Claw-free Graphs. IV. Decomposition theorem

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Abstract

A graph is *claw-free* if no vertex has three pairwise nonadjacent neighbours. In this series of papers we give a structural description of all claw-free graphs. In this paper, we achieve a major part of that goal; we prove that every claw-free graph either belongs to one of a few basic classes, or admits a decomposition in a useful way.

1 Introduction

Let G be a graph. (All graphs in this paper are finite and simple.) If $X \subseteq V(G)$, the subgraph $G|X$ *induced on* X is the subgraph with vertex set X and edge-set all edges of G with both ends in X. ($V(G)$ and $E(G)$ denote the vertex- and edge-sets of G respectively.) We say that $X \subseteq V(G)$ is a *claw in* G if $|X| = 4$ and $G|X$ is isomorphic to the complete bipartite graph $K_{1,3}$. We say G is *claw-free* if no $X \subseteq V(G)$ is a claw in G. Our objective in this series of papers is to show that every claw-free graph can be built starting from some basic classes by means of some simple constructions.

For instance, one of the first things we shall show is that if G is claw-free, and has an induced subgraph that is a line graph of a (not too small) cyclically 3-connected graph, then either the whole graph G is a line graph, or G admits a decomposition of one of two possible types. That suggests that we should investigate which other claw-free graphs do not admit either of these decompositions; and that turns out to be a good question, because at least when $\alpha(G) \geq 4$ there is a nice answer. (We denote the size of the largest stable set of vertices in G by $\alpha(G)$.) All claw-free graphs G with $\alpha(G) \geq 4$ that do not admit either of these decompositions can be explicitly described, and fall into a few basic classes; and all connected claw-free graphs G with $\alpha(G) \geq 4$ can be built from these basic types by simple constructions. (When $\alpha(G) \leq 3$ the situation becomes more complicated; there are both more basic types and more decompositions required, as we shall explain.)

There is a difference between a "decomposition theorem" and a "structure theorem", although they are closely related. In this paper we prove a decomposition theorem for claw-free graphs; we show that they all either belong to a few basic classes or admit certain decompositions. But this can be refined into a structure theorem that is more informative; for instance, every connected claw-free graph G with $\alpha(G) \geq 4$ has the same overall "shape" as a line graph, and more or less can be regarded as a line graph with "strips" substituted for some of the vertices. For reasons of space, that development, and its application to several open questions about claw-free graphs, is postponed to a future paper.

2 Trigraphs

To facilitate converting this decomposition theorem to a structure theorem, it is very helpful (indeed, necessary, as far as we can see) to work with slightly more general objects than graphs, that we call "trigraphs". In a graph, every pair of vertices are either adjacent or nonadjacent, but in a trigraph, some pairs may be "undecided". For our purposes, we may assume that this set of undecided pairs is a matching. Thus, let us say a *trigraph* G consists of a finite set $V(G)$ of vertices, and a map $\theta_G : V(G)^2 \to \{1, 0, -1\}$, satisfying:

- for all $v \in V(G)$, $\theta_G(v, v) = 0$
- for all distinct $u, v \in V(G)$, $\theta_G(u, v) = \theta_G(v, u)$
- for all distinct $u, v, w \in V(G)$, at most one of $\theta_G(u, v), \theta_G(u, w) = 0$.

We call θ_G the *adjacency function* of G. For distinct u, v in $V(G)$, we say that u, v are *strongly adjacent* if $\theta_G(u, v) = 1$, *strongly antiadjacent* if $\theta_G(u, v) = -1$, and *semiadjacent* if $\theta_G(u, v) = 0$. We say that u, v are *adjacent* if they are either strongly adjacent or semiadjacent, and *antiadjacent* if they are either strongly antiadjacent or semiadjacent. Also, we say u is adjacent to v and u is a *neighbour* of v if u, v are adjacent (and a *strong neighbour* if u, v are strongly adjacent); u is antiadjacent to v and u is an *antineighbour* of v if u, v are antiadjacent. We denote by $F(G)$ the set of all pairs $\{u, v\}$ such that $u, v \in V(G)$ are distinct and semiadjacent. Thus a trigraph G is a graph if $F(G) = \emptyset$.

For a vertex a and a set $B \subseteq V(G) \setminus \{a\}$ we say that a is *complete* to B or *B-complete* if a is adjacent to every vertex in B; and that a is *anticomplete* to B or *B-anticomplete* if a has no neighbour in B. For two disjoint subsets A and B of $V(G)$ we say that A is *complete*, respectively *anticomplete*, to B, if every vertex in A is complete, respectively anticomplete, to B. (We sometimes say A is B-complete, or the pair (A, B) is complete, meaning that A is complete to B.) Similarly, we say that a is *strongly complete* to B if a is strongly adjacent to every member of B, and so on.

Let G be a trigraph. A *clique* in G is a subset $X \subseteq V(G)$ such that every two members of X are adjacent, and a *strong clique* is a subset such that every two of its members are strongly adjacent. A set $X \subseteq V(G)$ is *stable* if every two of its members are antiadjacent, and *strongly stable* if every two of its members are strongly antiadjacent. We define $\alpha(G)$ to be the maximum cardinality of a stable set.

If $X \subseteq V(G)$, we define the trigraph $G|X$ *induced on* X as follows. Its vertex set is X, and its adjacency function is the restriction of θ_G to X^2. Isomorphism for trigraphs is defined in the natural way, and if G, H are trigraphs, we say that G *contains* H and H is an *induced subtrigraph* of G if there exists $X \subseteq V(G)$ such that H is isomorphic to $G|X$.

A *claw* is a trigraph with four vertices a_0, a_1, a_2, a_3, such that $\{a_1, a_2, a_3\}$ is stable and a_0 is complete to $\{a_1, a_2, a_3\}$. If $X \subseteq V(G)$ and $G|X$ is a claw, we often loosely say that X is a claw; and if no induced subtrigraph of G is a claw, we say that G is *claw-free*. Thus, our object here is to obtain a decomposition theorem for claw-free trigraphs.

An induced subtrigraph $G|X$ of G is said to be a *path from u to v* if $|X| = n$ for some $n \geq 1$, and X can be ordered as $\{p_1, \ldots, p_n\}$, satisfying

- $p_1 = u$ and $p_n = v$
- p_i is adjacent to p_{i+1} for $1 \leq i < n$, and
- p_i is antiadjacent to p_j for $1 \leq i, j \leq n$ with $i + 2 \leq j$.

We say it has *length* $n - 1$. (Thus it has length 0 if and only if $u = v$.) It is often convenient to describe such a path by the sequence p_1-p_2-$\cdots$-p_n. Note that the sequence is uniquely determined by the set $\{p_1, \ldots, p_n\}$ and the vertices u, v, because $F(G)$ is a matching.

A *hole* in G is an induced subtrigraph C with n vertices for some $n \geq 4$, whose vertex set can be ordered as $\{c_1, \ldots, c_n\}$, satisfying (reading subscripts modulo n)

- c_i is adjacent to c_{i+1} for $1 \leq i \leq n$, and
- c_i is antiadjacent to c_j for $1 \leq i, j \leq n$ with $j \neq i - 1, i, i + 1$.

Again, it is often convenient to describe C by the sequence c_1-c_2-$\cdots$-c_n-c_1, and we say it has *length* n. The sequence is uniquely determined by a knowledge of $V(C)$, up to choice of the first term and up to reversal. An *n-hole* means a hole of length n. A *centre* for a hole C is a vertex in $V(G) \setminus V(C)$ that is adjacent to every vertex of the hole. A hole C is *dominating* in G if every vertex in $V(G) \setminus V(C)$ has a neighbour in C.

3 The main theorem

In this section we state our main theorem, but first we need a number of further definitions. A clique with cardinality three is a *triangle*. A *triad* in a trigraph G means a set of three vertices of G, pairwise antiadjacent. Let us explain the decompositions that we shall use in the main theorem.

The first is that G admits "twins". Two strongly adjacent vertices of a trigraph G are called *twins* if (apart from each other) they have the same neighbours in G, and the same antineighbours, and if there are two such vertices, we say "G admits twins". If $X \subseteq V(G)$ is a strong clique and every vertex in $V(G) \setminus X$ is either strongly complete or strongly anticomplete to X, we call X a *homogeneous set*. Thus, G admits twins if and only if some homogeneous set has more than one member.

For the second decomposition, let A, B be disjoint subsets of $V(G)$. The pair (A, B) is called a *homogeneous pair* in G if A, B are strong cliques, and for every vertex $v \in V(G) \setminus (A \cup B)$, v is either strongly A-complete or strongly A-anticomplete and either strongly B-complete or strongly B-anticomplete. (This is related to, but not the same as, the standard definition of "homogeneous pair", due to Chvatal and Sbihi [5]; it was convenient for us to modify their definition a little.) Let (A, B) be a homogeneous pair, such that A is neither strongly complete nor strongly anticomplete to B, and at least one of A, B has at least two members. In these circumstances we call (A, B) a *W-join*. A homogeneous pair (A, B) is *nondominating* if some vertex of $G \setminus (A \cup B)$ has no neighbour in $A \cup B$ (and *dominating* otherwise); and it is *coherent* if the set of all $(A \cup B)$-complete vertices in $V(G) \setminus (A \cup B)$ is a strong clique.

Next, suppose that V_1, V_2 is a partition of $V(G)$ such that V_1, V_2 are nonempty and V_1 is strongly anticomplete to V_2. We call the pair (V_1, V_2) a *0-join* in G.

Next, suppose that V_1, V_2 is a partition $V(G)$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

- $A_i, V_i \setminus A_i \neq \emptyset$ for $i = 1, 2$
- $A_1 \cup A_2$ is a strong clique, and
- $V_1 \setminus A_1$ is strongly anticomplete to V_2, and V_1 is strongly anticomplete to $V_2 \setminus A_2$.

In these circumstances, we say that (V_1, V_2) is a 1-*join*.

Next, suppose that V_0, V_1, V_2 are disjoint subsets with union $V(G)$, and for $i = 1, 2$ there are subsets A_i, B_i of V_i satisfying the following:

- $V_0 \cup A_1 \cup A_2$ and $V_0 \cup B_1 \cup B_2$ are strong cliques, and V_0 is strongly anticomplete to $V_i \setminus (A_i \cup B_i)$ for $i = 1, 2$;
- for $i = 1, 2$, $A_i \cap B_i = \emptyset$ and A_i, B_i and $V_i \setminus (A_i \cup B_i)$ are all nonempty; and
- for all $v_1 \in V_1$ and $v_2 \in V_2$, either v_1 is strongly antiadjacent to v_2, or $v_1 \in A_1$ and $v_2 \in A_2$, or $v_1 \in B_1$ and $v_2 \in B_2$.

We call the triple (V_0, V_1, V_2) a *generalized* 2-*join*, and if $V_0 = \emptyset$ we call the pair (V_1, V_2) a 2-*join*. (This is closely related to, but not exactly the same as, what has been called a 2-join in other papers.)

We use one more decomposition, the following. Let (V_1, V_2) be a partition of $V(G)$, such that for $i = 1, 2$ there are strong cliques $A_i, B_i, C_i \subseteq V_i$ with the following properties:

- V_1, V_2 are both nonempty;
- for $i = 1, 2$ the sets A_i, B_i, C_i are pairwise disjoint and have union V_i;
- if $v_1 \in V_1$ and $v_2 \in V_2$, then v_1 is strongly adjacent to v_2 unless either $v_1 \in A_1$ and $v_2 \in A_2$, or $v_1 \in B_1$ and $v_2 \in B_2$, or $v_1 \in C_1$ and $v_2 \in C_2$; and in these cases v_1, v_2 are strongly antiadjacent.

In these circumstances we say that G is a *hex-join* of $G|V_1$ and $G|V_2$. Note that if G is expressible as a hex-join as above, then the sets $A_1 \cup B_2, B_1 \cup C_2$ and $C_1 \cup A_2$ are three strong cliques with union $V(G)$, and consequently no trigraph with four pairwise antiadjacent vertices is expressible as a hex-join.

Next, we list some basic classes of trigraphs. First some convenient terminology. If H is a graph and G is a trigraph, we say that G is an *H-trigraph* if $V(G) = V(H)$, and for all distinct $u, v \in V(H)$, if u, v are adjacent in H then they are adjacent in G, and if u, v are nonadjacent in H then they are antiadjacent in G.

- **Line trigraphs.** Let H be a graph, and let G be a trigraph with $V(G) = E(H)$. We say that G is a *line trigraph* of H if for all distinct $e, f \in E(H)$:
  - if e, f have a common end in H then they are adjacent in G, and if they have a common end of degree at least three in H, then they are strongly adjacent in G
  - if e, f have no common end in H then they are strongly antiadjacent in G.

  We say that $G \in \mathcal{S}_0$ if G is isomorphic to a line trigraph of some graph. It is easy to check that any line trigraph is claw-free.

- **Trigraphs from the icosahedron.** The *icosahedron* is the unique planar graph with twelve vertices all of degree five. For $k = 0, 1, 2, 3$, $icosa(-k)$ denotes the graph obtained from the icosahedron by deleting k pairwise adjacent vertices. We say $G \in \mathcal{S}_1$ if G is a claw-free $icosa(0)$-trigraph, $icosa(-1)$-trigraph or $icosa(-2)$-trigraph. (We prove in 5.1 and 5.2 below that for $k = 0, 1$, every claw-free $icosa(-k)$-trigraph G satisfies $F(G) = \emptyset$ and therefore is a graph; and every claw-free $icosa(-2)$-trigraph G satisfies $|F(G)| \leq 2$.)

- **The graphs $\mathcal{S}_2$.** Let G be the trigraph with vertex set $\{v_1, \ldots, v_{13}\}$, with adjacency as follows. v_1-$\cdots$-v_6 is a hole in G of length 6. Next, v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5 and possibly to v_7; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; v_{12} is adjacent to $v_2, v_3, v_5, v_6, v_9, v_{10}$; and v_{13} is adjacent to $v_1, v_2, v_4, v_5, v_7, v_8$. No other pairs are adjacent, and all adjacent pairs are strongly adjacent except possibly for v_7, v_8 and v_9, v_{10}. (Thus the pair v_7v_8 may be strongly adjacent, semiadjacent or strongly antiadjacent; the pair v_9v_{10} is either strongly adjacent or semiadjacent.) We say $H \in \mathcal{S}_2$ if H is isomorphic to $G \setminus X$, where $X \subseteq \{v_7, v_{11}, v_{12}, v_{13}\}$.

