

# $K_4$ -free graphs with no odd holes

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### **Abstract**

All  $K_4$ -free graphs with no odd hole and no odd antihole are three-colourable, but what about  $K_4$ -free graphs with no odd hole? They are not necessarily three-colourable, but we prove a conjecture of Ding that they are all four-colourable. This is a consequence of a decomposition theorem for such graphs; we prove that every such graph either has no odd antihole, or belongs to one of two explicitly-constructed classes, or admits a decomposition.

# 1 Introduction

All graphs in this paper are finite and have no loops or multiple edges. A *hole* in a graph is an induced cycle of length at least four, and an *antihole* is an induced subgraph isomorphic to the complement of a cycle of length at least four. As usual we denote by  $\chi(G)$  the chromatic number of  $G$  and by  $\omega(G)$  the clique number. Recently [2] we were able to prove the “strong perfect graph conjecture” of Berge [1], the following:

**1.1** *If a graph  $G$  has no odd holes and no odd antiholes, then  $\chi(G) = \omega(G)$ .*

A graph is said to be *perfect* if every induced subgraph has chromatic number equal to clique number; and so 1.1 implies that graphs with no odd holes or antiholes are perfect. Since odd holes and odd antiholes do not satisfy the conclusion of 1.1, none of them can be left out from the hypothesis of the theorem. However, it is possible that the hypotheses can be relaxed and we could still deduce that  $\chi(G)$  is bounded by some function of  $\omega(G)$ , where the function does not depend on  $G$ , of course. Gyarfás [4] conjectured:

**1.2 Conjecture.** *For each integer  $k \geq 0$  there is a least integer  $g(k)$  such that every graph  $G$  with no odd hole and with  $\omega(G) = k$  satisfies  $\chi(G) \leq g(k)$ .*

Clearly  $g(i) = i$  for  $i \leq 2$ , but  $g(3) \geq 4$  since the complement of a cycle of length seven is not 3-colourable, and Ding [3] conjectured that  $g(3) = 4$ . We prove Ding’s conjecture. For a graph  $F$  we say that a graph is  $F$ -free if it has no induced subgraph isomorphic to  $F$ , and for a family  $\mathcal{F}$  we say that a graph is  $\mathcal{F}$ -free if it has no subgraph isomorphic to a member of  $\mathcal{F}$ . Our main result is:

**1.3** *Every  $K_4$ -free graph with no odd hole is 4-colourable.*

We deduce 1.3 from a decomposition theorem 3.1 for  $K_4$ -free graphs with no odd holes. The decomposition theorem requires a number of definitions before it can be formulated, and so we postpone its statement until Section 3. Let us remark that our decomposition theorem is not completely satisfactory in that it only applies to non-perfect graphs. It would be nice to have an analogous result for  $K_4$ -free perfect graphs, but that remains open.

There is a conjectured strengthening of 1.2 due to C. T. Hoàng and C. McDiarmid [5], the following.

**1.4 Conjecture.** *For every graph  $G$  with no odd hole and with at least two vertices, there is a partition  $(V_1, V_2)$  of  $V(G)$  such that every maximum clique of  $G$  meets both  $V_1$  and  $V_2$ .*

Our result 1.3 shows that 1.4 is true for all  $K_4$ -free graphs.

## 2 Harmonious cutsets

The *length* of a path or cycle is the number of edges in it, and we say a path or cycle is *even* or *odd* depending whether its length is even or odd. If  $A, B \subseteq V(G)$  are disjoint, we say that  $A$  is *complete* to  $B$  if every vertex in  $A$  is adjacent to every vertex in  $B$ , and  $A$  is *anticomplete* to  $B$  if no vertex in  $A$  is adjacent to a vertex in  $B$ . (We say a vertex  $v$  is complete to a set  $B$  if  $\{v\}$  is complete to

$B$ , and the same for anticomplete.) If  $X \subseteq V(G)$ ,  $G|X$  denotes the subgraph of  $G$  induced on  $X$ , and  $G \setminus X$  denotes the graph obtained by deleting  $X$ , that is, the subgraph induced on  $V(G) \setminus X$ . A *cutset* in a graph  $G$  is a set  $X \subseteq V(G)$  such that  $G \setminus X$  has at least two components. A cutset  $X$  is *harmonious* if  $X$  can be partitioned into disjoint sets  $X_1, X_2, \dots, X_k$  such that:

- for all  $i, j \in \{1, 2, \dots, k\}$ , if  $P$  is an induced path with one end in  $X_i$  and the other end in  $X_j$ , then  $P$  is even if  $i = j$  and odd otherwise, and
- if  $k \geq 3$ , then  $X_1, \dots, X_k$  are pairwise complete to each other.

Thus the first condition implies that each  $X_i$  is a stable set.

**2.1** *Let  $X$  be a harmonious cutset in a graph  $G$ , let  $C_1, C_2$  be a partition of  $V(G) \setminus X$  into two nonempty sets that are anticomplete to each other, and for  $t = 1, 2$  let  $G_t$  be  $G|(C_t \cup X)$ . If  $G_1, G_2$  have no odd hole then  $G$  has no odd hole.*

We omit the (easy) proof since we do not need the result, which is included just to motivate the concept of harmonious cutset. 2.1 implies that if we understand all graphs with no odd hole and no harmonious cutset, then by repeatedly piecing them together on harmonious cutsets we can “construct” all graphs with no odd hole. However, this does not really count as a construction. If  $G, X, G_1, G_2$  are as above, and we wish to view this as a construction of  $G$  from things that we already understand, we need to know not only that  $G_1, G_2$  have no odd hole, but that the cutset  $X$  of  $G$  will be harmonious. This can be stated as a property of the pairs  $(G_1, X)$  and  $(G_2, X)$ ; but we need to have constructions for the pairs  $(G_1, X)$  and  $(G_2, X)$ , not just for  $G_1, G_2$ , before we can claim to have a construction for  $G$ . We have not yet resolved this issue.

Let us return to the colouring problem.

**2.2** *Let  $X$  be a harmonious cutset in a graph  $G$ , let  $C_1, C_2$  be a partition of  $V(G) \setminus X$  into two nonempty sets that are anticomplete to each other, and for  $t = 1, 2$  let  $G_t$  be  $G|(C_t \cup X)$ . If  $G_1, G_2$  are 4-colourable then  $G$  is 4-colourable.*

**Proof.** Let  $X_1, X_2, \dots, X_k$  be as in the definition of a harmonious cutset. By hypothesis both  $G_1$  and  $G_2$  are 4-colourable. Let  $t \in \{1, 2\}$ , and let  $c$  be a 4-colouring of  $G_t$  (using colours 1, 2, 3, 4, and so  $c$  is a map into  $\{1, 2, 3, 4\}$ ). We say that a vertex  $v \in X$  is *c-compliant* if  $c(v) = i$ , where  $i$  is the index such that  $v \in X_i$ . We claim

(1)  $G_t$  has a 4-colouring  $c_t$  such that every vertex of  $X$  is  $c_t$ -compliant.

To prove this claim let  $c$  be a 4-colouring of  $G_t$  that maximizes the number of  $c$ -compliant vertices. We will show that  $c$  is as desired. To this end, suppose for a contradiction that  $v \in X$  is not  $c$ -compliant, say  $v \in X_i$  and  $c(v) = j$ , where  $i \neq j$ . Let  $H$  be the component containing  $v$  of the subgraph of  $G_t$  induced by vertices coloured  $i$  or  $j$ . We claim that no vertex of  $H$  in  $X$  is  $c$ -compliant. For let  $u \in V(H) \cap X$ , and let  $P$  be an induced path of  $H$  joining  $u, v$ . Now  $c(u) = c(v)$  (that is,  $c(u) = j$ ) if and only if  $P$  has even length, from the definition of  $H$ ; but  $P$  has even length if and only if  $u, v$  belong to the same member of  $\{X_1, \dots, X_k\}$  (that is,  $u \in X_i$ ), since  $X$  is harmonious. Consequently  $c(u) = j$  if and only if  $u \in X_i$ , and so  $u$  is not  $c$ -compliant. This proves that no vertex of  $H$  in  $X$  is  $c$ -compliant.

Let  $c'$  be the colouring obtained from  $c$  by swapping the colours  $i$  and  $j$  for every vertex of  $H$ . Then  $v$  is  $c'$ -compliant. Since no vertex of  $H$  is  $c$ -compliant, it follows that more vertices in  $X$  are  $c'$ -compliant than are  $c$ -compliant, contrary to our choice of  $c$ . This proves (1).

Now the colourings  $c_1$  and  $c_2$  can be combined to produce a 4-colouring of  $G$ , as desired.  $\blacksquare$

It is easy to see that a graph with a harmonious cutset has either what is called an even pair, an odd pair, or a clique cutset (we omit the definitions of these standard terms, which we do not need any more), and one could eliminate the use of 2.2 by using these three things instead, and three corresponding theorems from the literature. The interested reader can easily work this out.

What follows is a lemma to make it easier to prove that a given cutset is harmonious.

**2.3** *Let  $G$  be a graph with no odd hole, let  $X$  be a cutset in  $G$ , and let  $X_1, \dots, X_k$  be a partition of  $X$  into stable sets, such that if  $k \geq 3$  then the sets  $X_1, \dots, X_k$  are pairwise complete. Suppose that for all nonadjacent  $a, b \in X$ , there is an induced path  $P$  joining  $a, b$ , with interior in  $V(G) \setminus X$ , such that  $P$  is even if some  $X_i$  contains both  $a, b$ , and odd otherwise. Then  $G$  admits a harmonious cutset.*

**Proof.** If some proper subset  $X'$  of  $X$  is a cutset, then  $X'$  and the sets  $X' \cap X_i$  ( $1 \leq i \leq k$ ) satisfy the hypotheses of the theorem and we may replace  $X$  by  $X'$ . We may therefore assume that  $X$  is a minimal cutset. Let  $C_1, \dots, C_t$  be the vertex sets of the components of  $G \setminus X$ ; thus every member of  $X$  has a neighbour in  $C_i$  for all  $i$  with  $1 \leq i \leq t$ .

(1) *Let  $a, b \in X$ . Every induced path between  $a, b$  with no internal vertex in  $X$  is even if some  $X_i$  contains both  $a, b$ , and odd otherwise.*

For we may assume that  $a, b$  are nonadjacent, since  $X_1, \dots, X_k$  are stable. By hypothesis, there is an induced path  $P$  joining  $a, b$ , with interior in  $V(G) \setminus X$ , such that  $P$  is even if some  $X_i$  contains both  $a, b$ , and odd otherwise. Since no internal vertex of  $P$  is in  $X$ , the interior of  $P$  is contained in one of  $C_1, \dots, C_t$ , say  $C_1$ . Now  $t > 1$ , so  $a, b$  both have neighbours in  $C_2$  from the minimality of  $X$ , and hence there is an induced path  $Q$  joining  $a, b$  with interior in  $C_2$ . Since the union of  $P, Q$  is an even hole, it follows that  $Q, P$  have the same parity. Now let  $R$  be any path with ends  $a, b$  and with interior disjoint from  $X$ . Then there exists  $j \in \{1, \dots, t\}$  such that the interior of  $R$  is a subset of  $C_j$ . Consequently one of  $P \cup R, Q \cup R$  is a hole, and since  $P, Q$  have the same parity, it follows that  $R$  also has the same parity. This proves (1).

