $C$. By 5.4, each $C \in \mathcal{C}$ has a (unique) home. For each $e \in E(H)$, let $\eta'(e)$ be the union of $\eta(e)$ and all $C \in \mathcal{C}$ with home $e$. Define $\eta'(e, v) = \eta(e, v)$ if $v \in V(H)$ is incident with $e$. For each $v \in V(H)$, define $\eta'(v)$ to be the union of all $C \in \mathcal{C}$ with home $v$, and for each triangle $D$ of $H$, let $\eta'(D)$ be the union of all $C \in \mathcal{C}$ with home $D$. It follows that $\eta'$ is an extended $H$-strip decomposition of $(G, Z)$. This proves 3.1. ■

# 6 The algorithm

So far, we have proved our main result, the description of the structure of the constricted pairs, but we have not shown how to test whether a given pair $(G, Z)$ is constricted. It would be nice to use the theorem to show that some simple algorithm works, but so far we have not been able to do this. The best we can see is just to convert the proof of the theorem to an algorithm. Thus we choose some frame $(K_1, W)$ for $(G, Z)$, by using the method of 5.1. (Or we find some induced tree containing at least three members of $Z$, and then we output this and stop.) This takes time $O(n^2)$, where $n = |V(G)|$. Then we make a sequence of frames $(K_i, W)$ for $(G, Z)$, for $i = 2, 3, \ldots$, at each step increasing the number of branches by at least one. Since every frame has at most $n$ branches, this process is iterated at most $n$ times. Thus we need to be able to do the following: given a frame $(K_i, W)$ for $(G, Z)$, either

- produce a larger frame $(K_{i+1}, W)$, or
- find some induced tree containing at least three members of $Z$, or
- find an extended $H$-strip decomposition of $(G, Z)$, where $H$ is obtained from $K_i$ by suppressing all vertices of degree two.

To achieve this, we adapt the proof of 5.3 and 5.4; we choose a connected $H$-strip structure $\eta$, and use the methods of 5.3 and 5.4 to produce either:

- a larger frame $(K_{i+1}, W)$, or
- some induced tree containing at least three members of $Z$, or
- some other connected $H$-strip structure $\eta'$ with $|\cup \eta'| > |\cup \eta|$, or
- an extended $H$-strip decomposition of $(G, Z)$.

This can be iterated at most $n$ times, since $|\cup\eta| \leq n$, and each iteration takes time $O(n^2)$ (with some care). To arrange this last assertion, let us remember slightly more than just a connected $H$-strip structure $\eta$; we will also record, for each $e \in E(H)$ and each $v \in \eta(e)$, some $e$-rung that contains $v$, and update this as we grow $\cup\eta$. It also helps the running time to maintain a list, for every vertex $x \in V(G) \setminus \cup\eta$ that has an attachment in $\cup\eta$, of the (at most one) edge $e \in E(H)$ such that all attachments of $x$ belong to $\eta(e)$, and the (at most two) vertices $v \in V(H)$ such that all attachments of $x$ lie in $N(v)$, and the (at most one) edge $e \in E(H)$ such that all attachments of $x$ lie in $M(e)$, and, if there is no such edge, the (at most one) triangle $D$ such that all attachments of $x$ lie in $N(D)$. The implementation is straightforward and we omit further details.

In summary, then, we have an algorithm with running time $O(|V(G)|^4)$, which, with input a graph $G$ and a subset $Z \subseteq V(G)$ with $|Z| \geq 2$, either outputs an induced tree in $G$ containing three members of $Z$, or outputs a graph $H$ and an extended $H$-strip decomposition of $(G, Z)$.

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