- **Long circular interval trigraphs.** Let Σ be a circle, and let $F_1, \ldots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$. Assume that no three of $F_1, \ldots, F_k$ have union Σ, and no two of $F_1, \ldots, F_k$ share an end-point. Now let $V \subseteq \Sigma$ be finite, and let G be a trigraph with vertex set V in which, for distinct $u, v \in V$,

- if $u, v \in F_i$ for some i then u, v are adjacent, and if also at least one of u, v belongs to the interior of F_i then u, v are strongly adjacent
- if there is no i such that $u, v \in F_i$ then u, v are strongly antiadjacent.

Such a trigraph G is called a *long circular interval trigraph.* We write $G \in \mathcal{S}_3$ if G is a long circular interval trigraph. ("Long" refers to the fact that no three of $F_1, \ldots, F_k$ have union Σ; in later papers we shall need to omit this condition.)

- **Modifications of $L(K_6)$.** Let H be a graph with seven vertices $h_1, \ldots, h_7$, in which h_7 is adjacent to h_6 and to no other vertex, h_6 is adjacent to at least three of $h_1, \ldots, h_5$, and there is a cycle with vertices h_1-h_2-$\cdots$-h_5-h_1 in order. Let $J(H)$ be the graph obtained from the line graph of H by adding one new vertex, adjacent precisely to those members of $E(H)$ that are not incident with h_6 in H. Then $J(H)$ is a claw-free graph. Let G be either $J(H)$ (regarded as a trigraph), or (in the case when h_4, h_5 both have degree two in H), the trigraph obtained from $J(H)$ by making the vertices $h_3h_4, h_1h_5 \in V(J(H))$ semiadjacent. Let $\mathcal{S}_4$ be the class of all such trigraphs G.

- **The trigraphs $\mathcal{S}_5$.** Let $n \geq 2$. Construct a trigraph G as follows. Its vertex set is the disjoint union of four sets A, B, C and $\{d_1, \ldots, d_5\}$, where $|A| = |B| = |C| = n$, say $A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$. Let $X \subseteq A \cup B \cup C$ with $|X \cap A|, |X \cap B|, |X \cap C| \leq 1$. Adjacency is as follows: A, B, C are strong cliques; for $1 \leq i, j \leq n$, a_i, b_j are adjacent if and only if $i = j$, and c_i is strongly adjacent to a_j if and only if $i \neq j$, and c_i is strongly adjacent to b_j if and only if $i \neq j$. Moreover

  - a_i is semiadjacent to c_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $b_i \in X$
  - b_i is semiadjacent to c_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $a_i \in X$
  - a_i is semiadjacent to b_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $c_i \in X$
  - no two of $A \setminus X$, $B \setminus X$, $C \setminus X$ are strongly complete to each other.

  Also, d_1 is strongly $A \cup B \cup C$-complete; d_2 is strongly complete to $A \cup B$, and either semiadjacent or strongly adjacent to d_1; d_3 is strongly complete to $A \cup \{d_2\}$; d_4 is strongly complete to $B \cup \{d_2, d_3\}$; d_5 is strongly adjacent to d_3, d_4; and all other pairs are strongly antiadjacent. Let the trigraph just constructed be G, and let $H = G|(V(G) \setminus X)$. Then H is claw-free; let $\mathcal{S}_5$ be the class of all such trigraphs H.

- **Near-antiprismatic trigraphs.** Let $n \geq 2$. Construct a trigraph as follows. Its vertex set is the disjoint union of three sets A, B, C, where $|A| = |B| = n + 1$ and $|C| = n$, say $A = \{a_0, a_1, \ldots, a_n\}, B = \{b_0, b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$. Adjacency is as follows. A, B, C are strong cliques. For $0 \leq i, j \leq n$ with $(i, j) \neq (0, 0)$, let a_i, b_j be adjacent if and only if $i = j$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. a_0, b_0 may be semiadjacent or strongly antiadjacent. All other pairs not mentioned so far are strongly antiadjacent. Now let $X \subseteq A \cup B \cup C \setminus \{a_0, b_0\}$ with $|C \setminus X| \geq 2$. Let all adjacent pairs be strongly adjacent except:

  - a_i is semiadjacent to c_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $b_i \in X$
  - b_i is semiadjacent to c_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $a_i \in X$

- a_i is semiadjacent to b_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $c_i \in X$

Let the trigraph just constructed be G, and let $H = G|(V(G) \setminus X)$. Then H is claw-free; let $\mathcal{S}_6$ be the class of all such trigraphs H. We call such a trigraph H *near-antiprismatic*, since making a_0, b_0 strongly adjacent would produce an antiprismatic trigraph.

- **Antiprismatic trigraphs.** Let us say a trigraph is *antiprismatic* if for every $X \subseteq V(G)$ with $|X| = 4$, X is not a claw and there are at least two pairs of vertices in X that are strongly adjacent. We give a structural description of such trigraphs elsewhere (for instance, the first two papers of this series [1, 2] describe all antiprismatic trigraphs that are graphs). Let $\mathcal{S}_7$ be the class of all antiprismatic trigraphs.

Now we can state the main result of this paper, the following.

3.1 *Let G be a claw-free trigraph. Then either*

- $G \in \mathcal{S}_0 \cup \dots \cup \mathcal{S}_7$, *or*
- *G admits either twins, a nondominating W-join, a 0-join, a 1-join, a generalized 2-join, or a hex-join.*

The proof is given in the final section of the paper. We postpone to future papers the problem of converting this decomposition theorem to a structure theorem.

4 More on decompositions

Before we begin the main proof, it is helpful to develop a few tools that will enable us to prove more easily that trigraphs are decomposable. First, here is a useful decomposition. Suppose that there is a partition (V_1, V_2, X) of $V(G)$ such that X is a strong clique, and $|V_1|, |V_2| \geq 2$, and V_1 is strongly anticomplete to V_2. In these circumstances we say that X is an *internal clique cutset.* This is not one of the decompositions used in the statement of the main theorem (indeed, it is not the inverse of a composition that preserves being claw-free, unlike the other decompositions we mentioned). Nevertheless, we win if we can prove that our trigraph admits an internal clique cutset, because of the following, proved in [3]. We say that a trigraph G is a *linear interval trigraph* if the vertices of G can be numbered $v_1, \dots, v_n$ such that for all i, j with $1 \leq i < j \leq n$, if v_i is adjacent to v_j then $\{v_i, v_{i+1}, \dots, v_{j-1}\}$ and $\{v_{i+1}, v_{i+2}, \dots, v_j\}$ are strong cliques. (Every such trigraph is a long circular interval trigraph, as may easily be checked.)

4.1 *Let G be a claw-free trigraph. If G admits an internal clique cutset, then either G is a linear interval trigraph, or G admits either a 1-join, a 0-join, a coherent W-join, or twins.*

For brevity, let us say that G is *decomposable* if it admits either a generalized 2-join, or a 1-join, or a 0-join, or a nondominating W-join, or twins, or an internal clique cutset, or a hex-join. There follow four lemmas that will speed up our recognition of decomposable trigraphs.

4.2 *Let G be a claw-free trigraph, and let $A, C \subseteq V(G)$ be disjoint, such that*

- *A is a strong clique*
- *if $C = \emptyset$ then $|A| > 1$*
- *every vertex in $V(G) \setminus (A \cup C)$ is strongly C-anticomplete, and either strongly A-complete or strongly A-anticomplete*
- *$|V(G) \setminus (A \cup C)| \geq 2$.*

Then G is decomposable.

Proof. If C is empty then $|A| > 1$ and any two members of A are twins. So we may assume that C is nonempty. If A is strongly anticomplete to C then G admits a 0-join, so we may assume that $a \in A$ and $c \in C$ are adjacent. Let Y be the set of vertices in $V(G) \setminus (A \cup C)$ that are A-complete, and let $Z = V(G) \setminus (A \cup C \cup Y)$. If $y_1, y_2 \in Y$, then since $\{a, c, y_1, y_2\}$ is not a claw, it follows that y_1, y_2 are strongly adjacent, and so Y is a strong clique. If Z is nonempty then $(A \cup C, Y \cup Z)$ is a 1-join, so we assume that Z is empty. But $|Y| \geq 2$ by hypothesis, and all members of Y are twins, and so G is decomposable. This proves 4.2. ■

4.3 *Let G be a claw-free trigraph, and let (A, B) be a homogeneous pair in G.*

- *If (A, B) is nondominating and at least one of A, B has cardinality > 1, then G admits twins or a nondominating W-join.*
- *If (A, B) is dominating and coherent, A is not strongly anticomplete to B, and $A \cup B \neq V(G)$, then G admits a hex-join.*

In either case G is decomposable.

Proof. Suppose that (A, B) is nondominating and at least one of A, B has cardinality > 1, say $|A| > 1$. If B is either strongly complete or strongly anticomplete to A then the elements of A are twins, and otherwise (A, B) is a nondominating W-join. Thus in this case G is decomposable.

Now suppose that (A, B) is dominating and coherent, A is not strongly anticomplete to B, and $A \cup B \neq V(G)$. Let $V = V(G) \setminus (A \cup B)$; thus $V \neq \emptyset$. Let X, Y be the sets of vertices in V that are strongly adjacent to A, and strongly adjacent to B, respectively. Since (A, B) is a homogeneous pair, every vertex in $V \setminus X$ is strongly antiadjacent to A, and similarly for B; since (A, B) is dominating, it follows that $X \cup Y = V$; and since (A, B) is coherent, $X \cap Y$ is a strong clique. We claim that $X \setminus Y$ is a strong clique; for suppose not. Let $u, v \in X \setminus Y$ be antiadjacent. Choose $a \in A$ and $b \in B$, adjacent (this is possible since A is not strongly anticomplete to B by hypothesis). Then $\{a, b, u, v\}$ is a claw, a contradiction. This proves that $X \setminus Y$ is a strong clique, and similarly so is $Y \setminus X$. Moreover, A and B are strong cliques, since (A, B) is a homogeneous pair. But then if we define $P_1 = X \setminus Y, P_2 = Y \setminus X$ and $P_3 = X \cap Y$, and $Q_1 = B, Q_2 = A, Q_3 = \emptyset$, we see that each of the sets $P_1, P_2, P_3, Q_1, Q_2, Q_3$ is a strong clique, and their union is $V(G)$, and P_i is strongly anticomplete to Q_j if $i = j$, and otherwise P_i is strongly complete to Q_j. Since $A, B \neq \emptyset$ and $V \neq \emptyset$, it follows that then G admits a hex-join. This proves 4.3. ■

We say a triple (A, C, B) is a *breaker* in G if it satisfies:

- A, B, C are disjoint nonempty subsets of $V(G)$, and A, B are strong cliques
- every vertex in $V(G) \setminus (A \cup B \cup C)$ is either strongly A-complete or strongly A-anticomplete, and either strongly B-complete or strongly B-anticomplete, and strongly C-anticomplete
- there is a vertex in $V(G) \setminus (A \cup B \cup C)$ with a neighbour in A and an antineighbour in B; there is a vertex in $V(G) \setminus (A \cup B \cup C)$ with a neighbour in B and an antineighbour in A; and there is a vertex in $V(G) \setminus (A \cup B \cup C)$ with an antineighbour in A and an antineighbour in B
- if A is strongly complete to B, then there do not exist adjacent $x, y \in V(G) \setminus (A \cup B \cup C)$ such that x is $A \cup B$-complete and y is $A \cup B$-anticomplete.

The reason for interest in breakers is that they allow us to deduce that our trigraph admits one of our decompositions, without having to figure out which one, in view of the following theorem.

4.4 *Let G be a claw-free trigraph. If G admits a breaker, then G admits either a 0-join, a 1-join, or a generalized 2-join.*

Proof. Let (A_1, C_1, B_1) be a breaker; let $V_1 = A_1 \cup B_1 \cup C_1$, let V_0 be the set of all vertices not in V_1 that are $A_1 \cup B_1$-complete, and let $V_2 = V(G) \setminus (V_1 \cup V_0)$. Let A_2 be the set of A_1-complete vertices in V_2, and B_2 the set of B_1-complete vertices in V_2. Let $C_2 = V_2 \setminus (A_2 \cup B_2)$. By hypothesis, A_2, B_2, C_2 are all nonempty. If C_1 is strongly anticomplete to $A_1 \cup B_1$, then G admits a 0-join, so from the symmetry we may assume that C_1 is not strongly anticomplete to A_1. Since $A_1 \cup C_1 \cup A_2 \cup V_0$ includes no claw, it follows that $A_2 \cup V_0$ is a strong clique. We claim that also $B_2 \cup V_0$ is a strong clique. For suppose not; then by the same argument, C_1 is strongly anticomplete to B_1. Let A' be the set of vertices in A_1 with a neighbour in C_1. Since $B_2 \neq \emptyset$ and we may assume that $(C_1 \cup A', V(G) \setminus (C_1 \cup A'))$ is not a 1-join, it follows that A' is not strongly anticomplete to B_1. Consequently some vertex $a \in A_1$ has a neighbour $b \in B_1$ and a neighbour $c \in C_1$; and since C_1 is anticomplete to B_1, it follows that $\{a, b, c, a_2\}$ is a claw (where $a_2 \in A_2$) a contradiction. This proves that $B_2 \cup V_0$ is a strong clique.

Suppose that V_0 is not strongly anticomplete to C_2, and choose $x \in V_0$ and $y \in C_2$, adjacent. By hypothesis, A_1 is not strongly complete to B_1; choose $a \in A_1$ and $b \in B_1$, antiadjacent. Then $\{x, y, a, b\}$ is a claw, a contradiction. It follows that V_0 is strongly anticomplete to C_2, and consequently (V_0, V_1, V_2) is a generalized 2-join. This proves 4.4. ■

Here is another shortcut, this time useful for handling hex-joins.