Let  $P$  be an induced path with both ends in  $X$ , and let its ends be  $v, v'$  say, where  $v \in X_i$  and  $v' \in X_{i'}$ . We must show that  $P$  is even if and only if  $i = i'$ . We proceed by induction on the length of  $P$ . If no internal vertex of  $P$  is in  $X$ , the claim follows from (1), so we may assume that there is an internal vertex  $u$  of  $P$  in  $X_j$  say. Let  $Q, Q'$  be the subpaths of  $P$  between  $v, u$  and between  $u, v'$  respectively. From the inductive hypothesis,  $Q$  is even if and only if  $i = j$ , and  $Q'$  is even if and only if  $i' = j$ . Now  $P$  is odd if and only if exactly one of  $Q, Q'$  is odd, that is, if exactly one of  $i, i'$  is equal to  $j$ . It follows that if  $P$  is odd then  $i \neq i'$ . For the converse, suppose that  $P$  is even; then either both  $i, i'$  are equal to  $j$  or both  $i, i'$  are different from  $j$ . In the first case  $i = i'$  as required. In the second case, if  $k \leq 2$  then  $i = i'$  as required, and if  $k \geq 3$  then  $i = i'$  since  $v, v'$  are nonadjacent. This proves that  $P$  is even if and only if  $i = i'$ , and so proves 2.3.  $\blacksquare$

For  $X$  as in 2.3, we call  $(X_1, \dots, X_k)$  the ‘‘corresponding colouring’’.

### 3 The main theorem

In this section we state the main result. If  $A, B \subseteq V(G)$  are disjoint, we say that  $A, B$  are *linked* if every member of  $A$  has a neighbour in  $B$ , and every member of  $B$  has a neighbour in  $A$ . We need to define two kinds of graphs.

We say a graph  $G$  is of  $T_{11}$  *type* if there is a partition of  $V(G)$  into eleven nonempty stable subsets  $W_1, \dots, W_{11}$ , such that (with index arithmetic modulo 11) for  $1 \leq i \leq 11$ ,  $W_i$  is anticomplete to  $W_{i+1} \cup W_{i+2}$  and complete to  $W_{i+3} \cup W_{i+4} \cup W_{i+5}$ .

We say that  $G$  is of *heptagram type* if there is a partition of  $V(G)$  into fourteen stable subsets  $W_1, \dots, W_7, Y_1, \dots, Y_7$ , where  $W_1, \dots, W_7$  are nonempty but  $Y_1, \dots, Y_7$  may be empty, satisfying the following (with index arithmetic modulo 7).

1. For  $1 \leq i \leq 7$ ,  $W_i$  is anticomplete to  $W_{i+3}$ .
2. For  $2 \leq i \leq 7$ ,  $W_i$  is complete to  $W_{i+2}$ , and  $W_1, W_3$  are linked.
3. For  $i \in \{3, 4, 6, 7\}$ ,  $W_i$  is complete to  $W_{i+1}$ ; for  $i = 1, 2, 5$ ,  $W_i, W_{i+1}$  are linked.
4. If  $v_i \in W_i$  for  $i = 1, 2, 3$ , and  $v_2$  is adjacent to  $v_1, v_3$ , then  $v_1$  is adjacent to  $v_3$ .
5. If  $v_i \in W_i$  for  $i = 1, 2, 3$ , and  $v_2$  is nonadjacent to  $v_1, v_3$ , then  $v_1$  is nonadjacent to  $v_3$ .
6. For  $1 \leq i \leq 7$ , every vertex in  $Y_i$  has a neighbour in each of  $W_i, W_{i+3}, W_{i-3}$  and has no neighbour in  $W_{i+1}, W_{i+2}, W_{i-1}, W_{i-2}$ .
7. For  $1 \leq i \leq 7$  and each  $y \in Y_i$ , let  $N_j$  be the set of neighbours of  $y$  in  $W_j$  for  $j = i, i+3, i-3$ ; then  $N_{i+3}$  is complete to  $N_{i-3}$ , and  $N_{i+3}$  is anticomplete to  $W_{i-3} \setminus N_{i-3}$ , and  $N_{i-3}$  is anticomplete to  $W_{i+3} \setminus N_{i+3}$ , and  $N_i$  is complete to  $W_{i+1} \cup W_{i-1}$ .
8. For  $1 \leq i \leq 7$ ,  $Y_i$  is complete to  $Y_{i+1}$  and anticomplete to  $Y_{i+2} \cup Y_{i+3}$ .
9. For  $1 \leq i \leq 7$ , if  $Y_i$  is not complete to  $W_{i+3} \cup W_{i-3}$  then  $W_{i-3} \cup W_{i+3}$  is complete to  $W_{i-2} \cup W_{i+2}$ , and  $Y_{i-1}, Y_{i+1}, Y_{i-3}, Y_{i+3}$  are all empty.
10. For  $1 \leq i \leq 7$ , at least one of  $Y_i, Y_{i+1}, Y_{i+2}$  is empty.

It is questionable whether the description given above of graphs of heptagram type really counts as an explicit construction. We return to this in the final section, where we give a more complicated but more explicit construction of the same class of graphs. We leave the reader to check that graphs of  $T_{11}$  type and graphs of heptagram type have no odd hole, are  $K_4$ -free, do not admit a harmonious cutset, and contain an antihole of length seven. (To check that graphs of heptagram type have no odd hole, we suggest the use of theorem 5.2 below.) Our main result is the converse, the following.

**3.1** *Let  $G$  be a  $K_4$ -free graph with no odd hole, and with no harmonious cutset, containing an antihole of length seven. Then  $G$  is either of heptagram type or of  $T_{11}$  type.*

This has the corollary mentioned earlier:

**3.2** *Every  $K_4$ -free graph with no odd hole is four-colourable.*

**Proof.** Let  $G$  be a  $K_4$ -free graph with no odd hole; we prove by induction on  $|V(G)|$  that  $G$  is four-colourable. If  $G$  admits a harmonious cutset, the result follows from 2.2 and the inductive hypothesis. If  $G$  contains no antihole of length seven, then it contains no odd hole or antihole, and therefore is perfect by 1.1 (or Tucker's earlier result [6]), and so is three-colourable. We may therefore assume that  $G$  satisfies the hypotheses of 3.1; but then, by 3.1,  $G$  is of one of the two types listed. It is easy to check that graphs of these two types are four-colourable. This proves 3.2.  $\blacksquare$

## 4 Graphs of $T_{11}$ type

Let  $X_1, \dots, X_n$  be disjoint subsets of  $V(G)$ ; by an *induced path of the form  $X_1 \cdots X_n$*  we mean an induced path  $x_1 \cdots x_n$  where  $x_i \in X_i$  for  $1 \leq i \leq n$  (and when some  $X_i$  is a singleton, say  $\{x\}$ , we sometimes write  $x$  instead of  $X_i$ ). We use analogous terminology for holes. Let  $T_{11}$  be the graph with vertex set  $w_1, \dots, w_{11}$ , in which for  $1 \leq i \leq 11$ ,  $w_i$  is nonadjacent to  $w_{i+1}, w_{i+2}$  and adjacent to  $w_{i+3}, w_{i+4}, w_{i+5}$ . (Throughout this section, index arithmetic is modulo 11.) In this section we show the following.

**4.1** *Let  $G$  be a  $K_4$ -free graph with no odd holes and no harmonious cutset. If  $G$  contains  $T_{11}$  as an induced subgraph then  $G$  is of  $T_{11}$  type.*

**Proof.** Since  $G$  contains  $T_{11}$  as an induced subgraph, we may choose eleven nonempty stable sets  $W_1, \dots, W_{11}$ , pairwise disjoint, such that for  $1 \leq i \leq 11$ ,  $W_i$  is anticomplete to  $W_{i+1}, W_{i+2}$  and complete to  $W_{i+3}, W_{i+4}, W_{i+5}$ . Choose them with maximal union, and let their union be  $W$ .

(1) *If  $v \in V(G) \setminus W$ , and  $a, b \in W$  are adjacent to  $v$ , then either  $a, b$  are adjacent or  $a, b \in W_i$  for some  $i \in \{1, \dots, 11\}$ .*

For suppose not; then from the symmetry we may assume that  $a \in W_1$  and  $b \in W_2 \cup W_3$ . Let  $N$  be the set of neighbours of  $v$  in  $W$ . By a  *$v$ -path* we mean an induced path in  $G|W$  with both ends in  $N$  and with no internal vertices in  $N$ . Since  $G$  has no odd hole, every odd  $v$ -path has length one. For  $1 \leq i \leq 11$  choose  $w_i \in W_i$ . Suppose first that  $b \in W_2$ . Since there is no  $v$ -path of the form  $a-W_4-W_{10}-b$ , it follows that  $N$  includes one of  $W_4, W_{10}$ ; and from the symmetry we may assume that  $W_4 \subseteq N$ . Since no three members of  $N$  are pairwise adjacent (since  $G$  is  $K_4$ -free) it follows that  $N$  is disjoint from  $W_7, W_8, W_9$ . Since there is no  $v$ -path of the form  $b-(W_5 \cup W_6)-(W_1 \cup W_{11})-w_4$  it follows that  $N$  includes one of  $W_5 \cup W_6, W_1 \cup W_{11}$ , and we claim we may assume the second. For if  $w_5 \notin N$  then the second statement holds anyway; and if  $w_5 \in N$  then  $W_2 \subseteq N$  (since there is no  $v$ -path of the form  $w_4-W_7-W_2-w_5$ ), and so there is symmetry between the pairs  $(W_1, W_2)$  and  $(W_5, W_4)$ , and we may assume that  $W_1 \cup W_{11} \subseteq N$  because of this symmetry. Thus, we may assume that  $W_1 \cup W_{11} \subseteq N$ . Since there is no  $v$ -path of the form  $a-w_9-W_3-w_{11}$  it follows that  $W_3 \subseteq N$ ; and since  $N$  includes no triangle within  $W_3 \cup W_6 \cup W_{11}$ , it follows that  $N \cap W_6 = \emptyset$ . There is no  $v$ -path of the form  $a-W_5-W_{10}-w_3$ , so  $N$  includes one of  $W_5, W_{10}$ , and from the symmetry exchanging  $W_1, W_3$  we may assume that  $W_5 \subseteq N$ . Since  $N$  includes no triangle within  $W_2 \cup W_5 \cup W_{10}$ , it follows that  $N$  is disjoint from  $W_{10}$ . Since there is no  $v$ -path of the form  $w_4-w_7-W_2-w_5$ , we deduce that  $W_2 \subseteq N$ , and so  $N$  is the union of  $W_i$  for  $i = 11, 1, 2, 3, 4, 5$ . But then  $v$  can be added to  $W_8$ , contradicting the maximality of  $W$ .

This proves that  $b \notin W_2$ , and more generally for  $1 \leq i \leq 11$ ,  $N$  is disjoint from one of  $W_i, W_{i+1}$ . Now  $b \in W_3$ , and so  $N$  is disjoint from  $W_{11}, W_2, W_4$ . But then there is a  $v$ -path  $a-w_4-w_{11}-b$ , a contradiction. This proves (1).

(2) Let  $X \subseteq V(G) \setminus W$  such that  $G|X$  is connected. If  $a, b \in W$  have neighbours in  $X$  then either  $a, b$  are adjacent or  $a, b \in W_i$  for some  $i \in \{1, \dots, 11\}$ .

For suppose not, and choose  $X$  minimal such that some such pair  $a, b$  violates (2). It follows that there is an induced path  $a-x_1-\dots-x_k-b$  where  $X = \{x_1, \dots, x_k\}$ . By (1),  $a, b$  have no common neighbour in  $X$ , and so  $k \geq 2$ . From the symmetry we may assume that  $a \in W_1$  and  $b \in W_2 \cup W_3$ . For  $1 \leq i \leq 11$  choose  $w_i \in W_i$ , choosing  $w_i \in \{a, b\}$  if possible. For  $1 \leq i \leq 11$ , the minimality of  $X$  implies that not all of  $w_i, w_{i+1}, w_{i+2}$  have neighbours in  $X$ , since then some two of them would be joined by a proper subpath of  $x_1-\dots-x_k$ . In particular, not all of  $w_6, w_7, w_8$  have neighbours in  $X$ ; say  $w_j$  does not, where  $6 \leq j \leq 8$ . Consequently  $w_j-a-x_1-\dots-x_k-b-w_j$  is a hole, and therefore  $k$  is odd.

Suppose first that  $b \in W_2$ . Since  $a-x_1-\dots-x_k-b-w_{10}-w_4-a$  is not an odd hole, we may assume from the symmetry that  $w_4$  has a neighbour in  $X$ . From the minimality of  $X$ ,  $w_4$  is adjacent to  $x_1$  and to no other member of  $X$ . Since not all  $w_{11}, w_1, w_2$  have neighbours in  $X$ , it follows that  $w_{11}$  has no neighbour in  $X$ . Since not all  $w_4, w_5, w_6$  have neighbours in  $X$ , there exists  $i \in \{5, 6\}$  such that  $w_i$  has no neighbour in  $X$ . But then  $w_4-x_1-\dots-x_k-b-w_i-w_{11}-w_4$  is an odd hole, a contradiction.

Thus  $b \notin W_2$ , so  $b \in W_3$ , and more generally for  $1 \leq i \leq 11$  at least one of  $W_i, W_{i+1}$  is anticomplete to  $X$ . In particular,  $w_{11}, w_2, w_4$  have no neighbour in  $X$ . Thus  $a-x_1-\dots-x_k-b-w_{11}-w_4-a$  is an odd hole, a contradiction. This proves (2).

Suppose that  $W \neq V(G)$ ; we shall prove that  $G$  admits a harmonious cutset. Choose  $C \subseteq V(G) \setminus W$  maximal such that  $G|C$  is connected. Let  $N$  be the set of vertices in  $W$  with neighbours in  $C$ . By (2) (and since  $11/4 < 3$ ),  $N \cap W_i$  is nonempty for at most three values of  $i \in \{1, \dots, 11\}$ , and  $N \cap W_i$  is complete to  $N \cap W_j$  for all distinct  $i, j \in \{1, \dots, 11\}$ . Thus by 2.3 it suffices to show that if  $a, b \in N \cap W_1$  then there is an even path joining  $a, b$  with interior in  $W \setminus N$ . But  $a, b$  have a common neighbour in  $W_j$  for  $j = 4, 5$ , and not both these belong to  $N$  by (2). This completes the proof of 4.1. ■

## 5 Heptagrams

In view of 4.1, to prove 3.1 it suffices to prove it for  $\{K_4, T_{11}\}$ -free graphs, and that is the main goal of the remainder of the paper.

If a graph  $G$  contains an antihole of length seven, then the vertices of that antihole can be numbered  $w_1, w_2, \dots, w_7$  in such a way that  $w_i$  is adjacent to  $w_j$  if and only if  $|i - j| \in \{1, 2, 5, 6\}$ . This motivates the following definition. We say that  $W = (W_1, W_2, \dots, W_7)$  is a *heptagram* in  $G$  if (here and later index arithmetic is modulo 7)

**(S1)** the sets  $W_1, W_2, \dots, W_7 \subseteq V(G)$  are disjoint, nonempty, and stable,

**(S2)** for  $1 \leq i \leq 7$ ,  $W_i$  is anticomplete to  $W_{i+3} \cup W_{i+4}$

(S3) for  $1 \leq i \leq 7$ , the sets  $W_i, W_{i+1}, W_{i+2}$  are pairwise linked

(S4) if  $u \in W_{i-1}, v \in W_i, w \in W_{i+1}$  and  $v$  is adjacent to both  $u$  and  $w$ , then  $u$  is adjacent to  $w$ ,

(S5) if  $u \in W_{i-1}, v \in W_i, w \in W_{i+1}$  and  $v$  is non-adjacent to both  $u$  and  $w$ , then  $u$  is non-adjacent to  $w$ , and

(S6) if  $u \in W_{i-1}, v \in W_i, w \in W_{i+1}, x \in W_{i+2}$ ,  $u$  is adjacent to  $w$  and  $v$  is adjacent to  $x$ , then either  $u$  is adjacent to  $v$  or  $w$  is adjacent to  $x$ .

If  $W = (W_1, \dots, W_7)$  is a heptagram in  $G$ , we also use  $W$  to denote the set  $W_1 \cup \dots \cup W_7$ . This mild abuse of notation should cause no confusion.

Let us explain briefly where these conditions came from. It is clear that (S1)–(S3) are designed to mimic the edge-structure of the antihole on seven vertices, but (S4)–(S6) are less natural. They arose from the following consideration. Let  $(W_1, \dots, W_7)$  satisfy (S1)–(S3), in a graph  $G$ . One can check that if (S4)–(S6) are also satisfied, then  $G|W$  has no odd hole (to prove this, use 5.3 below); and also the converse holds, that is, if  $G|W$  has no odd hole then (S4)–(S6) hold, provided all the graphs  $G|W_i \cup W_{i+1}$  are connected.

Our strategy to prove 3.1 is to choose a heptagram  $W$  in  $G$  with  $W$  maximal, and to analyze how the remainder of  $G$  attaches to  $W$ . But first, in this section we study the internal structure of a heptagram. We begin with:

**5.1** *Let  $(W_1, W_2, \dots, W_7)$  be a heptagram in a graph  $G$ . For  $1 \leq i \leq 7$ , if  $W_i$  is complete to  $W_{i+1}$ , then  $W_i$  is complete to  $W_{i+2}$  and  $W_{i-1}$  is complete to  $W_{i+1}$ .*

**Proof.** Let  $u \in W_i$  and  $w \in W_{i+2}$ , and let  $v \in W_{i+1}$  be a neighbour of  $w$ . (This exists by (S3).) Since  $W_i$  is complete to  $W_{i+1}$ , it follows that  $v$  is adjacent to both  $u, w$ ; and so  $u$  is adjacent to  $w$  by (S4). This proves that  $W_i$  is complete to  $W_{i+2}$ . The second assertion follows by symmetry. This proves 5.1. ■

**5.2** *Let  $(W_1, W_2, \dots, W_7)$  be a heptagram in a graph  $G$ . For  $1 \leq i \leq 7$  either  $W_i$  is complete to  $W_{i+1}$  or  $W_{i+2}$  is complete to  $W_{i+3}$ .*

**Proof.** From the symmetry we may assume  $i = 1$ .

(1) *Let  $w_i \in W_i$  for  $i = 1, 3, 4$ . Then  $w_3$  is adjacent to one of  $w_1, w_4$ .*

For suppose not. By (S3),  $w_1$  has a neighbour  $w_2 \in W_2$ ; by (S4),  $w_2, w_3$  are nonadjacent, and so by (S5),  $w_2, w_4$  are nonadjacent. By (S3) again,  $w_2$  has a neighbour  $n_3 \in W_3$ ; by (S4),  $w_1, n_3$  are adjacent, and by (S4) again,  $n_3, w_4$  are nonadjacent. Again by (S3),  $w_4$  has a neighbour  $n_2 \in W_2$ ; by (S5),  $n_2, w_3$  are adjacent, and so by (S4),  $n_2, w_1$  are nonadjacent. But then  $w_1, n_2, n_3, w_4$  violate (S6). This proves (1).

To prove the theorem, suppose that  $w_i \in W_i$  for  $1 \leq i \leq 4$ , say, and  $w_1, w_2$  are nonadjacent, and  $w_3, w_4$  are nonadjacent. By (1),  $w_1, w_3$  are adjacent, and similarly so are  $w_2, w_4$ ; but then (S6) is violated. This proves 5.2. ■

**5.3** Let  $(W_1, W_2, \dots, W_7)$  be a heptagram in a graph  $G$ . Then there exists  $t \in \{1, \dots, 7\}$  such that  $W_{j-1}$  is complete to  $W_{j+1}$  for all  $j \in \{1, \dots, 7\} \setminus \{t\}$ , and  $W_j$  is complete to  $W_{j+1}$  for all  $j \in \{t-3, t-2, t+1, t+2\}$ . Consequently, for all  $i \in \{1, \dots, 7\}$ , if  $u \in W_{i-2}$  and  $v \in W_{i+2}$ , then

- $u, v$  have common neighbours in  $W_{i-3}$ , in  $W_i$  and in  $W_{i+3}$ , and
- there is a path of the form  $u-W_{i-1}-W_{i+1}-v$ .

**Proof.** The first assertion follows from 5.2 and 5.1, and the others follow from this and (S3). This proves 5.3. ■

## 6 Y-vertices

Until the end of section 8, where we complete the proof of 3.1,  $G$  is a  $\{K_4, T_{11}\}$ -free graph with no odd hole, containing an antihole of length seven. Consequently there is a heptagram in  $G$ , say  $W = (W_1, \dots, W_7)$ ; and let us choose the heptagram with  $W_1 \cup \dots \cup W_7$  maximal. (We call this the “maximality” of  $W$ .) Again,  $W$  is fixed until the end of section 8.

We say that  $y \in V(G) \setminus W$  is a *Y-vertex* or a *Y-vertex of type  $t$*  if the following hold, where  $N_i$  denotes the set of neighbours of  $y$  in  $W_i$  for  $1 \leq i \leq 7$ :

- $N_t, N_{t+3}, N_{t-3}$  are nonempty, and  $N_i = \emptyset$  for  $i = t-2, t-1, t+1, t+2$
- $N_{t-3}$  is complete to  $N_{t+3}$ , and  $N_{t-3}$  is anticomplete to  $W_{t+3} \setminus N_{t+3}$ , and  $N_{t+3}$  is anticomplete to  $W_{t-3} \setminus N_{t-3}$
- $N_t$  is complete to  $W_{t+1} \cup W_{t+2} \cup W_{t-1} \cup W_{t-2}$

The main result of this section is the following:

**6.1** Let  $v \in V(G) \setminus W$ . Then one of the following holds:

- $v$  is a *Y-vertex*, or
- let  $N$  be the set of neighbours of  $v$  in  $W$ ; then  $N \cap W_i$  is nonempty for at most two values of  $i \in \{1, \dots, 7\}$ , and if there are two such values,  $i$  and  $j$  say, then  $j \in \{i-2, i-1, i+1, i+2\}$  and  $N \cap W_i$  is complete to  $N \cap W_j$ .

**Proof.** Let  $N_i = N \cap W_i$  and  $M_i = W_i \setminus N_i$  for  $1 \leq i \leq 7$ . Let

$$I = \{i \in \{1, \dots, 7\} : N_i \neq \emptyset\}.$$

By a *v-path* we mean an induced path of  $G|W$  such that its ends are in  $N$  and its internal vertices are not in  $N$ . Since  $G$  has no odd hole, every odd *v-path* has length one. Since  $G$  is  $K_4$ -free, no three members of  $N$  are pairwise adjacent (briefly,  $N$  is triangle-free).

(1) For  $1 \leq i \leq 7$ , not all  $i, i+1, i+2, i+3$  belong to  $I$ .

For suppose that  $1, 2, 3, 4 \in I$  say, and choose  $n_i \in N_i$  for  $1 \leq i \leq 4$ . By 5.2, either  $n_1, n_2$  are

adjacent or  $n_3, n_4$  are adjacent, and we may assume the first by the symmetry. Since  $N$  is triangle-free,  $\{n_1, n_2, n_3\}$  is not a triangle, and so (S4) implies that  $n_2, n_3$  are nonadjacent. By 5.2  $W_1$  is complete to  $W_7$ , so by 5.1  $W_2$  is complete to  $W_7$ ; and so  $N_7 = \emptyset$  since  $N$  is triangle-free; and by 5.2 again,  $W_4$  is complete to  $W_5$ . Choose  $w_7 \in W_7$  adjacent to  $n_2$ ; and choose  $n_5 \in W_5$  and  $w_6 \in W_6$ , both adjacent to  $w_7$ . By 5.3,  $n_3, n_5$  are adjacent, and since  $n_3-n_5-w_7-n_2$  is not a  $v$ -path, it follows that  $n_5 \in N_5$ . Since  $N_2, N_3, N_4, N_5 \neq \emptyset$ , the argument earlier in this paragraph implies that  $n_3, n_4$  are nonadjacent, and  $N_6 = \emptyset$ . Now  $n_3$  is nonadjacent to both  $n_2, n_4$ , and so (S5) implies that  $n_2, n_4$  are nonadjacent. By 5.3,  $n_4-w_6-w_7-n_2$  is a  $v$ -path, a contradiction. This proves (1).

(2)  $|I| \leq 4$ .