4.5 *Let G be a claw-free trigraph, and let A, B, C be disjoint nonempty strong cliques. Suppose that every vertex in $V(G) \setminus (A \cup B \cup C)$ is strongly complete to two of A, B, C and strongly anticomplete to the third. Suppose also that one of A, B, C has cardinality > 1, and $A \cup B \cup C \neq V(G)$. Then G admits either a hex-join, or a nondominating W-join, or twins.*

Proof. Let $V_1 = A \cup B \cup C$, and $V_2 = V(G) \setminus V_1$. Let A_2 be the set of vertices in V_2 that are anticomplete to A, and define B_2, C_2 similarly. If A_2, B_2, C_2 are strong cliques, then G is the hex-join of $G|V_1$ and $G|V_2$, so we may assume that there exist antiadjacent $u, v \in A_2$. For $w \in A$ and $x \in B \cup C$, $\{x, w, u, v\}$ is not a claw, and so w, x are strongly antiadjacent; and consequently A is strongly anticomplete to $B \cup C$. Thus (B, C) is a homogeneous pair, and it is nondominating since A is nonempty; so by 4.3 we may assume that $|B|, |C| = 1$, and therefore $|A| > 1$ by hypothesis, and yet every two members of A are twins. This proves 4.5. ■

5 The icosahedron

Our first main goal is to prove that claw-free trigraphs that include a "substantial" line trigraph either are line trigraphs or are decomposable. To make this theorem as useful as possible, we want to weaken the meaning of "substantial" as far as we can; and on the borderline where the theorem is just about to become false, there are two situations where the theorem is false in a way we can handle. It is convenient to deal with them first before we embark on line trigraphs in general. We do one in this section and the other in the next, and then start on line trigraphs proper in the section after that. Some general notation; if G is a trigraph and $v \in V(G)$, we denote by $N_G(v)$ the union of $\{v\}$ and the set of all neighbours of v in G, and by $N^*_G(v)$ the union of $\{v\}$ and the set of all strong neighbours of v in G. (Sometimes we abbreviate these to $N(v), N^*(v)$ when the dependence on G is clear.)

In this section we study the icosahedron and some of its subgraphs. We begin by proving the assertions of the previous section about *icosa*$(-k)$ for $k = 0, 1, 2$.

5.1 *Let G be a claw-free icosa(0)-trigraph or a claw-free icosa(-1)-trigraph. Then $F(G) = \emptyset$.*

Proof. Let H be a graph obtained from the icosahedron by deleting one vertex, and let G be a claw-free H-trigraph. We must show that $F(G) = \emptyset$. Number $V(H)$ as $\{v_1, \ldots, v_{11}\}$, where for $1 \le i < j \le 10$, v_i is adjacent to v_j if either $j - i \le 2$ or $j - i \ge 8$, and v_{11} is adjacent to v_1, v_3, v_5, v_7, v_9, and all other pairs are nonadjacent in H. We recall that $V(G) = V(H)$, and every pair of vertices that are adjacent in H are adjacent in G, and every pair that are are nonadjacent in H are antiadjacent in G. We show first that all pairs that are nonadjacent in H are strongly antiadjacent in G. From the symmetry, it suffices to check three pairs, namely v_2v_{11}, v_1v_7 and v_1v_6. Since $\{v_2, v_{11}, v_4, v_{10}\}$ is not a claw, v_{11} is strongly antiadjacent to v_2 in G; since $\{v_1, v_3, v_7, v_{10}\}$ is not a claw, v_1 is strongly antiadjacent to v_7; and since $\{v_6, v_1, v_4, v_8\}$ is not a claw, v_1 is strongly antiadjacent to v_6. This proves that all pairs that are nonadjacent in H are strongly antiadjacent in G.

Next we claim that all pairs that are adjacent in H are strongly adjacent in G. Again, from the symmetry it suffices to check four pairs, namely $v_1v_{11}, v_1v_2, v_1v_3, v_2v_{10}$. Since $\{v_3, v_4, v_1, v_{11}\}$ is not a claw, v_1, v_{11} are strongly adjacent; since $\{v_3, v_5, v_1, v_2\}$ is not a claw, v_1, v_2 are strongly adjacent; since $\{v_{11}, v_7, v_1, v_3\}$ is not a claw, v_1, v_3 are strongly adjacent; and since $\{v_1, v_{11}, v_2, v_{10}\}$ is not a claw, v_2, v_{10} are strongly adjacent. This proves that all pairs that are adjacent in H are strongly adjacent in G. Consequently $F(G) = \emptyset$.

Next we assume that H is the icosahedron and G is a claw-free H-trigraph. Again we must show that $F(G) = \emptyset$. Suppose not, and choose $v \in V(G)$ such that some member of $F(G)$ does not contain v. Then deleting v from G yields a claw-free $H \setminus \{v\}$-trigraph G' with $F(G') \neq \emptyset$, a contradiction to what we proved before. This proves 5.1. ■

Next we need a similar statement for *icosa*(-2). This graph has ten vertices, and they can be labelled as

$$\{a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, e, f\},$$

where its edges are the pairs $a_ic_i, b_ic_i, a_id_i, b_id_i, c_id_i, a_ie, d_ie, b_if, d_if$ for $i = 1, 2$, together with a_1a_2, b_1b_2 and ef.

5.2 *Let H be icosa(-2), and let G be a claw-free H-trigraph. Label the vertices of H as above. Then $F(G) \subseteq \{a_1b_1, a_2b_2\}$.*

Proof. We claim first that every pair of vertices that is adjacent in H is strongly adjacent in G. To show this, it suffices from the symmetry to check the pairs

$$a_1c_1, a_1a_2, a_1d_1, a_1e, c_1d_1, d_1e, ef.$$

Because $\{d_1, f, a_1, c_1\}$ is not a claw, it follows that a_1, c_1 are strongly adjacent; and the other six pairs follows similarly, since the sets

$$\{e, f, a_1, a_2\}, \{e, d_2, a_1, d_1\}, \{d_1, b_1, a_1, e\}, \{a_1, a_2, c_1, d_1\}, \{f, b_2, d_1, e\}, \{d_1, c_1, e, f\}$$

are not claws, respectively. Now to check the pairs that are nonadjacent in H, it suffices to check the pairs

$$c_1e, c_1a_2, c_1c_2, c_1d_2, d_1a_2, d_1d_2, a_1f, a_1b_2, a_1b_1.$$

Since $\{e, c_1, a_2, f\}$ is not a claw, c_1, e are strongly antiadjacent. Similarly the next seven pairs listed are strongly antiadjacent, because the sets

$$\{a_2, c_1, e, c_2\}, \{c_2, c_1, a_2, b_2\}, \{d_2, c_1, a_2, b_2\}, \{d_1, a_2, c_1, f\}, \{d_1, d_2, a_2, b_2\}, \{f, a_1, b_1, d_2\}, \{b_2, a_1, b_1, c_2\}$$

respectively are not claws. (The last pair a_1b_1 cannot be shown strongly adjacent this way.) This proves 5.2. ∎

Frequently we assume that our current claw-free trigraph G has an induced subtrigraph H that we know, and we wish to enumerate all the possibilities for the neighbour set in $V(H)$ of vertices in $V(G) \setminus V(H)$. And having done so, then we try to figure out the adjacencies between the vertices in $V(G) \setminus V(H)$. To aid with that, here are three trivial facts that are used so often that it is worth stating them explicitly. (All three proofs are obvious and we omit them.)

5.3 *Let G be a claw-free trigraph, and let $v \in V(G)$; then $N_G(v)$ includes no triad.*

5.4 *Let G be claw-free, let $X \subseteq V(G)$, and let $v \in V(G) \setminus X$. Then there is no path of length 2 in $G|X$ with middle vertex in $N_G(v)$ and with both ends not in $N^*_G(v)$.*

5.5 *Let G be claw-free, and let $X \subseteq V(G)$. Let $u, v \in V(G) \setminus X$ have a common neighbour $a \in X$ and a common antineighbour $b \in V(H)$. If a, b are distinct and adjacent then u, v are strongly adjacent.*

The icosahedron is claw-free, and in this section we study claw-free trigraphs which contain it (or most of it) as an induced subtrigraph.

5.6 *Let G be claw-free, containing an icosa(-1)-trigraph. Then either $G \in \mathcal{S}_1$, or two vertices of G are twins, or G admits a 0-join. In particular, either $G \in \mathcal{S}_1$, or G is decomposable.*

Proof. Since G contains an $icosa(-1)$-trigraph, 5.1 implies that there exist disjoint strong cliques $V_1, \ldots, V_{11}, V_{12}$ in $V(G)$, such that

- $V_1, \ldots, V_{11}$ are nonempty (possibly $V_{12} = \emptyset$)
- for $1 \le i < j \le 10$, V_i, V_j are strongly complete if either $j - i \le 2$ or $j - i \ge 8$, and otherwise V_i, V_j are strongly anticomplete
- for $1 \le i \le 10$ and $j \in \{11, 12\}$, if i, j are both odd or both even then V_i, V_j are strongly complete, and otherwise they are strongly anticomplete.

Let W be the union of $V_1, \ldots, V_{12}$, and choose these cliques with W maximal. Suppose first that $W = V(G)$. If some V_i has at least two members, then they are twins, so we may assume that $|V_i| = 1$ for $1 \le i \le 11$ and $|V_{12}| \le 1$; but then $G \in \mathcal{S}_1$. We may therefore assume that $W \ne V(G)$. If $V(G) \setminus W$ is strongly anticomplete to W then G admits a 0-join, so we may assume that there exists $v \in V(G) \setminus W$ such that $N(v) \cap W \ne \emptyset$. Let $N = N_G(v) \cap W$ and $N^* = N^*_G(v) \cap W$.

Suppose first that there exists $v_{11} \in N \cap V_{11}$. For $v_1 \in V_1$ and $v_5 \in V_5$, 5.4 applied to the path v_1-v_{11}-v_5 tells us that at least one of $v_1, v_5 \in N^*$, and so N^* includes one of V_1, V_5. (We will need this argument many times, and we speak of "5.4 applied to V_1-V_{11}-V_5" or "5.4 with V_1-V_{11}-V_5" for brevity.) Similarly N^* includes at least one of every antiadjacent pair of sets in the list V_1, V_3, V_5, V_7, V_9, and so we may assume that $V_1, V_3, V_5 \subseteq N^*$, from the symmetry. From 5.3, $N \cap V_8, N \cap V_{12} = \emptyset$. Suppose that $N \cap V_7, N \cap V_9$ are both nonempty. Then 5.3 implies that N is disjoint from V_2, V_4, V_6, V_{10}; 5.4 applied to V_2-V_1-V_9 implies that $V_9 \subseteq N^*$, and similarly $V_7 \subseteq N^*$, and 5.4 applied to V_2-V_1-V_{11} implies that $V_{11} \subseteq N^*$, and so v can be added to N_{11}, contrary to the maximality of W. Hence from the symmetry we may assume that $N \cap V_9 = \emptyset$. By 5.4 with V_2-V_1-V_9, it follows that $V_2 \subseteq N^*$, and by 5.3, it follows that $N \cap V_6 = \emptyset$. By 5.4 with V_6-V_7-V_9, $N \cap V_7 = \emptyset$, and by 5.4 with V_4-V_5-V_7, $V_4 \subseteq N^*$. By 5.3, $N \cap V_{10} = \emptyset$, and by 5.4 with V_{11}-V_5-V_6, $V_{11} \subseteq N^*$. But then v can be added to V_3, contrary to the maximality of W. This proves that $N \cap V_{11} = \emptyset$. If $V_{12} \ne \emptyset$, then from the symmetry between $V_1, \ldots, V_{12}$, it follows that $N \cap V_i = \emptyset$ for $1 \le i \le 12$, a contradiction. Thus $V_{12} = \emptyset$.

Suppose next that $N \cap V_1 \ne \emptyset$. By 5.4 with V_{11}-V_1-V_2, $V_2 \subseteq N^*$, and similarly $V_{10} \subseteq N^*$. By 5.4 with V_3-V_1-V_9, one of $V_3, V_9 \subseteq N^*$, and from the symmetry we may assume that $V_3 \subseteq N^*$. By 5.4 with V_4-V_3-V_{11}, $V_4 \subseteq N^*$. By 5.3, N is disjoint from V_6, V_7, V_8. By 5.4 with V_6-V_5-V_{11} and with V_8-V_9-V_{11}, N is disjoint from V_5, V_9; and by 5.4 with V_1-V_3-V_5, $V_{11} \subseteq N^*$. But then v can be added to V_2, contrary to the maximality of W.

Hence N is disjoint from V_1, and similarly from V_3, V_5, V_7, V_9. By 5.4 with V_1-V_2-V_4 and V_2-V_4-V_5, it follows that either N^* includes $V_2 \cup V_4$ or N is disjoint from $V_2 \cup V_4$; and the same holds for all adjacent pairs of $V_2, V_4, V_6, V_8, V_{10}$. Since N is nonempty, it follows that $N^* = V_2 \cup V_4 \cup V_6 \cup V_8 \cup V_{10}$. But then v can be added to V_{12}, contrary to the maximality of W. This proves 5.6. ■

5.6 handles claw-free trigraphs that contain $icosa(-1)$-trigraphs; next we need to consider $icosa(-2)$.

5.7 *Let G be a claw-free trigraph containing an $icosa(-2)$-trigraph. Then either $G \in \mathcal{S}_1$, or G is decomposable.*

Proof. Since G contains an $icosa(-2)$-trigraph, we may choose ten disjoint nonempty strong cliques $A_1, B_1, C_1, A_2, B_2, C_2, D_1, D_2, E, F$ in G, satisfying:

- The following pairs are strongly complete: A_1A_2, B_1B_2, EF, and for $i = 1, 2$, the pairs $A_iC_i, B_iC_i, A_iD_i, B_iD_i, C_iD_i, A_iE, D_iE, B_iF, D_iF$.
- The pairs A_1B_1 and A_2B_2 are not strongly complete (but not necessarily anticomplete).
- All remaining pairs are strongly anticomplete.

Let us choose such a set of cliques with maximal union W say. Suppose first that $W = V(G)$. Then (A_1, B_1) is a homogeneous pair, nondominating since $C_2 \neq \emptyset$, and so by 4.3 we may assume $|A_1| = |B_1| = 1$, and similarly $|A_2| = B_2| = 1$. If one of the other six cliques has cardinality > 1, say X, then the members of X are twins and the theorem holds. If all ten cliques have cardinality 1 then $G \in \mathcal{S}_1$, as required. So we may assume that $W \neq V(G)$. If W is strongly anticomplete to $V(G) \setminus W$, then G admits a 0-join, so we may assume that there exists $v \in V(G) \setminus W$ with $N \neq \emptyset$, where $N = N_G(v) \cap W$. Let $N^* = N^*_G(v) \cap W$.

(1) *At least one of $N \cap C_1, N \cap C_2$ is nonempty.*

For suppose that $N \cap C_i = \emptyset$ for $i = 1, 2$. Suppose first that $N \cap A_1 \neq \emptyset$. Then 5.4 (with C_1-A_1-A_2 and A_1-A_2-C_2) implies that $A_2, A_1 \subseteq N^*$. 5.4 (with C_1-A_1-E) implies that $E \subseteq N^*$. Suppose in addition that $N \cap (B_1 \cup B_2) \neq \emptyset$. Then from the symmetry, $B_1 \cup B_2 \cup F \subseteq N^*$; and 5.3 (with A_1, B_1, D_2 and A_2, B_2, D_1) implies that N is disjoint from D_1, D_2, contrary to 5.4 (with D_1-E-D_2). So $N \cap (B_1 \cup B_2) = \emptyset$. 5.4 (with D_1-E-D_2) implies that N^* includes one of D_1, D_2, say D_1; 5.4 (with C_1-D_1-F) implies that $F \subseteq N^*$; 5.4 (with B_1-F-D_2) implies that $D_2 \subseteq N$; but then v can be added to E, contrary to the maximality of W. This proves that N is disjoint from A_1, and by symmetry from B_1, A_2, B_2. 5.4 (with A_1-D_1-B_1) implies that $N \cap D_1 = \emptyset$, and by symmetry $N \cap D_2 = \emptyset$; and then 5.4 (with D_1-E-D_2 and D_1-F-D_2) implies that N is disjoint from E, F. But then $N = \emptyset$, a contradiction. This proves (1).

(2) *Both $N \cap C_1, N \cap C_2$ are nonempty.*

For suppose not; then from (1) and the symmetry, we may assume that $N \cap C_1 \neq \emptyset$ and $N \cap C_2 = \emptyset$. Suppose first that $N \cap A_2 \neq \emptyset$. Then 5.4 (with A_1-A_2-C_2 and with E-A_2-C_2) implies that $A_1, E \subseteq N^*$. 5.3 (with A_2, C_1, F) implies that $N \cap F = \emptyset$. 5.4, applied in turn to the triples A_2-E-F; C_2-B_2-F; C_2-D_2-F; D_1-E-D_2; C_1-D_1-F implies that $A_2 \subseteq N^*$; $N \cap B_2 = \emptyset$; $N \cap D_2 = \emptyset$; $D_1 \subseteq N^*$, and $C_1 \subseteq N^*$. But then v can be added to A_1, contrary to the maximality of W. This proves that $N \cap A_2 = \emptyset$, and by symmetry $N \cap B_2 = \emptyset$. 5.4 (with A_2-D_2-B_2) implies that $N \cap D_2 = \emptyset$. 5.4 (with A_1-C_1-B_1) implies that N^* meets one of A_1, B_1. (Recall that A_1 is not necessarily strongly anticomplete to B_1, so we cannot deduce that N^* includes one of A_1, B_1). 5.4 (with D_1-A_1-A_2 if N meets A_1, and D_1-B_1-B_2 otherwise) implies that $D_1 \subseteq N^*$. Suppose first that N is disjoint from both E, F. Then 5.4 (with B_1-D_1-E and A_1-D_1-F) implies that $B_1, A_1 \subseteq N^*$, and 5.4 (with A_2-A_1-C_1) implies that $C_1 \subseteq N^*$. But then v can be added to C_1, contradicting the maximality of W. Hence N is not disjoint from both E, F, and from the symmetry we may assume that $N \cap E \neq \emptyset$. 5.4 (with A_2-E-F) implies that $F \subseteq N^*$, and from symmetry $E \subseteq N^*$. 5.4 (with A_1-E-D_2) implies $A_1 \subseteq N^*$, and by symmetry $B_1 \subseteq N^*$; and 5.4 (with C_1-A_1-A_2) implies that $C_1 \subseteq N^*$. Then v can be added to D_1, contrary to the maximality of W. This proves (2).

From (2) and 5.3, N is disjoint from E, F. Since A_1, B_1 are not strongly complete, 5.3 (with A_1-B_1-C_2) implies that $A_1 \cup B_1 \not\subseteq N$; and so 5.4 (with A_1-D_1-F if $A_1 \not\subseteq N$, and B_1-D_1-E otherwise) implies that $D_1 \cap N = \emptyset$. Similarly $D_2 \cap N = \emptyset$. Since A_1, B_1 are not strongly complete, 5.4 (with A_1-C_1-B_1) implies that N meets at least one of A_1, B_1, say A_1. Then 5.4 (with D_1-A_1-A_2) implies $A_2 \subseteq N^*$, and by symmetry $A_1 \subseteq N^*$. Similarly, if $N \cap (B_1 \cup B_2) \neq \emptyset$, then $B_1 \cup B_2 \subseteq N^*$, contrary to 5.3 (with A_2, B_2, C_1), and so $N \cap (B_1 \cup B_2) = \emptyset$. Then G contains an $icosa(-1)$-trigraph (choose one vertex from each of the ten cliques, choosing neighbours of v from C_1, C_2, and such that for $i = 1, 2$ the representatives of A_i, B_i are nonadjacent; and take v as the eleventh vertex). Then the theorem holds by 5.6. This proves 5.7. ■

Next we need to consider deleting from the icosahedron two vertices at distance two. This is a case of what we call an "XX-configuration". Let J be a graph with ten vertices

$$a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2,$$

where the following pairs are adjacent:

$$d_1d_2, \text{ and } a_ib_i, a_ic_i, b_ic_i, b_id_i, c_id_i, b_ib_3, c_ic_3, d_ib_3, d_ic_3 \text{ for } i = 1, 2,$$

and possibly the edge a_1a_2. Let H be a claw-free J-trigraph. We call any such trigraph H an *XX-configuration.*

We need the following lemma:

5.8 *Let J be as above, with vertices labelled as above, and let H be a claw-free J-trigraph. Then $F(H) \subseteq \{a_1a_2, d_1d_2\}$.*

The proof is straightforward and we leave it to the reader.

5.9 *Let G be a claw-free trigraph containing an XX-configuration. Then either $G \in \mathcal{S}_1 \cup \mathcal{S}_2$, or G is decomposable.*

Proof. Since G contains an XX-configuration, by 5.8 we may choose fourteen disjoint subsets

$$A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, D_1, D_2, E_1, E_2, F,$$

with the following properties:

- all fourteen sets are strong cliques except possibly A_3; and the ten sets

  $$A_1, A_2, B_1, B_2, B_3, C_1, C_2, C_3, D_1, D_2$$

  are nonempty

- the pairs

  $$A_iB_i, A_iC_i, B_iC_i, B_iB_3, C_iC_3, B_iD_i, C_iD_i, B_3D_i, C_3D_i$$

  are strongly complete for $i = 1, 2$; E_1 is strongly complete to $B_1, B_3, D_1, D_2, C_3, C_2$; E_2 is strongly complete to $B_2, B_3, D_1, D_2, C_3, C_1$; and F is strongly complete to $A_1, B_1, C_1, A_2, B_2, C_2$. All other pairs of the fourteen subsets named are strongly anticomplete, with the possible exception of $D_1D_2, A_1A_2, A_1A_3, A_2A_3$

- D_1 is not strongly anticomplete to D_2.

Consequently we may choose these fourteen sets with maximal union W say.

Suppose first that $W = V(G)$. Any two vertices of B_1 are twins in G, and the same holds for $B_2, B_3, C_1, C_2, C_3, E_1, E_2, F$, and so we may assume that these sets all have cardinality at most one (and therefore the first six of them have cardinality exactly one.) Moreover, (D_1, D_2) is a homogeneous pair, nondominating since $A_1 \neq \emptyset$, and so by 4.3, we may assume that D_1, D_2 both have cardinality 1. Now every vertex not in $A_1 \cup A_2 \cup A_3$ is either strongly A_1-complete or strongly A_1-anticomplete, and either strongly A_2-complete or strongly A_2-anticomplete, and strongly A_3-anticomplete. Also, if $x, y \in V(G) \setminus A_1 \cup A_2 \cup A_3$ and x is $A_1 \cup A_2$-complete and y is $A_1 \cup A_2$-anticomplete, then $x \in F$ and $y \in B_3 \cup C_3 \cup D_1 \cup D_2 \cup E_1 \cup E_2$, and so x, y are not adjacent. Consequently if $A_3 \neq \emptyset$ then (A_1, A_3, A_2) is a breaker, and the theorem holds by 4.4, so may assume that $A_3 = \emptyset$. Then (A_1, A_2) is a homogeneous pair, nondominating since $D_1 \neq \emptyset$, and therefore by 4.3 we may assume that $|A_i| = 1$ for $i = 1, 2$. But then $G \in \mathcal{S}_2$, and the theorem holds.

We may therefore assume that $W \neq V(G)$. If W is strongly anticomplete to $V(G) \setminus W$ then G admits a 0-join, so we may assume that there exists $v \in V(G) \setminus W$ with $N \neq \emptyset$, where $N = N_G(v) \cap W$. Let $N^* = N^*_G(v) \cap W$.

First assume that $N \cap B_3, N \cap C_3 \neq \emptyset$. By 5.3, N is disjoint from $A_1 \cup A_2 \cup A_3$. By 5.4 (with B_1-B_3-B_2), N^* includes one of B_1, B_2, and we may assume that it includes B_1 from the symmetry. By 5.3, $N \cap B_2 = \emptyset$. By 5.4 applied in turn to B_2-B_3-D_1, A_1-B_1-B_3, A_1-B_1-E_1, A_1-F-B_2 we deduce that $D_1 \subseteq N^*$, $B_3 \subseteq N^*$, $E_1 \subseteq N^*$, and $N \cap F = \emptyset$. Suppose that $N \cap C_1$ is nonempty. By 5.3, $N \cap C_2 = \emptyset$; by 5.4 applied in turn to C_1-C_3-C_2, C_3-C_1-A_1 and C_2-C_3-E_2, we deduce that $C_1 \subseteq N^*$, $C_3 \subseteq N^*$, and $E_2 \subseteq N^*$; but then v can be added to D_1, contrary to the maximality of W. Thus $N \cap C_1 = \emptyset$. By 5.4, applied to D_2-C_3-C_1, C_1-C_3-C_2, B_2-D_2-C_3, and C_1-E_2-B_2, we deduce that $D_2 \subseteq N^*$, $C_2 \subseteq N^*$, $C_3 \subseteq N^*$, and $N \cap E_2 = \emptyset$; but then v can be added to E_1, contrary to the maximality of W.

So we may assume that N is disjoint from one of B_3 and C_3, say C_3. Next assume that N meets both D_1 and D_2. By 5.4 (with B_3-D_1-C_3), $B_3 \subseteq N^*$. By 5.4 (with B_1-D_1-C_3), $B_1 \subseteq N^*$, and similarly $B_2 \subseteq N^*$. By 5.3, N is disjoint from $A_1 \cup A_2 \cup A_3$. By 5.4 (with A_1-B_1-D_1), $D_1 \subseteq N^*$, and similarly $D_2 \subseteq N^*$. By 5.4 (with A_1-C_1-C_3), $N \cap C_1 = \emptyset$, and similarly $N \cap C_2 = \emptyset$. By 5.4 applied to A_i-B_i-E_i for $i = 1, 2$ and to C_1-F-C_2, we deduce that $E_1, E_2 \subseteq N^*$ and $N \cap F = \emptyset$. But then v can be added to B_3, contrary to the maximality of W.

So we may assume that N is disjoint from both C_3 and D_2 say. Choose $d_1 \in D_1$ and $d_2 \in D_2$, adjacent. Suppose that $d_1 \in N$. By 5.4, applied in turn to B_1-d_1-C_3, B_3-d_1-C_3, C_1-d_1-d_2, and A_1-C_1-C_3 we deduce that $B_1 \subseteq N^*$, $B_3 \subseteq N^*$, $C_1 \subseteq N^*$ and $A_1 \subseteq N^*$. By 5.3, $N \cap (B_2 \cup C_2) = \emptyset$ and $N \cap (A_2 \cup A_3) = \emptyset$. By 5.4 applied to B_2-B_3-D_1, E_1-B_3-D_2, B_2-E_2-C_3 and F-C_1-C_3, we deduce that $D_1 \subseteq N^*$, $E_1 \subseteq N^*$, $N \cap E_2 = \emptyset$, and $F \subseteq N^*$. But then v can be added to B_1, contrary to the maximality of W.

Hence $d_1 \notin N$. Suppose next that $N \cap B_3$ is nonempty. By 5.4 (with d_1-B_3-B_2), $B_2 \subseteq N^*$, and similarly $B_1 \subseteq N^*$. By 5.4 (with d_1-B_1-A_1), $A_1 \subseteq N^*$, and similarly $A_2 \subseteq N^*$. By 5.3, A_1 is complete to A_2, and for the same reason, N is disjoint from $A_3 \cup C_1 \cup C_2 \cup D_1$. But then G contains an $icosa(-1)$-trigraph (choose one vertex from each of the eight sets $A_1, A_2, B_1, B_2, B_3, C_1, C_2, C_3$, together with d_1, d_2, v), and the theorem holds by 5.6.

So we may assume that $N \cap B_3 = \emptyset$. By 5.4 (with B_3-D_1-C_3), $N \cap D_1 = \emptyset$, and so N is disjoint from all four of B_3, C_3, D_1, D_2. By 5.4 (with B_3-E_i-C_3 for $i = 1, 2$) it follows that N is disjoint from

E_1, E_2. If N intersects none of B_1, B_2, C_1, C_2, then 5.4 (with B_1-F-B_2) implies that $N \cap F = \emptyset$, and then v can be added to A_3, contrary to the maximality of W. So we may assume from the symmetry that N meets B_1. By 5.4 (with C_1-B_1-B_3), $C_1 \subseteq N^*$, and similarly $B_1 \subseteq N^*$; by 5.4 (with B_3-B_1-A_1), $A_1 \subseteq N^*$; and by 5.4 (with B_3-B_1-F), $F \subseteq N^*$. If N intersects either B_2 or C_2, then similarly it includes $A_2 \cup B_2 \cup C_2$, and by 5.3, N is disjoint from A_3, but then v can be added to F, contrary to the maximality of W. So we may assume that N is disjoint from $B_2 \cup C_2$. But then v can be added to A_1, contrary to the maximality of W. This proves 5.9. ■

6 The second line trigraph anomaly

Now we handle the second peculiarity that will turn up when we come to treat line trigraphs. Let J be a graph with eleven vertices

$$a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2,$$

and the following edges: for $i = 1, 2$, $\{a_i, b_i, c_i, d_i\}$ are cliques, and so are $\{b_1, b_2, b_3\}$ and $\{c_1, c_2, c_3\}$; d_1, d_2 are nonadjacent, and every other pair of a_3, b_3, c_3, d_1, d_2 are adjacent; and there are no other edges except possibly a_1a_2. We call any claw-free J-trigraph a *YY-configuration.* We begin with a lemma:

6.1 *Let J be as above, with vertices labelled as above, and let H be a claw-free J-trigraph. Then $F(H) \subseteq \{d_1d_2, a_1a_2, a_1a_3, a_2a_3\}$.*

The proof is straighforward (analogous to that of 5.1), and we leave it to the reader. The main result of this section is the following.