For (1) implies that  $|I| \leq 5$ ; suppose that  $|I| = 5$ . From (1) again we may assume that  $I = \{1, 2, 4, 5, 7\}$ . Choose  $n_1 \in N_1$ . If  $n_1$  has a neighbour in  $N_2$  and one in  $N_7$ , then by (S4) there is a triangle in  $N$ , a contradiction. Thus we may assume that  $n_1$  is anticomplete to  $N_2$ . By 5.2,  $W_3$  is complete to  $W_4$ , and  $W_6$  to  $W_7$ . Choose  $n_2 \in N_2$ . If  $n_2$  has a neighbour  $w_1 \in M_1$ , then since  $W_1$  is complete to  $W_6$  by 5.1, there is a  $v$ -path of the form  $n_2-w_1-W_6-n_1$ , a contradiction. This proves that  $n_2$  is anticomplete to  $M_1$ . Choose  $n'_1 \in W_1$  adjacent to  $n_2$ ; it follows that  $n'_1 \in N_1$ . Since  $n'_1 \in N_1$  and has a neighbour in  $N_2$ , it follows from our previous argument that  $n'_1$  is anticomplete to  $N_7$ . By 5.2,  $W_2$  is complete to  $W_3$ , and  $W_5$  to  $W_6$ . Choose  $n_7 \in N_7$ . Now  $n_1$  has a neighbour in  $W_2$ , necessarily in  $M_2$ ; let  $w_2$  be such a neighbour. Similarly let  $w_7 \in M_7$  be adjacent to  $n'_1$ . Choose  $n_4 \in N_4$ . If  $n_4$  is anticomplete to  $N_5$ , then since  $W_5$  is complete to  $W_7$  by 5.1, and  $n_4$  has a neighbour (say  $w_5$ ) in  $W_5$ ,  $n_4-w_5-w_7-n_5$  is a  $v$ -path (where  $n_5 \in N_5$ ), a contradiction. Thus we may choose  $n_5 \in N_5$  adjacent to  $n_4$ . Choose  $w_3 \in W_3$  and  $w_6 \in W_6$ . Now  $n_2, w_7$  are adjacent by (S4). If  $n_2, n_7$  are nonadjacent, then  $n_2-w_7-w_6-n_7$  is a  $v$ -path, a contradiction. Thus  $n_2, n_7$  are adjacent, and so by (S5),  $n_1, n_7$  are adjacent. By (S4),  $n_7, w_2$  are adjacent. By (S5),  $n'_1, w_2$  are adjacent, and similarly  $n_1, w_7$  are adjacent. By (S4),  $w_7, w_2$  are adjacent. But then the subgraph induced on

$$\{v, w_3, w_7, n_7, n_4, n'_1, n_1, n_5, n_2, w_2, w_6\}$$

is isomorphic to  $T_{11}$  (and these eleven vertices are written in the appropriate order), a contradiction. This proves (2).

(3)  $|I| \leq 3$ .

For suppose not; then  $|I| = 4$  by (2), and we may assume that  $1, 4 \in I$ . By 5.3, there is a path of the form  $N_1-W_7-W_5-N_4$ . Since this is not a  $v$ -path, it follows that one of  $N_5, N_7 \neq \emptyset$ , and from the symmetry we may assume that  $5 \in I$ . Suppose that  $6 \in I$ , and so  $I = \{1, 4, 5, 6\}$ . If  $N_4$  is not complete to  $N_5$  there is a  $v$ -path of the form  $N_5-W_7-W_2-N_4$ , a contradiction, so  $N_4$  is complete to  $N_5$ . Choose  $n_6 \in N_6$ . Since  $N_4$  is complete to  $N_5$  and  $N$  is triangle-free, it follows from (S4) that  $n_6$  has no neighbour in  $N_5$ ; and consequently  $n_6$  is adjacent to some  $w_5 \in M_5$ . But then by 5.3 there is a  $v$ -path of the form  $N_5-W_3-w_5-n_6$ , a contradiction. This proves that  $6 \notin I$ , and similarly  $3 \notin I$ , and so from the symmetry we may assume that  $2 \in I$ , and therefore  $I = \{1, 2, 4, 5\}$ .

In this case we will show that we can add  $v$  to  $W_3$ , forming a heptagram  $W'$ , contrary to the maximality of  $W$ . Define  $W'_i = W_i$  for  $1 \leq i \leq 7$  with  $i \neq 3$ , and  $W'_3 = W_3 \cup \{v\}$ ; and let  $W' = (W'_1, \dots, W'_7)$ . We must check that  $W'$  satisfies (S1)–(S6). The first three are clear. Since  $W$

satisfies (S4)–(S6), in order to check that  $W'$  satisfies (S4)–(S6), it suffices from the symmetry to show that:

1.  $N_2$  is complete to  $N_4$
2.  $N_4$  is anticomplete to  $M_5$
3.  $M_2$  is anticomplete to  $M_4$
4.  $M_4$  is complete to  $N_5$
5. if  $M_2 \neq \emptyset$  then  $N_4$  is complete to  $N_5$
6. every vertex in  $W_6$  is either anticomplete to  $M_4$  or complete to  $N_5$ .

Let us prove these statements. For the first, if  $n_2 \in N_2$  and  $n_4 \in N_4$  are nonadjacent, choose  $w_i \in W_i$  for  $i = 6, 7$ , adjacent; then by 5.2,  $n_4, w_6$  are adjacent and so are  $n_2, w_7$ , and therefore  $n_4-w_6-w_7-n_2$  is a  $v$ -path, a contradiction.

For the second, suppose that  $n_4 \in N_4$  is adjacent to  $w_5 \in M_5$ . Choose  $n_1 \in N_1$  and  $w_7 \in W_7$  adjacent to both  $n_1, w_5$  (this is possible by 5.3); then  $n_4-w_5-w_7-n_1$  is a  $v$ -path, a contradiction.

For the third statement, suppose that  $w_2 \in M_2$  and  $w_4 \in M_4$  are adjacent. Choose  $n_1 \in N_1$  and  $n_5 \in N_5$ . Since  $n_1-w_2-w_4-n_5$  is not a  $v$ -path, we may assume that  $n_1, w_2$  are nonadjacent, and indeed  $w_2$  has no neighbour in  $N_1$ . Choose  $w_1 \in W_1$  adjacent to  $w_2$  (necessarily in  $M_1$ ), and choose  $w_7 \in W_7$  adjacent to  $w_1$ . By (S4),  $w_2, w_7$  are adjacent, and by (S5),  $n_1, w_7$  are adjacent. Choose  $n_4 \in N_4$ ; by 5.3,  $n_4, w_2$  are adjacent, since  $w_2, n_1$  are not adjacent. But then  $n_1-w_7-w_2-n_4$  is a  $v$ -path, a contradiction.

For the fourth statement, suppose that  $w_4 \in M_4$  and  $n_5 \in N_5$  are nonadjacent. Choose  $w_6 \in W_6$  adjacent to  $w_4$ ; then (S5) implies that  $n_5, w_6$  are adjacent. Choose  $n_2 \in N_2$ ; by 5.3,  $n_2, w_4$  are adjacent. But then  $n_2-w_4-w_6-n_5$  is a  $v$ -path, a contradiction.

For the fifth statement, suppose that  $w_2 \in M_2$ ,  $n_4 \in N_4$  and  $n_5 \in N_5$ , where  $n_4, n_5$  are nonadjacent. By 5.3,  $w_2, n_4$  are adjacent. By 5.3, there exists  $w_7 \in W_7$  adjacent to both  $w_2, n_5$ ; but then  $n_4-w_2-w_7-n_5$  is a  $v$ -path, a contradiction.

Finally, for the last statement, suppose that  $w_6 \in W_6$  is adjacent to  $w_4 \in M_4$  and nonadjacent to  $n_5 \in N_5$ . Choose  $n_1 \in N_1$ . By (S5),  $n_5, w_4$  are adjacent, and by 5.3,  $w_6, n_1$  are adjacent; but then  $n_1-w_6-w_4-n_5$  is a  $v$ -path, a contradiction.

This proves that  $W'$  is a heptagram, contrary to the maximality of  $W$ . This completes the proof of (3).

(4) *If  $|I| = 3$  then the first outcome of the theorem holds.*

For suppose first that  $I = \{1, 2, 3\}$ , and choose  $n_i \in N_i$  for  $i = 1, 2, 3$ . Since  $N$  is triangle-free, we may assume from (S4) that  $n_1, n_2$  are nonadjacent. Choose  $w_4 \in W_4$  and  $w_6 \in W_6$ , adjacent; then by 5.3,  $n_2-w_4-w_6-n_1$  is a  $v$ -path, a contradiction.

Thus  $I$  does not consist of three consecutive integers (modulo seven), and so we may assume that  $1, 4 \in I$ . Since there is no  $v$ -path of the form  $N_4-W_5-W_7-N_1$ , 5.3 implies one of  $N_5, N_7$  is nonempty, and from the symmetry we may assume that the former. Thus  $I = \{1, 4, 5\}$ . By the same argument,  $N_4$  is anticomplete to  $M_5$ , and  $N_5$  is anticomplete to  $M_4$ . If  $N_4$  is not complete to  $N_5$ , 5.3 implies that

there is a  $v$ -path of the form  $N_5-W_7-W_2-N_4$ , a contradiction. Thus  $N_4$  is complete to  $N_5$ . Suppose that  $N_1$  is not complete to  $W_2$ , and choose  $n_1 \in N_1$  and  $w_2 \in W_2$ , nonadjacent. Choose  $w_7 \in W_7$  adjacent to  $w_2$ ; then (S5) implies that  $n_1, w_7$  are adjacent. But by 5.3,  $w_2, n_4$  are adjacent, and so  $n_1-w_7-w_2-n_4$  is a  $v$ -path, a contradiction. Thus  $N_1$  is complete to  $W_2$  and therefore to  $W_3$ , by (S4). Similarly  $N_1$  is complete to  $W_7, W_6$ . But then  $v$  is a Y-vertex of type 1, and the first statement of the theorem holds. This proves (4).

(5) If  $|I| = 2$  then the second outcome of the theorem holds.

For then we may assume that  $I = \{1, t\}$  where  $t \in \{2, 3, 4\}$ . If  $t = 4$ , there is a  $v$ -path of the form  $N_4-W_5-W_7-N_1$ , a contradiction. Thus  $t \in \{2, 3\}$ . Suppose there exist  $n_1 \in N_1$  and  $n_t \in N_t$ , nonadjacent. Choose  $w_6 \in W_6$  adjacent to  $n_1$ . By 5.3, there exists  $w_4 \in W_4$  adjacent to both  $n_t, w_6$ ; but then  $n_1-w_6-w_4-n_t$  is a  $v$ -path, a contradiction. Thus  $N_1$  is complete to  $N_t$  and the second outcome of the theorem holds. This proves (5).

From (2)–(5), we may assume that  $|I| \leq 1$ ; but then the second outcome of the theorem holds. This proves 6.1. ■

## 7 V-vertices

Let  $1 \leq t \leq 7$ . A *tail*, or *tail of type  $t$* , is an induced path  $v_1-\cdots-v_k$  with the following properties:

- $k \geq 1$  is odd, and  $v_1, \dots, v_k \in V(G) \setminus W$
- $v_1$  has a neighbour in  $W_{t-3}$  and a neighbour in  $W_{t+3}$ , and  $W_{t-3}, W_{t+3}$  are anticomplete to  $\{v_2, \dots, v_k\}$
- $W_{t-1}, W_{t+1}$  and at least one of  $W_{t-2}, W_{t+2}$  are anticomplete to  $\{v_1, \dots, v_k\}$
- $v_k$  has a neighbour in  $W_t$ , and  $W_t$  is anticomplete to  $\{v_1, \dots, v_{k-1}\}$
- for  $j = t-3, t+3$  let  $N_j$  be the set of neighbours of  $v_1$  in  $W_j$ ; then  $N_{t-3}$  is complete to  $N_{t+3}$ ,  $N_{t-3}$  is anticomplete to  $W_{t+3} \setminus N_{t+3}$ , and  $N_{t+3}$  is anticomplete to  $W_{t-3} \setminus N_{t-3}$
- every neighbour of  $v_k$  in  $W_t$  is complete to each of  $W_{t-2}, W_{t-1}, W_{t+1}, W_{t+2}$ .