6.2 *Let G be a claw-free trigraph containing a YY-configuration. Then G is decomposable.*

Proof. Since there is a YY-configuration in G, by 6.1 we may choose nine strong cliques A^i_j ($1 \leq i, j \leq 3$), with the following properties (for $1 \leq i \leq 3$, A^i denotes $A^i_1 \cup A^i_2 \cup A^i_3$, and A_i denotes $A^1_i \cup A^2_i \cup A^3_i$):

- these nine sets are nonempty and pairwise disjoint
- for $1 \leq i, j, i', j' \leq 3$, if $i \neq i'$ and $j \neq j'$ then A^i_j is strongly anticomplete to $A^{i'}_{j'}$
- for $1 \leq j \leq 3$, A_j is a strong clique
- for $i = 1, 2$, A^i is a strong clique
- A^3_1 and A^3_2 are not strongly complete to A^3_3
- for $1 \leq j \leq 3$, let S_j be the set of all vertices that are strongly anticomplete to A_j and strongly complete to the other two of A_1, A_2, A_3; then S_1 is not strongly complete to S_2 (and consequently S_1, S_2 are nonempty)
- subject to these conditions, the union W of the sets A^i_j ($1 \leq i, j \leq 3$) is maximal.

(To see this, take a YY-configuration, with vertices $a_1, a_2, \dots$ as before, and let $A_j^1 = \{b_j\}, A_j^2 = \{c_j\}, A_j^3 = \{a_j\}$ for $j = 1,2,3$; then d_1, d_2 belongs to S_2, S_1 respectively.)

Let $Z = V(G) \setminus (W \cup S_1 \cup S_2 \cup S_3)$, and for $i = 1,2$, let H_i be the set of vertices in A_i^3 that are strongly antiadjacent to A_3^3. Choose $s_1 \in S_1$ and $s_2 \in S_2$, antiadjacent.

(1) *Every vertex in $W \cup S_1 \cup S_2 \cup S_3$ with a neighbour in Z belongs to $H_1 \cup H_2$.*

For let $v \in Z$, and let N, N^* be respectively the intersections of $N_G(v), N_G^*(v)$ with $W \cup S_1 \cup S_2 \cup S_3$. We will show that

$$N \cap (W \cup S_1 \cup S_2 \cup S_3) \subseteq H_1 \cup H_2.$$

Assume for a contradiction that $s_1, s_2 \in N$. We claim that $A_3^1 \subseteq N^*$. For suppose not. 5.4 (with A_3^1-s_2-$(A_1^2 \cup A_1^3)$) implies that $A_1^2 \cup A_1^3 \subseteq N^*$, and similarly $A_2^2 \cup A_2^3 \subseteq N^*$. Since A_1^3 is not strongly complete to A_3^3, 5.3 (with A_1^3, A_3^3, A_2^2) implies that $A_3^3 \not\subseteq N$. 5.4 (with A_1^1-s_2-A_3^3 and A_2^1-s_1-A_3^3) implies that $A_1^1, A_2^1 \subseteq N^*$, and then three applications of 5.3 imply that $N \cap A_3 = \emptyset$. But then $v \in S_3$, a contradiction. This proves our claim that $A_3^1 \subseteq N^*$, and similarly $A_3^2 \subseteq N^*$. Suppose that $A_3^3 \not\subseteq N^*$. Then for $1 \le i, j \le 2$, 5.4 (with A_3^3-A_3^i-A_j^i) implies that $A_j^i \subseteq N^*$, and 5.3 implies that N is disjoint from A_1^3, contrary to 5.4 (with A_1^3-s_2-A_3^3). Thus $A_3^3 \subseteq N^*$. Since v cannot be added to A_3^3, N meets one of the sets A_j^i where $1 \le i, j \le 2$, and from the symmetry we may assume that $N \cap A_1^1 \neq \emptyset$. 5.3 implies that N is disjoint from A_2^2, A_2^3. If N meets A_2^1, then similarly N is disjoint from A_1^2, A_1^3, and 5.4 (with A_1^2-A_1^1-A_2^1) implies that $A_2^1 \subseteq N^*$, and similarly $A_1^1 \subseteq N^*$; but then v can be added to A_3^1, a contradiction. Thus $N \cap A_2^1 = \emptyset$. By 5.4 (with A_2^1-A_1^1-$(A_1^2 \cup A_1^3)$), $A_1^2 \cup A_1^3 \subseteq N^*$, and by 5.4 (with A_1^1-A_1^2-A_2^2), $A_1^1 \subseteq N$; but then $v \in S_2$, a contradiction. This completes the case when $s_1, s_2 \in N$.

Next assume (for a contradiction) that $s_1 \in N$ and $s_2 \notin N$. Suppose first that $A_2^1 \not\subseteq N^*$. 5.4 (with A_2^1-s_1-$(A_3^2 \cup A_3^3)$) implies that $A_3^2 \cup A_3^3 \subseteq N^*$; 5.4 (with s_2-A_3^1-A_2^1) implies that $N \cap A_3^1 = \emptyset$; 5.4 (with A_3^1-s_1-A_2^3) implies $A_2^3 \subseteq N^*$; 5.3 (with A_1^2, A_3^3, A_2^3) implies that $N \cap A_1^2 = \emptyset$; and this contradicts 5.4 (with A_1^2-A_3^2-A_3^1). This proves that $A_2^1 \subseteq N^*$. Similarly $A_2^2 \subseteq N^*$. If $A_2^3 \not\subseteq N^*$, then 5.4 (with A_2^3-A_2^i-A_j^i) implies that $A_j^i \subseteq N^*$, for $i = 1,2$ and $j = 1,3$; and then 5.3 implies that N is disjoint from both A_3^3, A_2^3, contrary to 5.4 (with A_3^3-s_1-A_2^3). Hence $A_2^3 \subseteq N^*$. Suppose that $N \cap (A_3^1 \cup A_3^2) = \emptyset$. Since v cannot be added to A_2^3, it follows that $N \cap (A_1^1 \cup A_1^2) \neq \emptyset$, and from the symmetry we may assume that $N \cap A_1^1 \neq \emptyset$. 5.4 (with $(A_1^2 \cup A_1^3)$-A_1^1-A_3^1) implies that $A_1^2 \cup A_1^3 \subseteq N^*$, and similarly $A_1^1 \subseteq N^*$, and 5.3 implies that $N \cap A_3^3 = \emptyset$; but then $v \in S_3$, a contradiction. Thus $N \cap (A_3^1 \cup A_3^2) \neq \emptyset$, and from the symmetry we may assume that $N \cap A_3^1 \neq \emptyset$. Suppose that $A_1^1 \not\subseteq N^*$. Then 5.4 (with A_1^1-A_3^1-$(A_3^2 \cup A_3^3)$) implies that $A_3^2 \cup A_3^3 \subseteq N^*$; three applications of 5.3 imply that $N \cap A_1 = \emptyset$; and 5.4 (with A_1^2-A_3^2-A_3^1) implies that $A_3^1 \subseteq N^*$; but then $v \in S_1$, a contradiction. This proves that $A_1^1 \subseteq N^*$. By 5.3, $N \cap A_j^i = \emptyset$ for $i = 2,3$ and $j = 1,3$; and 5.4 (with A_1^2-A_1^1-A_3^1) implies that $A_3^1 \subseteq N^*$. But then v can be added to A_2^1, a contradiction. This completes the case when $s_1 \in N$ and $s_2 \notin N$.

We deduce that $s_1 \notin N$, and similarly $s_2 \notin N$. 5.4 (with s_1-A_3-s_2) implies that $N \cap A_3 = \emptyset$. Suppose that $N \cap (A_1^1 \cup A_1^2) \neq \emptyset$. Then 5.4 (with A_3^1-A_1^1-A_1^2 and A_3^2-A_1^2-$(A_1^1 \cup A_1^3)$) implies that $A_1 \subseteq N^*$. Also 5.4 (with s_2-A_1^1-A_2^1) implies that $A_2^1 \subseteq N^*$, and so similarly $A_2 \subseteq N^*$ and therefore $v \in S_3$, a contradiction. This proves that $N \cap (A_1^1 \cup A_1^2) = \emptyset$, and similarly $N \cap (A_2^1 \cup A_2^2) = \emptyset$. 5.4 (with A_1^1-S_2-A_3^2) implies that $N \cap S_2 = \emptyset$, and similarly $N \cap S_1 = N \cap S_3 = \emptyset$. 5.4 (with A_j^1-$(A_j^3 \setminus H_j)$-A_3^3) implies that $N \cap A_j^3 \subseteq H_j$ for $j = 1,2$. Consequently $N \subseteq H_1 \cup H_2$. This proves (1).

(2) *Let* $v \in (W \setminus (H_1 \cup H_2)) \cup S_1 \cup S_2 \cup S_3$. *If* $v \in A_1 \cup S_2 \cup S_3$ *then* v *is strongly complete to* H_1*, and otherwise* v *is strongly anticomplete to* H_1*. An analogous statement holds for* H_2*.*

For if $v \in A_1 \cup S_2 \cup S_3$ then v is strongly complete to H_1, and if $v \in A_3 \cup S_1 \cup A_2^1 \cup A_2^2$ then v is anticomplete to H_1, so we may assume that $v \in A_2^3$. Let $a_2^2 \in A_2^2$. Since $v \notin H_2$, v has a neighbour $a_3^3 \in A_3^3$; and if v also has a neighbour $h_1 \in H_1$, then $\{v, h_1, a_2^2, a_3^3\}$ is a claw, a contradiction. Thus v is strongly anticomplete to H_1. This proves (2).

We claim that there do not exist adjacent $x, y \in (W \setminus (H_1 \cup H_2)) \cup S_1 \cup S_2 \cup S_3$ such that x is $H_1 \cup H_2$-complete and y is $H_1 \cup H_2$-anticomplete. For suppose that such x, y exist. By (2), $x \in S_3$, and $y \in A_3$; but then x, y are strongly antiadjacent, a contradiction. If $Z \neq \emptyset$, then (H_1, Z, H_2) is a breaker, by (1) and (2), and the theorem holds by 4.4. We may therefore assume that $Z = \emptyset$. Now S_1, S_2, S_3 are strong cliques by 5.5, and so G is the hex-join of $G|W$ and $G|(S_1 \cup S_2 \cup S_3)$. This proves 6.2. ■

7 Line graphs

Our next goal is to prove that if a trigraph G is claw-free and contains an induced subtrigraph which is a line trigraph of some graph H, and H is sufficiently nondegenerate, then either G itself is a line trigraph or it is decomposable. It is helpful first to weaken slightly what we mean by a line trigraph.

If H is a graph and $e, f \in E(H)$, we say that e, f are *cousins* if they have no common end in H, and there is an edge xy of H such that e is incident with x and f is incident with y and x, y both have degree two in H. Let H be a graph, and let G be a trigraph with $V(G) = E(H)$. We say that G is a *weak line trigraph* of H if for all distinct $e, f \in E(H)$:

- if e, f have a common end in H then they are adjacent in G, and if they have a common end of degree at least three in H, then they are strongly adjacent in G
- if e, f have no common end in H then they are antiadjacent in G, and if they are not cousins in H then they are strongly antiadjacent in G.

We remark:

7.1 *Let* G *be a claw-free trigraph with* $\alpha(G) \geq 3$ *and* $|V(G)| \geq 7$*. If* G *is a weak line trigraph of some graph* H*, then either* $G \in \mathcal{S}_0$ *or* G *is decomposable.*

Proof. We may assume that G is not decomposable, and that H has no vertex of degree zero. If there do not exist any pair of cousins in $E(H)$ that are semiadjacent in G, then G is a line trigraph of H as required, so we suppose that $a, b \in E(H)$ are cousins that are semiadjacent in G. Let v_1, v_2, v_3, v_4 be vertices of H such that $a = v_1v_2$, $b = v_3v_4$, there is an edge $c = v_2v_3$, and v_2, v_3 both have degree two in H.

Suppose first that c has no neighbours in G except a, b. Let A, B, C, D be respectively the sets of edges of H different from a, b, c that are incident with v_1 and not v_4, v_4 and not v_1, neither of v_1, v_4, and both v_1, v_4 respectively. Since G is not decomposable, it follows that $(\{a\}, \{c\}, \{b\})$ is not a breaker, and so one of A, B, C is empty. Suppose that $C = \emptyset$. Then $\{a\}, \{b\}, D$ are strong cliques, and so are $A, B, \{c\}$; and each of the first triple is strongly complete to two of the second triple and anticomplete to the third, and so G is expressible as a hex-join, a contradiction. Thus $C \neq \emptyset$, and so we may assume that $A = \emptyset$. But then $(\{a, b, c\} \cup D, B \cup C)$ is a 1-join, a contradiction.

Thus c has a neighbour in G different from a, b; and so c is semiadjacent to some cousin of c. Hence we may assume that there is a vertex v_5 of H adjacent to v_4, so that v_4 has degree two, and c, d are semiadjacent in G where $d = v_4v_5$; and $v_5 \neq v_2, v_3, v_4$. Let A, B, C, D be respectively the sets of edges of H different from a, b, c, d that are incident with v_1 and not v_5, v_5 and not v_1, neither of v_1, v_5, and both v_1, v_5 respectively. (Thus if $v_5 = v_1$ then $A = B = \emptyset$.) Suppose that $v_5 = v_1$; then $C \neq \emptyset$ since $\alpha(G) \geq 3$, and so $(\{a, b, c, d\}, C \cup D)$ is a 1-join, a contradiction. Thus $v_1 \neq v_5$. Since $(\{a\}, \{b, c\}, \{d\})$ is not a breaker, one of A, B, C is empty. Suppose that $C = \emptyset$. Since (A, B) is a nondominating homogeneous pair, we may assume by 4.3 that $|A|, |B| \leq 1$. Since $|V(G)| \geq 7$, it follows that $D \neq \emptyset$, and so G is expressible as a hex-join, with the six cliques $A \cup \{a\}, B \cup \{d\}, \{b, c\}, \emptyset, \emptyset, D$, a contradiction. Thus $C \neq \emptyset$, and we may therefore assume that $A = \emptyset$. But then $(\{a, b, c, d\} \cup D, B \cup C)$ is a 1-join, a contradiction. This proves 7.1. ■

In this paper, a *separation* of a graph H means a pair (A, B) of subsets of $V(H)$, such that $A \cup B = V(H)$ and $A \setminus B$ is anticomplete to $B \setminus A$. A *k-separation* means a separation (A, B) such that $|A \cap B| \leq k$, and a separation (A, B) is *cyclic* if both $H|A, H|B$ contain cycles. We say that H is *cyclically* 3*-connected* if it is 2-connected and not a cycle, and there is no cyclic 2-separation. (For instance, the complete bipartite graph $K_{2,3}$ is cyclically 3-connected, but the graph obtained from K_4 by deleting an edge is not. This differs slightly from the definition used in [4].)

A *branch-vertex* of a graph H means a vertex with at least three neighbours; and, if a graph H is cyclically 3-connected, a *branch* of H means a path B of H with distinct ends, both branch-vertices, such that no internal vertex of B is a branch-vertex. (The reason for insisting that H is cyclically 3-connected is because of our convention that all "paths" are induced subgraphs, and that is not our intention for branches; but no conflict arises when H is cyclically 3-connected.) A graph H is *robust* if:

- H is cyclically 3-connected,
- $|V(H)| \geq 7$, and
- $|V(H) \setminus V(B)| \geq 4$ for every branch B.

There is an analogue of 5.1 and 5.2 for line trigraphs, as follows.

7.2 *Let H be a robust graph. Let $L(H)$ be its line graph, and let G be a claw-free $L(H)$-trigraph. Then either G is a weak line trigraph of H, or G contains an XX-configuration or a YY-configuration.*

Proof. Thus $V(G) = E(H)$, and for all distinct $e, f \in E(H)$, if e, f share an end in H then they are adjacent in G, and if e, f are disjoint in H then they are antiadjacent in G. We must check that either G contains an XX-configuration or a YY-configuration, or

- if e, f share an end in H that has degree at least three in H, then e, f are strongly adjacent in G
- if e, f are disjoint in H and not cousins then they are strongly antiadjacent in G.

For the first claim, let $t \in V(H)$ be incident with $e_1, \dots, e_k$ say, where $k \geq 3$, and suppose that e_1, e_2 are antiadjacent in G. For $1 \leq i \leq k$ let e_i have ends t, t_i. For $3 \leq i \leq k$, if $g \in E(H)$ is incident with t_i and different from e_i, then since $\{g, e_i, e_1, e_2\}$ is not a claw in G, it follows that g is incident in H with one of t_1, t_2; and so t_i has no neighbours in H except t and possible t_1, t_2. Since H is cyclically 3-connected, each of $t_3, \dots, t_k$ is adjacent to both of t_1, t_2. For $3 \leq i \leq k$ let the three edges of H incident with t_i be $e_i = t_i t, f_i = t_i t_1$ and $g_i = t_i t_2$. Now $(\{t, t_1, \dots, t_k\}, V(H) \setminus \{t, t_3, \dots, t_k\})$ is a 2-separation of H, and so either $V(H) = \{t, t_1, \dots, t_k\}$, or $H \setminus \{t, t_3, \dots, t_k\})$ is a path between t_1, t_2. Suppose the first holds. Then $k \geq 6$ since $|V(H)| \geq 7$; and then G contains a YY-configuration (take the vertices called

$$a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2$$

in the definition of a YY-configuration to be

$$f_6, g_6, e_5, f_3, g_3, e_3, f_4, g_4, e_4, e_1, e_2$$

respectively). Now suppose the second holds, that is, $H \setminus \{t, t_3, \dots, t_k\})$ is a path between t_1, t_2. Let a_1, a_2 be the edges of this path incident with t_1, t_2 respectively. Then $k \geq 5$, since there are at least four vertices of H not in the branch between t_1, t_2, and again G contains a YY-configuration (take the same bijection as before, except use a_1, a_2 in place of f_6, g_6). This proves the first claim.

For the second claim, let e, f be disjoint edges of H, and suppose they are adjacent (and therefore semiadjacent) in G. Since H is robust, there is a cycle C of H of length at least five, containing e, f. Let e_1, e_2 be the two edges of C that share an end with e, and define f_1, f_2 similarly. Since $\{e, e_1, e_2, f\}$ is not a claw in G, f is strongly adjacent in G to one of e_1, e_2, and therefore f shares an end with one of e_1, e_2 in H. Hence we may assume that $e_2 = f_2$. Let C have vertices $c_1\text{-}\cdots\text{-}c_k\text{-}c_1$ in order, where e_1 is c_1c_2, e is c_2c_3, e_2 is c_3c_4, f is c_4c_5, and f_1 if c_5c_6 (where $c_6 = c_1$ if $k = 5$). If e, f belong to the same branch of H, then so does e_2, and therefore c_3, c_4 both have degree two in H and e, f are cousins as required; so we may assume that e, f do not belong to the same branch of H, and therefore $(\{c_2, c_3, c_4, c_5\}, V(H) \setminus \{c_3, c_4\})$ is not a 2-separation of H. Hence we may assume that c_3 is adjacent in H to some vertex $x \neq c_2, c_3, c_4, c_5$. If $x \neq c_1$ then $\{e, e_1, c_3x, f\}$ is a claw in G, a contradiction, and so $x = c_1$. Since H is cyclically 3-connected, and no branch contains all vertices except three, it follows that

$$(\{c_1, c_2, c_3, c_4, c_5\}, V(H) \setminus \{c_2, c_3, c_4\})$$

is not a 2-separation, and so one of c_2, c_3, c_4 has a neighbour $y \in V(H) \setminus \{c_1, c_2, c_3, c_4, c_5\}$. We have already seen that c_3 has no such neighbour, and if c_2, y are adjacent then $\{e, c_2y, c_1c_3, f\}$ is a claw in G, a contradiction; and so c_4, y are adjacent. Since $\{f, f_1, c_4y, e\}$ is not a claw, it follows that $y = c_6$. If c_2, c_5 are adjacent then G contains an XX-configuration (take the vertices called

$$a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2$$

in the definition of an XX-configuration to be

$$c_1c_k, c_6c_7, c_1c_3, c_4c_6, e_2, e_1, f_1, c_2c_5, e, f$$

respectively), so we assume not. Since the neighbours c_1, c_3 of c_2 are adjacent and H is cyclically 3-connected, it follows that c_2 has a neighbour $z \neq c_1, c_3$; and since $\{e, c_2z, c_1c_3, f\}$ is not a claw in G, z is incident with f. Hence c_2, c_4 are adjacent, and similarly so are c_3, c_5. But then again G contains an XX-configuration, since exchanging c_2, c_3 puts us back in the previous case. This proves 7.2. ■

7.3 *Let H be a robust graph, and let $X \subseteq E(H)$, satisfying the following:*

(Z1) *there do not exist three pairwise nonadjacent edges in X*

(Z2) *there do not exist distinct vertices t_1, t_2, t_3, t_4 of H, such that t_i is adjacent to t_{i+1} for $i = 1, 2, 3$, and the edge t_2t_3 belongs to X, and the other two edges t_1t_2, t_3t_4 do not belong to X.*

Then one of the following holds:

- *There is a subset $Y \subseteq V(H)$ with $|Y| \leq 2$ such that X is the set of all edges of H incident with a vertex in Y.*
- *There are vertices $s_1, s_2, s_3, t_1, t_2, t_3, u_1, u_2 \in V(H)$, all distinct except that possibly $t_1 = t_2$, such that the following pairs are adjacent in H: s_it_i, s_iu_1, s_iu_2 for $i = 1, 2, 3$, and s_1s_3. Moreover, X contains exactly six of these ten edges, the six not incident with s_1.*
- *There is a subgraph J of H isomorphic to a subdivision of K_4 (let its branch-vertices be $v_1, \ldots, v_4$, and let $B_{i,j}$ denote the branch between v_i, v_j); and $B_{2,3}, B_{3,4}, B_{2,4}$ all have length 1, $B_{1,2}, B_{1,3}$ have length 2, and $B_{1,4}$ has length ≥ 2. Moreover, the edges of J in X are precisely the five edges of $B_{1,2}, B_{1,3}$ and $B_{2,3}$.*

Proof. Since H is cyclically 3-connected, we have:

(1) *No vertex of H of degree 2 is in a triangle.*

(2) *If there is a vertex $y \in V(H)$ such that every edge in X is incident with y, then the theorem holds.*

For suppose y is such a vertex; let N be the set of neighbours v of y such that the edge $yv \in X$, and M the remaining neighbours of y. If $M = \emptyset$ or $N = \emptyset$ then the first statement of the theorem holds, so we assume that there exist $m \in M$ and $n \in N$. The only edge in X incident with n is ny, and by **(Z2)**, there is no edge in $E(H) \setminus X$ incident with n except possibly nm. Since n has degree ≥ 2, it follows that n has degree 2 and is in a triangle, contrary to (1). This proves (2).

(3) *If there exist two vertices y_1, y_2 of H such that every edge in X is incident with one of y_1, y_2, then the theorem holds.*

For let us choose y_1, y_2 with the given property, adjacent if possible. For $i = 1, 2$, let N_i be the set of all neighbours $v \in V(H) \setminus \{y_1, y_2\}$ of y_i such that the edge $y_iv \in X$, and let M_i be the other neighbours of y_i in $V(H) \setminus \{y_1, y_2\}$. If M_1, M_2 are both empty, then the first statement of the theorem holds, so we may assume that there exists $m_1 \in M_1$. By (2) we may assume that there exists $n_1 \in N_1$. Let a be any neighbour of n_1 different from y_1. If $an_1 \in X$ then $a = y_2$, since every edge in X is incident with one of y_1, y_2; and if $an_1 \notin X$ then $a = m_1$, by **(Z2)** applied to m_1-y_1-n_1-a. In particular, if $n_1 \notin N_2$ then n_1 has degree 2 and belongs to a triangle, contrary to (1). It follows that $N_1 \subseteq N_2$. Suppose that $|M_1| > 1$. Then no vertex in N_1 has a neighbour in M_1, and therefore every vertex in N_1 has degree 2. Since H is cyclically 3-connected, it follows that $N_1 = \{n_1\}$; and so every edge in X is incident with one of n_1, y_2. From the choice of y_1, y_2 it follows that y_1, y_2 are adjacent, and so n_1 belongs to a triangle, contrary to (1). This proves that $M_1 = \{m_1\}$. If there exist distinct

$u,v \in N_1$ both nonadjacent to m_1, then $(\{u,v,y_1,y_2\}, V(H)\setminus\{u,v\})$ is a 2-separation of G contradicting that H is robust. Thus every vertex in N_1 is adjacent to m_1 except possibly one. Moreover, $(N_1\cup\{y_1,y_2,m_1\}, V(H)\setminus(N_1\cup\{y_1\}))$ is a 2-separation of H, and so either $N_1\cup\{y_1,y_2,m_1\} = V(H)$, or $H\setminus(N_1\cup\{y_1\}))$ is a path of length >1 between m_1,y_2. In the first case, it follows that $|N_1|\geq 4$ since $|V(H)|\geq 7$, and the second statement of the theorem holds. Thus we assume the second case applies. Let P be the path $H\setminus(N_1\cup\{y_1\}))$. By hypothesis, at least four vertices of H do not belong to $V(P)$, and so $|N_1|\geq 3$. Let x be the neighbour of y_2 in P; then $x\neq m_1$. Choose $n_1'\in N_1$ adjacent to m_1; then from **(Z2)** applied to x-y_2-n_1'-m_1 we deduce that the edge xy_2 belongs to X. But then again the second statement of the theorem holds. This proves (3).

(4) *If there are three edges in X forming a cycle of length 3, then there is a fourth edge in X incident with a vertex of this cycle.*

For suppose that y_1,y_2,y_3 are vertices such that $y_1y_2, y_2y_3, y_3y_1 \in X$, and for $i=1,2,3$ no other edge in X is incident with y_i. Since H is cyclically 3-connected and $|V(H)|\geq 7$, it follows that there are two edges between $\{y_1,y_2,y_3\}$ and $V(H)\setminus\{y_1,y_2,y_3\}$, with no common end. But then both these edges belong to $E(H)\setminus X$, and **(Z2)** is violated. This proves (4).

(5) *There do not exist $Y\subseteq V(H)$ with $|Y|=3$ and $y_4\in V(H)\setminus Y$, such that every two members of Y are joined by an edge in X, and every other edge in X is incident with y_4.*

For let $Y=\{y_1,y_2,y_3\}$, and suppose first that there is a matching of size 2 consisting of edges of $H\setminus\{y_4\}$, each with one end in Y and the other not in this set. These two edges therefore do not belong to X, and so **(Z2)** is violated. Thus there is no such matching. Consequently, there is a vertex y_5 such that every edge of H with one end in Y and the other not in this set is incident with one of y_4,y_5. It follows that $(Y\cup\{y_4,y_5\}, V(H)\setminus Y)$ is a 2-separation of H, and therefore $H\setminus Y$ is a path between y_4,y_5, contrary to the hypothesis. This proves (5).