We see that every Y-vertex forms a 1-vertex path that is a tail of length zero, and for every tail of length zero, its unique vertex is a Y-vertex, by 6.1, and so we may regard tails as a generalization of Y-vertices. If  $v_1-\cdots-v_k$  is a tail, we say it is a *tail for  $v_1$* . If  $1 \leq t \leq 7$ , a vertex  $v \in V(G) \setminus W$  with neighbours in  $W_{t-3}$  and in  $W_{t+3}$ , and anticomplete to  $W_j$  for  $j = t-2, t-1, t, t+1, t+2$ , is called a *hat of type  $t$* . If  $v_1, \dots, v_k$  is a tail of type  $t$ , and has length greater than zero, then  $v_1$  is a hat of type  $t$ . We say a vertex  $v \in V(G) \setminus W$  is a *V-vertex of type  $t$*  if there is a tail of type  $t$  for  $v$ . Thus, every V-vertex of type  $t$  is either a Y-vertex of type  $t$  or a hat of type  $t$ .

Before we go on, let us give some idea where we are going. If every vertex in  $V(G) \setminus W$  is a V-vertex, then since every tail only contains one V-vertex it follows that every tail has length zero, and so every vertex in  $V(G) \setminus W$  is a Y-vertex, and we shall deduce that the graph is of heptagram

type. On the other hand, if some vertex in  $V(G) \setminus W$  is not a V-vertex, we shall prove that  $G$  admits a harmonious cutset.

If  $X \subseteq V(G)$ , we define  $N(X)$  to be the set of vertices in  $V(G) \setminus X$  with a neighbour in  $X$ . Here is a nice property of tails:

**7.1** *Let  $X \subseteq V(G) \setminus W$ , such that  $G|X$  is connected and contains no tail of  $G$ . Then there exists  $i \in \{1, \dots, 7\}$  such that  $N(X) \cap W \subseteq W_{i-1} \cup W_i \cup W_{i+1}$ .*

**Proof.** Suppose this is false, and choose a minimal counterexample  $X$ . Consequently there exists  $i \in \{1, \dots, 7\}$  such that  $N_i, N_{i+3}$  are both not anticomplete to  $X$ , and we may therefore assume that  $N(X) \cap W_1, N(X) \cap W_4 \neq \emptyset$ . Choose a minimal path from  $W_4$  to  $W_1$  with interior in  $X$ , say  $n_4-v_1-\dots-v_k-n_1$ . From the minimality of  $X$ , it follows that  $X = \{v_1, \dots, v_k\}$ , and from 6.1 it follows that  $k > 1$ . From the minimality of  $X$ ,  $W_1$  is anticomplete to  $\{v_1, \dots, v_{k-1}\}$ , and  $W_4$  is anticomplete to  $\{v_2, \dots, v_k\}$ . Suppose first that  $k$  is even. Then by 5.3,  $n_1, n_4$  have a common neighbour  $w_j \in W_j$  for  $j = 2, 3, 6$ , and since  $G$  has no odd hole, it follows that  $w_2, w_3, w_6$  each are adjacent to one of  $v_1, \dots, v_k$ . But each of  $v_1, v_k$  is nonadjacent to one of  $w_2, w_3$ , by 6.1, and so one of  $w_2, w_3$  is joined to  $w_6$  by a path with interior a proper subpath of  $v_1, \dots, v_k$ , contrary to the minimality of  $X$ . This proves that  $k$  is odd. Since there is no odd hole of the form

$$n_4-v_1-\dots-v_k-n_1-W_7-W_5-n_4,$$

it follows that some vertex of  $W_5 \cup W_7$  is adjacent to one of  $v_1, \dots, v_k$ , and from the symmetry we may assume this vertex is in  $W_5$ . From the minimality of  $X$ ,  $\{v_2, \dots, v_k\}$  is anticomplete to  $W_5$ , and so  $v_1$  has a neighbour in  $W_5$ . By 6.1, and since  $G|X$  contains no tail of  $G$  and hence  $X$  contains no Y-vertex, it follows that  $v_1$  is a hat of type 1. We will prove that  $v_1, \dots, v_k$  is a tail.

From the minimality of  $|X|$ ,  $W_2$  and  $W_7$  are both anticomplete to  $\{v_1, \dots, v_{k-1}\}$ . Suppose that  $v_k$  has a neighbour  $n_2 \in W_2$  say. Then by 6.1,  $v_k$  is a hat of type 5, and so  $W_7$  is anticomplete to  $X$ , and the minimality of  $X$  implies that  $W_6$  is anticomplete to  $X$ . If  $n_2, n_4$  are adjacent then  $n_4-v_1-\dots-v_k-n_2-n_4$  is an odd hole, and if  $n_2, n_4$  are nonadjacent then there is an odd hole of the form

$$n_4-v_1-\dots-v_k-n_2-W_7-W_6-n_4,$$

in either case a contradiction. This proves that  $v_k$  has no neighbour in  $W_2$ , and so  $X$  is anticomplete to  $W_2$ , and similarly to  $W_7$ . Now  $v_1$  is anticomplete to both  $W_3, W_6$ , and from the minimality of  $X$ , at least one of  $W_3, W_6$  is anticomplete to  $X \setminus \{v_1\}$ , and so at least one of  $W_3, W_6$  is anticomplete to  $X$ . We have therefore verified that  $v_1, \dots, v_k$  satisfies the first four conditions in the definition of a tail.

To verify the fifth condition, let  $N_i$  be the set of neighbours of  $v_1$  in  $W_i$  for  $i = 4, 5$ . By 6.1,  $N_4$  is complete to  $N_5$ . If  $w_4 \in N_4$  is adjacent to some  $w_5 \in W_5 \setminus N_5$ , then there is an odd hole of the form

$$w_4-v_1-\dots-v_k-n_1-W_7-w_5-w_4,$$

a contradiction. Similarly  $N_5$  is anticomplete to  $W_4 \setminus N_4$ , and this verifies the fifth condition.

To verify the sixth and last condition, let  $w_1 \in W_1$  be adjacent to  $v_k$ . If  $w_1$  is nonadjacent to some  $w_2 \in W_2$ , choose  $w_7 \in W_7$  adjacent to  $w_2$ ; then (S5) implies that  $w_1, w_7$  are adjacent, and so by 5.3 there is an odd hole

$$n_4-v_1-\dots-v_k-w_1-w_7-w_2-n_4,$$

a contradiction. Thus  $w_1$  is complete to  $W_2$ , and therefore to  $W_3$  by (S4), and similarly to  $W_7, W_6$ . This verifies the sixth condition.

Consequently  $v_1, \dots, v_k$  is a tail in  $G|X$ , a contradiction. Thus there is no such  $X$ . This proves 7.1. ■

**7.2** *Let  $U$  be the set of all vertices in  $V(G) \setminus W$  that are not V-vertices. For  $1 \leq t \leq 7$ , there is no path  $x_1 \cdots x_k$  in  $G$  satisfying the following:*

- $x_1$  is either a hat or Y-vertex of type  $t$
- $x_2, \dots, x_{k-1} \in U$
- $x_k \in V(G) \setminus W$  has a neighbour in  $W_{t+1} \cup W_{t-1}$ , and
- $x_k$  is not a Y-vertex of type  $t+1$  or  $t-1$ .

**Proof.** For suppose there is, and choose  $k$  minimum such that for some  $t$  there is such a path. We may assume that  $t = 1$ , and  $x_1$  is either a hat or a Y-vertex of type 1, and  $x_k \in V(G) \setminus W$  has a neighbour in  $W_2$ , and  $x_2, \dots, x_{k-1} \in U$ , and  $x_k$  is not a Y-vertex of type 2 or 7. Let  $X = \{x_1, \dots, x_k\}$ . From the minimality of  $k$ ,  $W_2, W_7$  are both anticomplete to  $X \setminus \{x_k\}$ . Choose  $w_2 \in W_2$  adjacent to  $x_k$ . Choose  $w_4 \in W_4$  adjacent to  $x_1$ , and also adjacent to  $w_2$  if possible. We claim that if  $x_1$  is a V-vertex, then  $w_2, w_4$  are adjacent; for if  $W_4$  is complete to  $W_5$  then  $x_1$  is complete to  $W_4$  (since  $x_1$  is a V-vertex), and if  $W_4$  is not complete to  $W_5$  then  $W_4$  is complete to  $W_2$  by 5.3. In either case it follows that  $w_2, w_4$  are adjacent.

(1)  $G|X$  contains a tail for  $x_k$  and a tail for  $x_1$ , and in particular  $x_1$  and  $x_k$  are V-vertices.

For suppose it contains no tail for  $x_k$ . By 7.1 applied to  $X \setminus \{x_1\}$  we deduce that  $W_5, W_6$  are anticomplete to  $X \setminus \{x_1\}$ . From 7.1,  $G|X$  contains a tail of  $G$ , and since  $X$  contains no V-vertex except possibly  $x_1$  and  $x_k$ , we may assume that  $G|X$  contains a tail for  $x_1$ . Thus  $x_1$  is a V-vertex, and so  $w_2, w_4$  are adjacent. Moreover, there exists  $j \leq k$  such that  $x_1 \cdots x_j$  is a tail for  $x_1$ . In particular,  $W_2$  is anticomplete to  $\{x_1, \dots, x_j\}$ , and so  $j < k$ .

Suppose that  $k$  is even. Since there is no odd hole of the form

$$x_1 \cdots x_k - w_2 - W_7 - W_5 - x_1,$$

it follows that  $x_k$  has a neighbour  $w_7 \in W_7$ . But then  $W_4$  is anticomplete to  $X \setminus \{x_1\}$  by 7.1, and so there is an odd hole of the form

$$x_1 \cdots x_k - w_7 - W_6 - W_4 - x_1,$$

a contradiction.

Thus  $k$  is odd. Since  $x_1 \cdots x_k - w_2 - w_4 - x_1$  is not an odd hole, we deduce that  $w_4$  has a neighbour in  $X \setminus \{x_1\}$ . From 7.1 applied to  $X \setminus \{x_1\}$ , we deduce that  $W_1, W_7$  are anticomplete to  $X \setminus \{x_1\}$ , and therefore  $j = 1$ , and so  $x_1$  is a Y-vertex. Choose  $w_1 \in W_1$  adjacent to  $x_1$ . Then  $w_1$  is complete to  $W_2$  from the definition of a Y-vertex, and in particular  $w_1, w_2$  are adjacent. But then  $x_1 \cdots x_k - w_2 - w_1 - x_1$  is an odd hole, a contradiction.

This proves that  $G|X$  contains a tail for  $x_k$ . In particular,  $x_k$  is either a hat or Y-vertex of type  $s$  say, where  $s = 5$  or  $6$ , and  $x_1$  has a neighbour in  $W_{s-1}$ . Thus there is symmetry between  $x_1$  and  $x_k$ , and since we have shown that  $G|X$  contains a tail for  $x_k$ , it follows that it also contains a tail for  $x_1$ . This proves (1).

(2)  $x_k$  is not a V-vertex of type 6.

For suppose it is; then it has neighbours in  $W_3$ . From the minimality of  $k$ ,  $W_7$  is anticomplete to  $X$ , and  $W_2$  is anticomplete to  $X \setminus \{x_k\}$ , and  $W_5$  is anticomplete to  $X \setminus \{x_1\}$ . Since there is no odd hole of the form

$$x_1 - \cdots - x_k - w_2 - W_7 - W_5 - x_1,$$

it follows that  $k$  is odd. Since  $w_4 - x_1 - \cdots - x_k - w_2 - w_4$  is not an odd hole, it follows that  $w_4$  has a neighbour in  $X \setminus \{x_1, x_k\}$ . By 7.1 applied to  $X \setminus \{x_1, x_k\}$ , it follows that  $W_1$  is anticomplete to  $X \setminus \{x_1, x_k\}$ . But by (1), some vertex  $w_1 \in W_1$  has a neighbour in a tail for  $x_1$  contained in  $x_1 - \cdots - x_k$ ;  $w_1$  is not adjacent to  $x_k$  since  $x_k$  is a V-vertex of type 6; and so  $w_1$  is adjacent to  $x_1$  and to none of  $x_2, \dots, x_k$ . Since  $x_1$  is a V-vertex,  $w_1$  is complete to  $W_2$  and in particular adjacent to  $w_2$ . But then  $w_1 - x_1 - \cdots - x_k - w_2 - w_1$  is an odd hole, a contradiction. This proves (2).