In view of (3),(4),(5), **(Z1)** and (for instance) Tutte's theorem [6], it follows that there is a set $Y\subseteq V(H)$ with $|Y|=5$ such that every edge in X has both ends in Y, and $H|(Y\setminus\{y\})$ has a 2-edge matching with both edges in X, for every vertex $y\in Y$. (We call this "criticality".) Criticality implies that among every three vertices in Y, some two are joined by an edge in X. Suppose that there is a 3-edge matching between $V(H)\setminus Y$ and Y. None of these three edges belongs to X, and so from **(Z2)** it follows that no two of y_1,y_2,y_3 are joined by an edge in X, contrary to criticality. We deduce that no such matching of size 3 exists. Consequently there is a set $Z\subseteq V(H)$ with $|Z|\leq 2$, such that every edge between Y and $V(H)\setminus Y$ is incident with a member of Z. By choosing Z with $Z\cup Y$ minimal, we deduce that every vertex in $Z\setminus Y$ has at least two neighbours in Y. Now $(Y\cup Z, (V(H)\setminus Y)\cup Z)$ is a 2-separation. Since $H|Y$ has a cycle, it follows that $H\setminus(Y\setminus Z)$ has no cycle; and consequently, either $Y\cup Z = V(H)$ (which implies that $|Z|=2$, since $|V(H)|\geq 7$), or $|Z|=2$ and $H\setminus(Y\setminus Z)$ is a path joining the two members of Z. Thus in either case, $|Z|=2$.

Suppose first that $Y\cap Z=\emptyset$. From the choice of Z minimizing $Y\cup Z$, it follows that we can write $Z=\{z_1,z_2\}$ and $Y=\{y_1,\ldots,y_5\}$ such that z_1y_1, z_2y_2, z_2y_3 are edges. By criticality, some two of y_1,y_2,y_3 are joined by an edge in X. From **(Z2)**, this edge is not y_1y_2 or y_1y_3, so it must be y_2y_3; that is, y_2,y_3 are adjacent and X contains the edge joining them. Consequently, by **(Z2)**, z_1,y_1 are both nonadjacent to both of y_2,y_3. Since z_1 has at least two neighbours in Y, we may assume that z_1 is adjacent to y_4; and so, by the symmetry between y_1, y_4 we deduce that y_4 is nonadjacent to y_2, y_3, and exchanging z_1, z_2 implies that $y_1y_4 \in X$, and z_2 is nonadjacent to y_1, y_4. Then $(\{z_1, y_1, y_4, y_5\}, V(H) \setminus \{z_1, y_5\})$ is a cyclic 2-separation of H, a contradiction.

So $Y \cap Z$ is nonempty, and in particular $Y \cup Z \neq V(H)$, since $|V(H)| \geq 7$. Consequently $H \setminus (Y \setminus Z)$ is a path P say, joining the two vertices in Z. Let $Z = \{z_1, z_2\}$ and $Y = \{y_1, \ldots, y_5\}$. Suppose first that $Z \not\subseteq Y$; then we may assume that $z_2 = y_4$ (since we have shown that $Y \cap Z$ is nonempty), and z_1 is adjacent to y_1, y_2, and P has length ≥ 2. By criticality, some two of y_1, y_2, y_4 are joined by an edge in X, and by **(Z2)** it must be y_1y_2; and therefore, by **(Z2)** again, y_4 is nonadjacent to y_1, y_2. Consequently, by criticality, y_4 is adjacent to y_3, y_5, and the edges $y_3y_4, y_4y_5 \in X$. Thus z_1 is nonadjacent to y_3, y_5. Since H is cyclically 3-connected, we may assume that y_2y_3, y_1y_5 are edges; and **(Z2)** implies they are both in X. Thus all edges of the cycle y_1-y_2-y_3-y_4-y_5-y_1 belong to X. But then the third statement of the theorem holds.

Finally, we may assume that $Z \subseteq Y$; but then $|V(H) \setminus V(P)| = 3$, contrary to the hypothesis. This proves 7.3. ■

We need a small lemma for the next proof.

7.4 *Let H be a cyclically 3-connected graph, and let B be a branch of H. Let $Y \subseteq V(B)$ with $|Y| \leq 2$, such that if $|Y| = 1$ then the member of Y is an internal vertex of B. Let e be an edge of H not in $E(B)$ and not incident with any vertex in Y. There is no $Z \subseteq V(H)$ with $|Z| \leq 2$ such that for every edge $f \in E(H)$, f has an end in Z if and only if either $f = e$ or f has an end in Y.*

Proof. Suppose Z is such a subset, and let N be the set of edges of H with an end in Y. Since $N \cup \{e\}$ is the set of edges with an end in Z, it follows that $N \neq \emptyset$, and therefore $Y \neq \emptyset$. Since $Y \subseteq V(B)$, it follows that $N \cap E(B) \neq \emptyset$, and therefore $Z \cap V(B) \neq \emptyset$. Let $z \in Z$ be incident with e. Since $e \notin E(B)$, z does not belong to the interior of B, and therefore is incident with an edge $e' \neq e$ and not in B. Hence $e' \in N$, and therefore is incident with a member of Y, say y; and consequently y is an end of B. There is an edge $e'' \neq e'$ incident with y and not in B, and since $e'' \in N$, it follows that $y \in Z$. But $y \neq z$ since e is not incident with any member of Y; and so $Z = \{y, z\}$, and $z \notin V(B)$ since H is cyclically 3-connected. Since y is an end of B, by hypothesis there is a second member $y' \in Y$. There is an edge incident with y' and not incident with y or z, a contradiction. This proves 7.4. ■

Let us say a graph H is a *theta* if it is cyclically 3-connected and has exactly two branch-vertices and three branches. If G is a trigraph, a subset $X \subseteq V(G)$ is *connected* if $X \neq \emptyset$ and there is no partition of X into two nonempty sets that are strongly anticomplete to each other. A *component* of a trigraph G is a maximal connected subset of $V(G)$. The earlier results of this section are combined with 5.9 and 6.2 to prove the following.

7.5 *Let H be a robust graph, and let G be a claw-free trigraph, containing an $L(H)$-trigraph. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$, or G is decomposable.*

Proof. We assume that $G \notin \mathcal{S}_1 \cup \mathcal{S}_2$, and G is not decomposable; and we shall prove that $G \in \mathcal{S}_0$. We may choose H with $|V(H)|$ maximum satisfying the hypotheses of the theorem (we call this the "maximality" of H). By hypothesis, $E(H) \subseteq V(G)$, and $G|E(H)$ is an $L(H)$-trigraph. By 7.2, 5.9 and 6.2, we may therefore assume that $G|E(H)$ is a weak line trigraph of H. For each $h \in V(H)$, let $D(h)$ denote the set of edges of H incident with h in H. We begin with:

(1) *Let $v \in V(G) \setminus E(H)$.*

- *There exists $Y \subseteq V(H)$ with $|Y| \leq 2$ such that $N(v) \cap E(H) = \bigcup(D(y) : y \in Y)$, and there is a branch B of H including Y.*
- *If $N^*(v) \cap E(H) \neq N(v) \cap E(H)$, then $|Y| = 2$, $Y = \{y, y'\}$ say, where y belongs to the interior of B, and y, y' are either adjacent or have a common neighbour in B, and the (unique) edge of H in $N(v) \cap E(H) \setminus N^*(v)$ is the edge of B incident with y that is not in the subpath of B between y and y'.*
- *If $|Y| = 2$ and the two members of Y are adjacent in H, joined by an edge q of H say, let H' be obtained from H by deleting the edge q and adding a new edge v with the same two ends as q; then $G|E(H')$ is an $L(H')$-trigraph.*

For let $N = N(v) \cap E(H)$ or $N^*(v) \cap E(H)$. Then $N \subseteq E(H)$, and satisfies the hypotheses of 7.3, by 5.3 and 5.4. Thus one of the three conclusions of 7.3 holds. Suppose that the second holds; then there are $s_1, s_2, s_3, t_1, t_2, t_3, u_1, u_2 \in V(H)$, all distinct except that possibly $t_1 = t_2$, such that the following pairs are adjacent in H: $s_i t_i, s_i u_1, s_i u_2$ for $i = 1, 2, 3$, and $s_1 s_3$. Moreover, N contains exactly six of these ten edges, the six not incident with s_1. Since $\{u_2 s_2, v, u_1 s_2, u_2 s_1\}$ is not a claw, it follows that $u_1 s_2 \in N^*(v)$, and similarly $u_1 s_3, u_2 s_2, u_2 s_3 \in N^*(v)$; since $\{u_1 s_2, v, s_2 t_2, u_1 s_1\}$ is not a claw, $s_2 t_2 \in N^*(v)$ and similarly $s_3 t_3 \in N^*(v)$; and since $\{v, u_1 s_1, u_2 s_2, s_3 t_3\}$ is not a claw, $u_1 s_1 \notin N(v)$, and similarly $u_2 s_1, s_1 t_1 \notin N(v)$. Thus each of these six edges that belong to N also belongs to $N^*(v)$, and the four that do not belong to N also do not belong to $N(v)$, except possibly for $s_1 s_3$. It follows that G contains a YY-configuration (take the vertices called

$$a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2$$

to be

$$s_1 t_1, s_2 t_2, s_3 t_3, s_1 u_1, s_2 u_1, s_3 u_1, s_1 u_2, s_2 u_2, s_3 u_2, s_1 s_3, v$$

respectively), and so by 6.2, we deduce that G is decomposable, a contradiction.

Suppose that the third conclusion of 7.3 holds. Then there is a subgraph J of H isomorphic to a subdivision of K_4 (let its branch-vertices be $v_1, \dots, v_4$, and let $B_{i,j}$ denote the branch between v_i, v_j); and $B_{2,3}, B_{3,4}, B_{2,4}$ all have length 1, $B_{1,2}, B_{1,3}$ have length 2, and $B_{1,4}$ has length ≥ 2. Moreover, the edges of J in N are precisely the five edges of $B_{1,2}, B_{1,3}$ and $B_{2,3}$. As in the previous case, it follows that the edges of $B_{1,2}$ and $B_{1,3}$ belong to $N^*(v)$, and the edges of $B_{1,4}, B_{2,4}, B_{3,4}$ are not in $N(v)$. But then G contains an XX-configuration (take the edges of J incident in J with one of $v_1, \dots, v_4$, together with v), and by 5.9, either G is decomposable, or it belongs to $\mathcal{S}_1 \cup \mathcal{S}_2$, again a contradiction.

Thus the first outcome of 7.3 holds (when $N = N(v) \cap E(H)$ and when $N = N^*(v) \cap E(H)$). Choose $Y, Z \subseteq V(H)$ with $|Y|, |Z| \leq 2$ such that $N(v) \cap E(H) = \bigcup(D(y) : y \in Y)$ and $N^*(v) \cap E(H) = \bigcup(D(y) : y \in Z)$.

Suppose that $N(v) \cap E(H) \neq N^*(v) \cap E(H)$. Choose $e_0 \in E(H)$ semiadjacent to v in G. Choose $y \in Y$ incident with e_0 in H. Let $D(y) = \{e_0, \dots, e_k\}$, and for $i = 0, \dots, k$ let x_i be the vertex of H different from y that is incident with e_i in H, and let B_i be the branch of H containing e_i. Thus

$k \geq 1$. Since H is cyclically 3-connected, x_1 has a neighbour different from y, x_0; let f be an edge of H incident with x_1 and not with x_0, y. Since G is claw-free, it follows that $f \in N^*(v)$, and so there exists $y' \in Y \setminus \{y\}$ incident with f. Hence $|Y| = 2$, and $Y = \{y, y'\}$, and if $x_1 \neq y'$ then x_1, y' are adjacent in H and x_1 has no neighbours in H except y, y' and possibly x_0. Suppose that $k = 1$. Then $B_0 = B_1$, and y is an internal vertex of B_0, and $x_0, x_1 \in V(B_0)$. Moreover, x_0, x_1 are nonadjacent, and so either $y' = x_1$ or x_1 has only two neighbours y, y'. In either case the result holds. We may therefore suppose (for a contradiction) that $k \geq 2$, and so $B_0, B_1, \ldots, B_k$ are all distinct. Now from the choice of Z,

$$D(y) \cup D(y') \setminus \{e_0\} = \bigcup (D(z) : z \in Z).$$

In particular, $y, x_0 \notin Z$, and since $k \geq 2$ and $e_i \in N^*(v)$ for $1 \leq i \leq k$, it follows that $k = 2$ and $Z = \{x_1, x_2\}$. From the symmetry between B_1 and B_2, we may assume that y' does not belong to B_1. Hence x_1, y' are adjacent in H and x_1 has no neighbours in H except y, y' and possibly x_0. In particular, x_1 has no neighbour in B_1 except y, and so x_1 is a branch-vertex. Thus x_1 is adjacent to x_0. If also y' does not belong to B_2, then similarly x_2 is a branch-vertex with neighbours set $\{y, y', x_0\}$, and so $(\{y, y', x_0, x_1, x_2\}, V(H) \setminus \{y, x_1, x_2\})$ is a 2-separation of H. Hence $H \setminus \{y, x_1, x_2\}$ is a branch of H between x_0, y', and it contains all except three vertices of H, a contradiction. Thus y' belongs to B_2. Since y' is adjacent to x_1, it follows that y' is a branch-vertex, and so y, y' are the ends of the branch B_2. Since every edge incident with y' belongs to $N^*(v)$ and so has an end in Z, and $|Z| \leq 2$, it follows that $y' \in Z$, and so $y' = x_2$. But then $(\{y, x_0, x_1, x_2\}, V(H) \setminus \{y, x_1\})$ is a 2-separation, and so $H \setminus \{y, x_1\}$ is a branch of H containing all its vertices except two, a contradiction.

This proves the second assertion of (1), and the third assertion follows. If $|Y| \leq 1$, or $|Y| = 2$ and some branch of H contains both members of Y, then the first assertion of (1) holds, so we assume (for a contradiction) that $Y = \{h_1, h_2\}$ say, and no branch of H contains both h_1, h_2. Consequently $N(v) \cap E(H) = N^*(v) \cap E(H)$. Let H' be the graph obtained from H by adding the edge v incident with both h_1, h_2. Then H' is robust (since h_1, h_2 do not belong to the same branch of H), and yet $G|E(H')$ is an $L(H')$-trigraph, a contradiction to the maximality of H. This proves (1).