(3)  $x_k$  is not a V-vertex of type 5.

For suppose it is, and so it has neighbours in  $W_1$ . By the minimality of  $k$ ,  $W_4, W_6$  are both anticomplete to  $X \setminus \{x_1\}$ . From the hole  $x_1 - \cdots - x_k - w_2 - w_4 - x_1$  we deduce that  $k$  is even. Choose  $w_5 \in W_5$  adjacent to  $x_1$ , and  $w_1 \in W_1$  adjacent to  $x_k$ . There is no odd hole of the form

$$x_1 - \cdots - x_k - w_2 - W_7 - w_5 - x_1,$$

and so  $w_5$  is not anticomplete to  $X \setminus \{x_1\}$ . Similarly  $w_1$  is not anticomplete to  $X \setminus \{x_k\}$ . By 7.1 applied to  $X \setminus \{x_1, x_k\}$ , not both  $w_1, w_5$  have neighbours in  $X \setminus \{x_1, x_k\}$ ; so from the symmetry we may assume that  $w_1$  is adjacent to  $x_1$  and not to  $x_2, \dots, x_{k-1}$ . In particular  $x_1$  is a Y-vertex. Since  $x_1 - \cdots - x_k - w_1 - x_1$  is not an odd hole, it follows that  $k = 2$ , and so  $w_5$  is adjacent to  $x_2$ ; and therefore  $x_2$  is also a Y-vertex.

Since  $x_1$  is a Y-vertex, it has a neighbour in  $W_1$  that is complete to  $W_2$ , and therefore  $G|(W_1 \cup W_2)$  is connected. Since  $x_2$  is a Y-vertex of type 5, its set of neighbours in  $W_1 \cup W_2$  is the vertex set of a component of  $G|(W_1 \cup W_2)$ ; and consequently  $x_2$  is complete to  $W_1 \cup W_2$ , and  $W_1$  is complete to  $W_2$ . Similarly  $x_1$  is complete to  $W_4 \cup W_5$  and  $W_4$  is complete to  $W_5$ . We claim that  $x_1$  is complete to  $W_1$ . For suppose that  $x_1$  is nonadjacent to some  $w_1 \in W_1$ . Then there is an odd hole of the form

$$x_1 - x_2 - w_1 - W_3 - w_4 - x_1,$$

a contradiction. This proves that  $x_1$  is complete to  $W_1$ , and similarly  $x_2$  is complete to  $W_5$ .

Define  $W'_6 = W_6 \cup \{x_1\}$ , and  $W'_7 = W_7 \cup \{x_2\}$ , and let  $W' = (W_1, \dots, W_5, W'_6, W'_7)$ . We claim that  $W'$  is a heptagram. We must check (S1)–(S6), but they are all obvious and we leave this to the reader. Thus  $W'$  is a heptagram, contrary to the maximality of  $W$ . This proves (3).

Since  $x_k$  is a V-vertex with a neighbour in  $W_2$ , and is not a Y-vertex of type 2, (1)–(3) are contradictory. Consequently there is no such path  $x_1, \dots, x_k$ . This proves 7.2. ■

We conclude this section with some more lemmas about V-vertices.

**7.3** For  $1 \leq i \leq 7$ , no two V-vertices of type  $i$  are adjacent.

**Proof.** Suppose that  $a, b$  are adjacent V-vertices of type 5 say. For  $j = 1, 2$ , let  $A_j, B_j$  be the set of neighbours in  $W_j$  of  $a, b$  respectively. Since  $G$  is  $K_4$ -free, and  $A_1$  is complete to  $A_2$ , it follows that  $A_1 \cup A_2 \neq B_1 \cup B_2$ . Since  $A_1 \cup A_2$  and  $B_1 \cup B_2$  are both vertex sets of components of  $G|(W_1 \cup W_2)$ , we deduce that  $A_j \cap B_j = \emptyset$  for  $j = 1, 2$ . Since  $G$  is  $K_4$ -free, and  $A_1$  is complete to  $A_2$ , some vertex of  $A_1 \cup A_2$  is not adjacent to  $b$ , and so  $A_j \cap B_j = \emptyset$  for  $j = 1, 2$ . In particular,  $W_1$  is not complete to  $W_2$ , and so  $W_1$  is complete to  $W_6$  by 5.3. Choose  $a_1 \in A_1, b_1 \in B_1$ , and  $w_6 \in W_6$ . Then  $w_6-a_1-a-b-b_1-w_6$  is a hole of length five, a contradiction. This proves 7.3.  $\blacksquare$

**7.4** For  $1 \leq i \leq 7$ , if  $a$  is a V-vertex of type  $i$ , and  $a$  is not complete to  $W_{i-3} \cup W_{i+3}$ , then  $W_{i-2} \cup W_{i+2}$  is complete to  $W_{i-3} \cup W_{i+3}$ .

**Proof.** We may assume that  $i = 5$  say. For  $j = 1, 2$ , let  $N_j$  be the set of neighbours of  $a$  in  $W_j$ , and let  $M_j = W_i \setminus N_j$ . Thus  $N_1$  is complete to  $N_2$ , and  $N_1$  is anticomplete to  $M_2$ , and  $M_1$  is anticomplete to  $N_2$ . By hypothesis  $M_1 \cup M_2 \neq \emptyset$ , and since each member of  $M_1$  has a neighbour in  $W_2$  (and therefore in  $M_2$ ), and vice versa, it follows that  $M_1, M_2 \neq \emptyset$ . Let  $w_3 \in W_3$ ; we will show that  $w_3$  is complete to  $W_1 \cup W_2$ . Suppose first that  $w_3$  is anticomplete to  $M_1$ . Then  $w_3$  has a neighbour in  $N_1$ , and so by (S5),  $w_3$  is complete to  $M_2$ . Yet  $w_3$  is anticomplete to  $M_1$ , and every vertex in  $M_2$  has a neighbour in  $M_1$ , contrary to (S4). This proves that  $w_3$  has a neighbour in  $M_1$ , say  $m_1$ . By (S5), since  $m_1$  is anticomplete to  $N_2$ , it follows that  $w_3$  is complete to  $N_2$ , and consequently complete to  $N_1$ , by (S4). Choose  $n_1 \in N_1$ ; then since  $n_1$  is anticomplete to  $M_2$ , (S5) implies that  $w_3$  is complete to  $M_2$ , and hence to  $M_1$ , by (S4). This proves our claim that  $w_3$  is complete to  $W_1 \cup W_2$ . We deduce that  $W_3$  is complete to  $W_1 \cup W_2$ , and similarly so is  $W_7$ . This proves 7.4.  $\blacksquare$

**7.5** For  $1 \leq i \leq 7$ , if  $a$  is a V-vertex of type  $i$ , and  $b$  is a V-vertex of type  $i + 1$ , then  $a, b$  are adjacent, and both are complete to  $W_{i-3}$ .

**Proof.** We may assume that  $i = 5$ , say. Let  $a, b$  be V-vertices of types 5 and 6 respectively, and let their tails be  $S, T$  respectively. For  $j = 1, 2$ , let  $A_j$  be the set of neighbours of  $a$  in  $W_j$ , and for  $j = 2, 3$ , let  $B_j$  be the set of neighbours of  $b$  in  $W_j$ . By 7.4, at least one of  $a, b$  is complete to  $W_2$ .

(1)  $a, b$  are adjacent.

For suppose  $a, b$  are nonadjacent. Since at least one of  $a, b$  is complete to  $W_2$ , they have a common neighbour  $w_2 \in W_2$ . Suppose first that  $S, T$  are disjoint and there is no edge between them. Then there is an induced path  $Q$  of odd length between  $a, b$  of the form

$$a-S-W_5-W_6-T-b,$$

and we can complete it to an odd hole via  $b-w_2-a$  (note that  $w_2$  has no neighbours in  $S \cup T$  except  $a, b$ ), a contradiction. Thus  $V(S) \cup V(T)$  induces a connected subgraph of  $G$ .

Now by 7.2,  $a$  is anticomplete to  $V(T) \setminus \{b\}$  and hence to  $V(T)$ , and similarly  $b$  is anticomplete to  $V(S)$ . Let  $X = V(S) \cup V(T) \setminus \{a, b\}$ . Since  $V(S) \cup V(T)$  induces a connected subgraph of  $G$ , it

follows that  $S, T$  both have positive length and  $G|X$  is connected. Since  $X$  contains no V-vertex, and  $N(X)$  has nonempty intersection with  $W_5, W_6$ , 7.2 implies that  $W_1, W_3$  have no neighbours in  $X$ . Choose  $a_1 \in A_1$ , and  $b_3 \in B_3$ . Since  $w_2$  is adjacent to  $a_1, b_3$ , (S4) implies that  $a_1, b_3$  are adjacent. But there is an induced path  $Q$  between  $a, b$  with interior in  $X$ , and it can be completed to holes via  $b-w_2-a$  and via  $b-b_3-a_1-a$ , and one of these is odd, a contradiction. This proves (1).

Suppose there exists  $a_2 \in W_2 \setminus B_2$ , say. Thus  $b$  is not complete to  $W_2$ , and so by 7.4,  $a$  is complete to  $W_1 \cup W_2$ , and in particular  $a_2 \in A_2$ . Choose  $b_3 \in B_3$ ; then  $a_2, b_3$  are nonadjacent since  $b$  is a V-vertex. Choose  $w_4 \in W_4$  adjacent to  $a_2$  and therefore to  $b_3$ , by (S5). Then  $a-b-b_3-w_4-a_2-a$  is a hole of length five, a contradiction. This proves that  $B_2 = W_2$ , and similarly  $A_2 = W_2$ , and hence proves 7.5.  $\blacksquare$

**7.6** For  $1 \leq i \leq 7$ , if  $a$  is a V-vertex of type  $i$ , and  $a$  is not complete to  $W_{i-3} \cup W_{i+3}$ , then there is no V-vertex of type  $j$  for  $j \in \{i-3, i-1, i+1, i+3\}$ .

**Proof.** We may assume that  $i = 5$ . By 7.5, there is no V-vertex of type 6, since  $a$  is not complete to  $W_2$ . Similarly there is none of type 4. Since no vertex in  $W_1$  is complete to  $W_2$ , there is no V-vertex of type 1, and similarly there is none of type 2. This proves 7.6.  $\blacksquare$

## 8 Attachments of the remaining vertices

In this section we complete the proof of 3.1. The main part of this proof is the next result.

**8.1** Let  $U$  be the set of all vertices in  $V(G) \setminus W$  that are not V-vertices. If  $U \neq \emptyset$  then  $G$  admits a harmonious cutset.

**Proof.** Suppose that  $U \neq \emptyset$ , and let  $X \subseteq U$  be maximal such that  $G|X$  is connected. Thus  $X \neq \emptyset$ , and  $N(X) \subseteq V(G) \setminus U$ . For  $1 \leq i \leq 7$ , let  $N_i = N(X) \cap W_i$ , let  $V_i$  be the set of all V-vertices of type  $i$ , and let  $P_i = N(X) \cap V_i$ . Let  $I = \{i \in \{1, \dots, 7\} : N_i \neq \emptyset\}$  and  $J = \{i \in \{1, \dots, 7\} : P_i \neq \emptyset\}$ . By 7.1 there exists  $t$  such that  $I \subseteq \{t-1, t, t+1\}$  and by 7.2 there exists  $t$  such that  $J \subseteq \{t, t+1\}$ .

(1) If  $1 \leq i \leq 7$  and  $a, b \in N_i$  then there is an induced even path joining  $a, b$  with interior in  $X$ .

Let  $Q$  be an induced path between  $a, b$  with interior in  $X$ . We will prove that  $Q$  is even. Let  $a, b \in W_3$  say; thus  $6, 7 \notin I$  and not both  $1, 5 \in I$ . From the symmetry we may assume that  $1 \notin I$ . If  $a, b$  have a common neighbour  $w_1 \in W_1$  then the claim holds, since  $w_1-a-Q-b-w_1$  is an even hole, so we assume not; and therefore  $W_1$  is complete to  $W_7$ , by 5.3. Choose  $a', b' \in W_1$  adjacent to  $a, b$  respectively. Thus  $a, b'$  are nonadjacent, and  $a', b$  are nonadjacent. Choose  $w_7 \in W_7$ ; then  $w_7-b'-b-Q-a-a'-w_7$  is a hole, and so  $Q$  is even. This proves (1).

(2) For  $1 \leq i \leq 7$ ,  $N_i$  is complete to  $N_{i+1}$ .

For suppose that  $i = 1$  say, and  $n_1 \in N_1$  and  $n_2 \in N_2$  are nonadjacent. Let  $Q$  be an induced

path between  $n_1, n_2$  with interior in  $X$ . By 7.1,  $4, 6 \notin I$ , and not both  $3, 7 \in I$  and we may assume that  $3 \notin I$ . Choose  $w_3 \in W_3$  adjacent to  $n_1$ ; then (S5) implies that  $n_2, w_3$  are adjacent. From the hole  $w_3-n_1-Q-n_2-w_3$  we deduce that  $Q$  is even. But there is a hole of the form

$$n_1-Q-n_2-W_4-W_6-n_1,$$

and it is odd, a contradiction. This proves (2).

(3) For  $1 \leq i \leq 7$ , every two members of  $P_i$  have the same neighbours in  $W_{i-3} \cup W_{i+3}$ , and  $P_i$  is complete to  $N_{i-3} \cup N_{i+3}$ .

For we may assume that  $i = 5$ , say, and we may assume that  $P_5 \neq \emptyset$ . For  $j = 1, 2$  let  $R_j$  be the set of vertices in  $W_j$  with a neighbour in  $X \cup P_5$ . We claim first that  $R_1$  is complete to  $R_2$ . For suppose that  $r_1 \in R_1$  and  $r_2 \in R_2$  are nonadjacent, and let  $Q$  be a path joining  $r_1, r_2$  with interior in  $X \cup P_5$ . It follows from 7.2 (since  $P_5 \neq \emptyset$ ) that  $X \cup P_5$  is anticomplete to  $W_4, W_6$ , and (by 7.1) anticomplete to at least one of  $W_3, W_7$ , say  $W_7$ . Consequently  $Q$  can be completed to a hole via  $r_2-W_7-r_1$  and via  $r_2-W_4-W_6-r_1$ , and one of these is odd, a contradiction. This proves that  $R_1$  is complete to  $R_2$ . Since each  $p_5 \in P_5$  is a V-vertex, and therefore its neighbour set in  $W_1 \cup W_2$  is the vertex set of a component of  $G|(W_1 \cup W_2)$ , it follows that each  $p_5 \in P_5$  is complete to  $R_1 \cup R_2$ . This proves (3).

We wish to prove that  $G$  admits a harmonious cutset, and henceforth we assume (for a contradiction) that it does not.

(4)  $J \neq \emptyset$ .

For suppose that  $J = \emptyset$ ; and we may assume that  $I \subseteq \{1, 2, 3\}$ . By (2),  $N_1$  is complete to  $N_2$ , and  $N_2$  to  $N_3$ , so if  $N_2 \neq \emptyset$  then  $N_1$  is complete to  $N_3$  by (S4), and by (1) and 2.3 applied to the cutset  $N_1 \cup N_2 \cup N_3$ , we deduce that  $G$  admits a harmonious cutset, a contradiction. We may therefore assume that  $N_2 = \emptyset$ . Let  $n_1 \in N_1$  and  $n_3 \in N_3$  be nonadjacent; and let  $Q$  be a path between them with interior in  $X$ . By 5.3 there is a hole of the form  $n_1-Q-n_3-W_4-W_6-n_1$ , so  $Q$  is odd. Thus it again follows from (1) and 2.3 that  $G$  admits a harmonious cutset, a contradiction. This proves (4).

(5)  $I \cap J = \emptyset$ .

For suppose that  $5 \in I \cap J$  say. By 7.1,  $1, 2 \notin I$ . Since  $5 \in J$ , 7.2 implies that  $4, 6 \notin I$  and  $1, 2, 3, 7 \notin J$ . Since  $5 \in I$ , 7.2 implies that  $4, 6 \notin J$ . Consequently  $I \subseteq \{3, 5, 7\}$  and  $J = \{5\}$ . By 7.1 not both  $3, 7 \in I$ , so we may assume that  $I \subseteq \{3, 5\}$ . We claim that  $P_5 \cup N_3 \cup N_5$  is a harmonious cutset (where  $(P_5 \cup N_3, N_5)$  is the corresponding colouring). We must check:

- if  $a, b \in P_5 \cup N_3$  then there is an induced even path joining them with interior disjoint from  $P_5 \cup N_3 \cup N_5$
- if  $a, b \in N_5$  then there is an induced even path joining them with interior disjoint from  $P_5 \cup N_3 \cup N_5$

- if  $a \in P_5 \cup N_3$  and  $b \in N_5$  then there is an induced odd path joining them with interior disjoint from  $P_5 \cup N_3 \cup N_5$ .

For the first, if  $a, b \in N_3$  this follows from (1), so we may assume that  $a \in P_5$ . But then  $a, b$  have a common neighbour in  $W_2$  by 7.4 and (3), and so the claim follows since  $2 \notin I$ . The second follows from (1). For the third, let  $a \in P_5 \cup N_3$  and  $b \in N_5$ , and we may assume that  $a, b$  are nonadjacent; then there is an induced path of the form  $a-W_1-W_6-b$  satisfying the claim. Consequently, 2.3 implies that  $G$  admits a harmonious cutset, a contradiction. This proves (5).

In view of (5), since the same conclusion holds for every choice of  $X$ , we may therefore assume that every tail has length zero, and therefore every V-vertex is a Y-vertex.

(6) *There exists  $t \in \{1, \dots, 7\}$  such that  $I \subseteq \{t-1, t, t+1\}$  and  $J \subseteq \{t-3, t+3\}$ .*

For we may assume that  $5 \in J$  say. By (5),  $5 \notin I$ ; and by 7.2,  $4, 6 \notin I$ ; and not both  $3, 7 \in I$ , say  $7 \notin I$ . But 7.2 implies that  $7, 1, 2, 3 \notin J$ , and not both  $4, 6 \in J$ . If  $4 \notin J$  then the claim holds with  $t = 2$ , so we may assume that  $4 \in J$ . By 7.2,  $3 \notin I$ , and now the claim holds with  $t = 1$ . This proves (6).

In view of (6) we henceforth assume that  $I \subseteq \{1, 2, 3\}$  and  $J \subseteq \{5, 6\}$ . We claim that  $N(X)$  is a cutset satisfying the hypotheses of 2.3, with corresponding colouring  $(N_2, N_1 \cup P_6, N_3 \cup P_5)$ . Certainly it is a cutset, and the three sets  $N_2, N_1 \cup P_6, N_3 \cup P_5$  are pairwise complete, by (1), (3) and 7.5. It suffices therefore (by the symmetry) to show that

- if  $a, b \in N_2$  then they are joined by an even induced path with interior disjoint from  $N(X)$ , and
- if  $a, b \in N_1 \cup P_6$  then they are joined by an even induced path with interior disjoint from  $N(X)$ .

The first is proved in (1). For the second, if  $a, b \in N_1$ , then again the claim follows from (1). If  $a, b \in P_6$ , then since they both have neighbours in  $W_6$  that are complete to  $W_5$ , there is an induced path between  $a, b$  of length two or four with interior in  $W_5 \cup W_6$ , satisfying the claim. If  $a \in N_1$  and  $b \in P_6$ , then  $b$  has a neighbour  $w_6 \in W_6$  that is complete to  $W_1$ , and so the path  $a-w_6-b$  satisfies the claim. This completes the proof of the two displayed statements above. Consequently, by 2.3, we deduce that  $G$  admits a harmonious cutset, a contradiction. This proves 8.1. ■

Finally we can prove our main decomposition theorem.

**Proof of 3.1.** Let  $G$  be a  $K_4$ -free graph with no odd hole, and with no harmonious cutset, containing an antihole of length seven. By 4.1 we may assume that  $G$  is  $T_{11}$ -free. Choose a maximal heptagram  $W = (W_1, \dots, W_7)$ . By 8.1, every vertex of  $G$  either belongs to  $W$  or is a V-vertex; and, since a tail contains only one V-vertex, it follows that every tail has length zero and so every V-vertex is a Y-vertex. For  $1 \leq i \leq 7$  let  $Y_i$  be the set of all Y-vertices of type  $i$ . We need to check the ten conditions in the definition of heptagram type. The first is clear; and by 5.3 we may assume that the second and third hold by renumbering  $W_1, \dots, W_7$ . Conditions 4–7 are clear. For the eighth condition, we see from 7.2 that  $Y_i$  is anticomplete to  $Y_{i+2}, Y_{i+3}$ , and from 7.5 that  $Y_i$  is complete to  $Y_{i+1}$ . The ninth condition follows from 7.4 and 7.6. For the tenth condition, suppose that  $y_{i-1} \in Y_{i-1}$ , and

$y_i \in Y_i$ , and  $y_{i+1} \in Y_{i+1}$ . Thus  $y_i$  is adjacent to  $y_{i-1}, y_{i+1}$ , and  $y_{i-1}, y_{i+1}$  are nonadjacent. But then there is an odd hole of the form

$$y_i - y_{i+1} - W_{i+1} - W_{i-1} - y_{i-1} - y_i,$$

a contradiction. This proves 3.1. ■

## 9 A more explicit construction

We hesitate to claim that our current definition of graphs of heptagram type is an “explicit construction”; it is certainly a helpful description, but the way the various hypotheses interact is not transparent. In this section we make it more explicit.

Let us say that  $G$  is of the *first heptagram type* if there exist  $t \geq 1$  and a partition of  $V(G)$  into ten stable sets

$$W_1, \dots, W_7, Y_2, Y_4, Y_7$$

where  $Y_4, Y_7$  may be empty but the other sets are nonempty, such that, with index arithmetic modulo seven:

- for  $1 \leq i \leq 7$ ,  $W_i$  is complete to  $W_{i+2}$  and anticomplete to  $W_{i+3}$
- for  $i \in \{3, 4, 6, 7\}$ ,  $W_i$  is complete to  $W_{i+1}$ , and for  $i = 1, 2$ ,  $W_i, W_{i+1}$  are linked; and every vertex in  $W_2$  is complete to one of  $W_1, W_3$
- for  $i = 4, 7$ , every vertex in  $Y_i$  is complete to  $W_{i+3} \cup W_{i-3}$ , has a neighbour in  $W_i$ , and has no neighbour in  $W_{i+1}, W_{i+2}, W_{i-1}, W_{i-2}$
- $Y_2, Y_4, Y_7$  are pairwise anticomplete
- there is a nonempty subset  $C \subseteq W_2$  such that  $C$  is complete to  $W_1 \cup W_3$ , and  $Y_2, C$  are linked, and  $Y_2$  is anticomplete to  $W_2 \setminus C$
- there exist partitions  $M_0, \dots, M_t$  of  $W_5$  and  $N_0, \dots, N_t$  of  $W_6$  where  $M_0, N_0$  may be empty but the other sets are nonempty, such that for  $1 \leq i \leq t$ ,  $M_i$  is complete to  $N_i$ ,  $M_i$  is anticomplete to  $W_6 \setminus N_i$ ,  $W_5 \setminus M_i$  is anticomplete to  $N_i$ , and  $M_0, N_0$  are linked (and consequently  $W_5, W_6$  are linked)
- there is a partition  $X_1, \dots, X_t$  of  $Y_2$  where  $X_1, \dots, X_t$  are all nonempty, such that for  $1 \leq i \leq t$ ,  $X_i$  is complete to  $M_i \cup N_i$ , and anticomplete to each of

$$W_5 \setminus M_i, W_6 \setminus N_i, W_7, W_1, W_3, W_4.$$

That completes the definition of the first heptagram type. Before we define the second, we need another definition. Let us say a triple  $(W_1, W_2, W_3)$  of disjoint stable subsets of  $V(G)$  is a *crescent* in  $G$  if the following hold:

- if  $v_i \in W_i$  for  $i = 1, 2, 3$ , and  $v_2$  is adjacent to  $v_1, v_3$ , then  $v_1$  is adjacent to  $v_3$
- if  $v_i \in W_i$  for  $i = 1, 2, 3$ , and  $v_2$  is nonadjacent to  $v_1, v_3$ , then  $v_1$  is nonadjacent to  $v_3$ .

We say that  $G$  is of the *second heptagram type* if there is a partition of  $V(G)$  into fourteen stable subsets  $W_1, \dots, W_7, Y_1, \dots, Y_7$ , where  $W_1, \dots, W_7$  are nonempty but  $Y_1, \dots, Y_7$  may be empty, such that (with index arithmetic modulo 7)

- for  $1 \leq i \leq 7$ ,  $W_i$  is anticomplete to  $W_{i+3}$
- for  $2 \leq i \leq 7$ ,  $W_i$  is complete to  $W_{i+2}$ , and the sets  $W_1, W_2, W_3$  are pairwise linked
- $(W_1, W_2, W_3)$  is a crescent, and if  $W_1$  is not complete to  $W_3$  then  $Y_2, Y_5, Y_6 = \emptyset$
- for  $i \in \{3, 4, 6, 7\}$ ,  $W_i$  is complete to  $W_{i+1}$ ;  $W_5, W_6$  are linked
- for  $1 \leq i \leq 7$ , every vertex in  $Y_i$  is complete to  $W_{i+3} \cup W_{i-3}$ , has a neighbour in  $W_i$ , and has no neighbour in  $W_{i+1}, W_{i+2}, W_{i-1}, W_{i-2}$
- for  $1 \leq i \leq 7$ , every vertex in  $W_i$  with a neighbour in  $Y_i$  is complete to  $W_{i+1} \cup W_{i-1}$
- for  $1 \leq i \leq 7$ ,  $Y_i$  is complete to  $Y_{i+1}$  and anticomplete to  $Y_{i+2} \cup Y_{i+3}$
- for  $1 \leq i \leq 7$ , at least one of  $Y_i, Y_{i+1}, Y_{i+2}$  is empty.

Then we have:

**9.1** *A graph is of heptagram type if and only if it is of either the first or second heptagram type.*

**Proof.** (A sketch, we leave the details to the reader.) Let  $G$  be of heptagram type, with notation as usual. Suppose first that some  $Y_i$  is not complete to  $W_{i-3} \cup W_{i+3}$ . Then we may assume that  $i = 2$ ; by 7.6  $Y_1, Y_3, Y_5, Y_6$  are empty; and if  $C$  denotes the set of vertices in  $W_2$  with neighbours in  $Y_2$ , then  $C$  is complete to  $W_1 \cup W_3$  and so (S4) implies that  $W_1$  is complete to  $W_3$ . By (S5), every vertex in  $W_2$  is complete to one of  $W_1, W_3$ . By 7.4 and 5.3,  $W_j$  is complete to  $W_{j+1}$  for  $j = 3, 7$ , and so 5.1 implies that  $W_j$  is complete to  $W_{j+2}$  for all  $j$ . Every two vertices in  $Y_2$  either have the same neighbours in  $W_5 \cup W_6$  or disjoint neighbour sets in  $W_5 \cup W_6$ . It follows that  $G$  is of the first heptagram type. On the other hand, if each  $Y_i$  is complete to  $W_{i-3} \cup W_{i+3}$ , then  $G$  is of the second type. ■

The two descriptions are more explicit than before, and the first heptagram type description is explicit and satisfactory; but there is still some degree of opacity in the description of the second type, due principally to the use of “crescents”. We need to transform the definition of a crescent into something transparent.

Let  $W_1, W_2, W_3$  be disjoint sets, and let  $f$  be a function from their union to the set of all integers, such that there do not exist  $w_i \in W_i$  ( $i = 1, 2, 3$ ) with  $f(w_1) = f(w_2) = f(w_3)$ . We define a graph  $H_f$  with vertex set  $W_1 \cup W_2 \cup W_3$  as follows.  $W_1, W_2, W_3$  are stable in  $H_f$ . For  $1 \leq i < j \leq 3$ , and all  $u \in W_i$  and  $v \in W_j$ , let  $u, v$  be adjacent if  $f(u) < f(v)$ , and nonadjacent if  $f(u) > f(v)$ ; if  $f(u) = f(v)$  then the adjacency between  $u$  and  $v$  is arbitrary. It is easy to check that  $(W_1, W_2, W_3)$  is a crescent in  $H_f$ . We prove in the next section that the converse is also true; if  $(W_1, W_2, W_3)$  is a crescent in  $G$ , then there is a function  $f$  as above such that  $H_f = G|(W_1 \cup W_2 \cup W_3)$ . This gives an explicit construction of all crescents, and hence can be used to convert our definition of the second heptagram type to an explicit construction.

## 10 Constructing a crescent

Let  $(W_1, W_2, W_3)$  be a partition of the vertex set of a graph  $G$ . We say the quadruple  $(G, W_1, W_2, W_3)$  is a *trident* if  $W_1, W_2, W_3$  are stable, and for all choices of  $w_i \in W_i$  for  $1 \leq i \leq 3$ ,  $w_1, w_2, w_3$  are not all pairwise adjacent and not all pairwise nonadjacent. How do we construct the most general trident? This will answer the crescent problem of the previous section, because if  $(W_1, W_2, W_3)$  is a partition of  $V(G)$ , and  $H$  is obtained from  $G$  by reversing all adjacencies between  $W_1$  and  $W_3$ , then  $(G, W_1, W_2, W_3)$  is a trident if and only if  $(W_1, W_2, W_3)$  is a crescent in  $H$ .

Let  $W_1, W_2, W_3$  be three disjoint sets with union  $W$  say, and let  $f$  be a function from  $W$  to the set of integers, such that there do not exist  $w_i \in W_i$  ( $1 \leq i \leq 3$ ) satisfying  $f(w_1) = f(w_2) = f(w_3)$ . Let  $G$  be a graph with vertex set  $W$  defined as follows. For  $1 \leq i \leq 3$ , let  $j = i + 1$  if  $i < 3$  and  $j = 1$  if  $i = 3$ ; then for all  $u \in W_i$  and  $v \in W_j$ , let  $u, v$  be adjacent if  $f(u) < f(v)$ , and nonadjacent if  $f(u) > f(v)$ , and either adjacent or nonadjacent if  $f(u) = f(v)$ . It is easy to check that  $(G, W_1, W_2, W_3)$  is a trident.

The result of this section is the converse: that every trident arises in this way from some appropriate function  $f$ . More precisely, let  $(G, W_1, W_2, W_3)$  be a trident. We say a function  $f$  from  $V(G)$  to the set of integers is a *certificate* for this trident if it satisfies the following:

- there do not exist  $w_1 \in W_1, w_2 \in W_2$  and  $w_3 \in W_3$  such that  $f(w_1) = f(w_2) = f(w_3)$ , and
- for all  $i, j \in \{1, 2, 3\}$  such that  $j - i = 1$  modulo 3, and all  $u \in W_i$  and  $v \in W_j$ , if  $f(u) < f(v)$  then  $u, v$  are adjacent, and if  $f(u) > f(v)$  then  $u, v$  are nonadjacent.

We shall prove:

### 10.1 Every trident admits a certificate.

**Proof.** Let  $(G, W_1, W_2, W_3)$  be a trident. We prove by induction on  $|V(G)|$  that  $(G, W_1, W_2, W_3)$  admits a certificate. If  $V(G) = \emptyset$  then the claim is true, so we may assume that  $V(G) \neq \emptyset$ . Below, all index arithmetic is modulo three.

(1) *There exists  $i \in \{1, 2, 3\}$  and  $v \in W_i$  such that  $v$  is adjacent to every member of  $W_{i+1}$ .*

For we may assume that  $W_1 \neq \emptyset$ . Choose  $w_1 \in W_1$  with as many neighbours in  $W_2$  as possible, and let  $N_2$  be the set of vertices in  $W_2$  adjacent to  $w_1$ . We may assume that some vertex  $w_2$  is nonadjacent to  $w_1$ . Similarly we may assume that some vertex  $w_3 \in W_3$  is nonadjacent to  $w_2$ . Since  $\{w_1, w_2, w_3\}$  is not a stable set it follows that  $w_1, w_3$  are adjacent. For  $n_2 \in N_2$ , since  $\{w_1, n_2, w_3\}$  is not a clique, it follows that  $n_2, w_3$  are nonadjacent, and so  $w_3$  is anticomplete to  $N_2$ . We may assume that there exists  $w'_1 \in W_1$  nonadjacent to  $w_3$ . For  $n_2 \in N_2 \cup \{w_2\}$ , since  $\{w'_1, n_2, w_3\}$  is not a stable set,  $w'_1$  is adjacent to  $n_2$ , and so  $w'_1$  is complete to  $N_2 \cup \{w_2\}$ . But then  $w'_1$  has more neighbours in  $W_2$  than  $w_1$ , contrary to the choice of  $w_1$ . This proves (1).

In view of (1), we may assume that some vertex in  $W_1$  is complete to  $W_2$ . Let  $A_1$  be the set of all vertices in  $W_1$  that are complete to  $W_2$ , and let  $A_3$  be the set of all vertices in  $W_3$  with a neighbour in  $A_1$ . For each  $a_3 \in A_3$ , since  $a_3$  is adjacent to some  $a_1 \in A_1$ , and  $a_1$  is adjacent to each  $w_2 \in W_2$ , and  $\{a_1, w_2, a_3\}$  is not a clique, it follows that  $a_3, w_2$  are nonadjacent, and so  $A_3$  is anticomplete to  $W_2$ . Also, for each  $w_1 \in W_1 \setminus A_1$ , since  $w_1$  has a non-neighbour  $w_2 \in W_2$ , and each  $a_3 \in A_3$  is

nonadjacent to  $w_2$ , and  $\{w_1, w_2, a_3\}$  is not a stable set, it follows that  $w_1, a_3$  are adjacent, and so  $A_3$  is complete to  $W_1 \setminus A_1$ . Let  $W' = V(G) \setminus (A_1 \cup A_3)$ ; then

$$(G|W', W_1 \setminus A_1, W_2, W_3 \setminus A_3)$$

is a trident, and since  $A_1 \neq \emptyset$ , it follows from the inductive hypothesis that there is a certificate,  $f'$  say, for this trident. Choose an integer  $n$  such that  $n < f'(v)$  for all  $v \in W'$ . Define a map  $f$  from  $W$  to the set of integers by setting  $f(v) = n$  if  $v \in A_1 \cup A_3$ , and  $f(v) = f'(v)$  otherwise. Then  $f$  is a certificate for  $(G, W_1, W_2, W_3)$  as required. This proves 10.1. ■

## References

- [1] C. Berge, “Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind”, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* 10 (1961), 114.
- [2] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, “The strong perfect graph theorem”, *Annals of Math.* 164 (2006), 51–229.
- [3] G. Ding, private communication.
- [4] A. Gyarfás, “Problems from the world surrounding perfect graphs”, *Proceedings of the International Conference on Combinatorial Analysis and its Applications*, (Pokrzywna, 1985), *Zastos. Mat.* 19 (1987), 413–441.
- [5] C. T. Hoàng and C. McDiarmid, “On the divisibility of graphs”, *Discrete Math.* 242 (2002), 145–156.
- [6] A. Tucker, “Critical perfect graphs and perfect 3-chromatic graphs”, *J. Combinatorial Theory Ser. B* 23 (1977), 143–149.