For each $v \in V(G) \setminus E(H)$, let $Y(v) \subseteq V(H)$ be the set Y described in (1). For each $v \in E(H)$, let $Y(v)$ be the set consisting of the two vertices of H incident with v in H. Make the following definitions:

- For each branch-vertex t of H, let $M(t) = \{v \in V(G) : Y(v) = \{t\}\}$.
- For each branch B with ends t_1, t_2 say, let $M(B) = \{v \in V(G) : Y(v) = \{t_1, t_2\}\}$.
- For each branch B and each end t of B, let

$$M(t, B) = \{v \in V(G) : Y(v) = \{t, h\} \text{ for some } h \text{ in the interior of } B\}.$$

- For each branch B with ends t_1, t_2 say, let

$$S(B) = \{v \in V(G) : \emptyset \neq Y(v) \subseteq V(B) \setminus \{t_1, t_2\}\}.$$

- Let $Z = \{v \in V(G) : Y(v) = \emptyset\}$.

From (1), we see that all these sets are pairwise disjoint (unless H is a theta, in which case all the sets $M(B)$ are equal), and have union $V(G)$.

(2) *Let B be a branch of H with ends t_1, t_2, let $v \in M(B)$, and let $u \in V(G)$ be adjacent to v. Then $Y(u)$ contains at least one of t_1, t_2.*

For $Y(v) = \{t_1, t_2\}$, and we may assume that $t_1, t_2 \notin Y(u)$. Suppose that $|Y(u)| \leq 1$. Let $B_1 \neq B$ be a branch incident with t_1 and with $V(B_1) \cap Y(u) = \emptyset$, with ends t_1, t_3 say. Let e_1 be the edge of B_1 incident with t_1, and let e_2 be any edge incident with t_2. Since $\{v, e_1, e_2, u\}$ is not a claw of G, we deduce that for every choice of e_2, either e_2 is incident with a member of $Y(u)$ or e_2 shares an end with e_1. Since there are at least three choices of e_2, and at most two of them share an end with e_1, and at most one is incident with a member of $Y(u)$, it follows that we have equality throughout; that is, t_2 has degree three, $|Y(u)| = 1$, $Y(u) = \{s\}$ say, and t_1, t_2 are adjacent (and consequently H is not a theta, and therefore $t_3 \neq t_2$), and the pairs t_2s, t_1t_3, t_2t_3 are adjacent. By exchanging t_1, t_2 we deduce also that t_1 has degree 3 and t_1, s are adjacent. Consequently H is a subdivision of K_4, and there is a branch of H with ends s, t_3. There are only two vertices of H not in this branch, contrary to hypothesis.

This proves that $|Y(u)| = 2$, say $Y(u) = \{s_1, s_2\}$. Let B' be a branch with $Y(u) \subseteq V(B')$. Since we have already seen that one of s_1, s_2 does not belong to B, it follows that $B' \neq B$. Suppose that B, B' share an end, say t_1, and let t_3 be the other end of B'. There is an edge e_1 of H incident with t_1, that belongs to neither of B, B'. Let e_2 be any edge incident with t_2; for each such choice, $\{v, u, e_1, e_2\}$ is not a claw in G. By choosing e_2 from B we deduce that t_1, t_2 are adjacent and therefore H is not a theta. It follows that for all choices of e_2, either e_2 has an end in $Y(u)$ (which, since H is not a theta, implies that e_2 is incident with t_3 and $t_3 \in Y(u)$), or e_2 shares an end with e_1. There is at most one choice for which the first occurs, and two for which the second occurs; and since t_2 has degree ≥ 3, we have equality throughout. More precisely, t_2 has degree 3, $t_3 \in Y(u)$, and the pairs t_1t_2, t_2t_3, t_2t_4 are adjacent, where e_1 has ends t_1, t_4. Moreover, no other choice of e_1 is possible, and so t_1 also has degree 3. Consequently H is a subdivision of K_4, and there is a branch P between t_3, t_4. By hypothesis, at least four vertices of H do not belong to P, and so B' has length ≥ 3. Let f_1 be an edge of B' incident with a vertex in $Y(u)$ but not incident with either of t_1, t_3 (this exists since B' has length ≥ 3 and one of its internal vertices is in $Y(u)$). Let f_2 be the edge of P incident with t_3. Then $\{u, v, f_1, f_2\}$ is a claw in G, a contradiction.

This proves that B, B' do not share an end, and so H is not a theta. We have already seen that one of s_1, s_2 is adjacent to one of t_1, t_2, say s_1, t_1 are adjacent. Consequently s_1 is an end of B'. Suppose that s_2 belongs to the interior of B'. Let e_1 be an edge incident with t_1, not in B and not incident with s_1; and let e_2 be any edge incident with t_2. Since $\{v, u, e_1, e_2\}$ is not a claw in G, it follows that for all choices of e_2, either e_2 is adjacent to s_1 or to an end of e_1. Consequently t_2 has degree 3, and t_2 is adjacent to s_1 and to both ends of e_1. Since this also holds for all choices of e_1, we deduce that t_1 also has degree 3. Let e_1 have ends t_1, t_3 say. Since H is cyclically 3-connected, it follows H is a subdivision of K_4 and t_3 is an end of B'. But then only two vertices of H do not belong to the branch B', contrary to hypothesis.

This proves that s_1, s_2 are both ends of B', and so $u \in M(B')$. Thus there is symmetry between u, v. Suppose that B has length 1, and let q be the edge of H incident with t_1, t_2. Let H' be the graph obtained from H by deleting q and adding a new edge v with the same ends t_1, t_2 as q. Then

H' is isomorphic to H, and by (1), $G|E(H')$ is an $L(H')$-trigraph, and so from (1) applied to H', there is a set $Y \subseteq V(H')$ with $|Y| \leq 2$ such that an edge of H' is adjacent to u in G if and only if it is incident in H' with a member of Y. But the edges of H' adjacent to u in G are precisely those with an end in $\{s_1, s_2\}$, together with the new edge v, and this contradicts 7.4. We may therefore assume that B has length > 1, and by symmetry we may assume the same for B'.

Let e_1 be the edge of B incident with t_1, and let e_2 be any edge of H incident with t_2. Since $\{v, u, e_1, e_2\}$ is not a claw in G, it follows that for all choices of e_2, either e_2 is incident in H with one of s_1, s_2, or it shares an end with e_1. Consequently t_2 has degree 3, and t_2 is adjacent to both s_1, s_2, and B has length 2. Similarly t_1, s_1, s_2 have degree 3, and B' has length 2, and s_1, s_2 are adjacent to both of t_1, t_2. But then $|V(H)| = 6$, a contradiction. This proves (2).

(3) *Let p_1-$\cdots$-p_k be a path of G such that $k \geq 2$, $p_1, p_k \notin Z$, and $p_2, \ldots, p_{k-1} \in Z$. Then either*

- *there is a branch B of H with ends t_1, t_2 say, such that p_1, p_k both belong to*
$$M(t_1) \cup M(t_2) \cup M(t_1, B) \cup M(t_2, B) \cup S(B),$$
or
- *$k = 2$, and $Y(p_1) \cap Y(p_2)$ contains a branch-vertex of H.*

For suppose first that $p_1 \in M(B)$ for some branch B. By (2), $k = 2$ and the second statement of the claim holds. So we may assume that p_1 does not belong to any $M(B)$, and the same for p_k. Since $p_1 \notin Z$, it follows that either $Y(p_1) = \{t_1\}$ for some branch-vertex t_1 of H, or there is a branch B_1 of H such that $Y(p_1) \subseteq V(B_1)$ and some internal vertex of B_1 belongs to $Y(p_1)$. Analogous statements hold for p_k. Suppose that $|Y(p_1)| = 1$ and $|Y(p_k)| = 1$, say $Y(p_1) = \{y_1\}$ and $Y(p_k) = \{y_2\}$. We claim that y_1, y_2 belong to the same branch of H. For suppose not. Then we may assume that p_i, p_j are strongly antiadjacent for $1 \leq i, j \leq k$ with $j \geq i + 2$. Let H' be the graph obtained from H by adding a new branch between y_1, y_2 with edges $p_1, \ldots, p_k$. Then H' is robust, and $G|E(H')$ is an $L(H')$-trigraph, contrary to the maximality of H. This proves that y_1, y_2 belong to the same branch of H; and so the first statement of the claim holds.

Thus we may assume that at least one of $|Y(p_1)|, |Y(p_k)| = 2$, say $|Y(p_1)| = 2$. Then $N(p_1) \cap E(H)$ is not a strong clique, and since p_2 is adjacent to p_1 and G contains no claw, it follows that p_2 has a strong neighbour in $N(p_1) \cap E(H)$, and in particular $p_2 \notin Z$. Thus $k = 2$.

Since $|Y(p_1)| = 2$, it follows that for some branch B_1 of H, $Y(p_1) \subseteq V(B_1)$ and some internal vertex of B_1 belongs to $Y(p_1)$. Let $Y(p_1) = \{y, y'\}$ say, where y' belongs to the interior of B_1. Next suppose that $|Y(p_2)| = 1$, say $Y(p_2) = \{z\}$. We may assume that $z \notin V(B_1)$, for otherwise the first statement of the claim holds. Let e' be an edge of B_1 incident with y' and not with y. Let e be an edge of H incident with y, not incident with z, and with no common end with e'. (This exists, since if y is an end of B_1 there are at least two edges incident with y and disjoint from e', and at most one of them is incident with z.) But then $\{p_1, p_2, e, e'\}$ is a claw in G, a contradiction. This proves that $|Y(p_2)| = 2$. Let $Y(p_2) = \{z, z'\}$ say, and let B_2 be a branch of H with $z, z' \in V(B_2)$ and with z' in the interior of B_2. We may assume that $B_2 \neq B_1$, for otherwise the first statement of the claim holds.

Suppose that $Y(p_1) \cap Y(p_2) \neq \emptyset$. It follows that $y = z$ is a common end of B_1, B_2. But then $p_1 \in M(y, B_1)$ and $p_2 \in M(y, B_2)$, and the second statement of the claim holds. We assume therefore that $Y(p_1) \cap Y(p_2) = \emptyset$.

If $p_2 \in E(H)$, then its ends in H are z, z', and therefore it has no end in $Y(p_1)$, a contradiction since p_1, p_2 are adjacent in G. Thus $p_2 \notin E(H)$, and similarly $p_1 \notin E(H)$. We claim that z, z' are nonadjacent in H. For suppose they are adjacent. Let q be the edge of B_2 joining them. Since $Y(p_1) \cap Y(p_2) = \emptyset$, it follows that q, p_1 are strongly antiadjacent in G. Let H' be the graph obtained from H by deleting q and replacing it by an edge p_2, joining the same two vertices z, z'. Then $G|E(H')$ is an $L(H')$-trigraph, by (1). Since H' is isomorphic to H, it follows from (1) applied to H' that there is a subset $Y \subseteq V(H')$ such that the set of members of $E(H')$ adjacent in G to p_1 equals the set of edges of H' with an end in Y. Now the set of members of $E(H')$ adjacent in G to p_1 equals $(N(p_1) \cap E(H)) \cup \{p_2\}$, since q is not adjacent to p_1 in G. Moreover, $N(p_1) \cap E(H)$ is the set of edges of H with an end in $Y(p_1)$, and since q has no end in $Y(p_1)$, this is equal to the set of edges of H' with an end in $Y(p_1)$. Consequently, the set of edges of H' with an end in Y equals the union of $\{p_2\}$ and the set of edges of H' with an end in $Y(p_1)$. But this is impossible, by 7.4. This proves that z, z' are nonadjacent, and similarly y, y' are nonadjacent.

Since y, y' are nonadjacent vertices of B_1 and y' is in the interior of B_1, there are edges e, e' of B_1 incident with y, y' respectively, such that e, e' have no end in common. Since $\{p_1, p_2, e, e'\}$ is not a claw in G, it follows that p_2 is adjacent in G to one of e, e', and so some vertex of $Y(p_2)$ belongs to $V(B_1)$. Since z' is an internal vertex of B_2, we deduce that B_1, B_2 have a common end z. Similarly their common end is y, and so $y = z$, contradicting that $Y(p_1) \cap Y(p_2) = \emptyset$. This proves (3).

(4) *Let $t \in V(H)$ be a branch-vertex. If $v_1, v_2 \in V(G)$ are distinct and antiadjacent in G, and $t \in Y(v_1) \cap Y(v_2)$, then there are distinct branches B_1, B_2, both of length ≥ 2, with $v_i \in M(B_i)$ $(i = 1, 2)$; and every vertex of $V(H)$ adjacent to t in H either belongs to one of B_1, B_2, or has degree 3 in H and is adjacent in H to all the ends of B_1, B_2.*

For since v_1, v_2 are antiadjacent in G, and $t \in Y(v_1) \cap Y(v_2)$ is a branch-vertex, it follows that $v_1, v_2 \notin E(H)$. By (1), there are branches B_1, B_2 of H, incident with t, such that $Y(v_i) \subseteq V(B_i)$ $(i = 1, 2)$. (If H is a theta, and some $Y(v_i)$ consists of the two branch-vertices, then we can choose any branch to be B_i; in this case, choose a shortest branch.) Let B_i have ends t, t_i $(i = 1, 2)$ say. Let x be adjacent in H to t, and not in $V(B_1) \cup V(B_2)$. Let $y \neq t$ be a second neighbour of x. Let e, f be the edges tx, xy of H. Since $\{e, f, v_1, v_2\}$ is not a claw in G, it follows that f is strongly adjacent in G to at least one of v_1, v_2, and in particular, $y \in Y(v_1) \cup Y(v_2)$. Since $Y(v_i) \subseteq V(B_i)$ $(i = 1, 2)$, we deduce that for some $i \in \{1, 2\}$, $y = t_i \in Y(v_i)$. If H is a theta, then x is the internal vertex of some branch of length 2; and since $v_i \in M(B_i)$, from the choice of B_i it follows that B_i has length ≤ 2. But then a branch of H contains all its vertices except two, contrary to the hypothesis. Thus, H is not a theta. Since no two branches have the same pair of ends, it follows that x is a branch-vertex; and since this holds for all choices of y, we deduce that x has degree 3 and is adjacent in H to both t_1, t_2, and $t_i \in Y(v_i)$ $(i = 1, 2)$. Moreover, B_1, B_2 are distinct. Suppose that say B_2 has length 1, and let q be the edge tt_2. Let H' be obtained from H by deleting q and adding a new edge v_2 incident with the same two vertices t, t_2. Then H' is isomorphic to H, and by (1) $G|E(H')$ is an $L(H')$-trigraph, and so by (1) applied to H', we may assume that there exists $Y \subseteq V(H') = V(H)$ with $|Y| \leq 2$, such that the set of edges of H' with an end in Y equals the set of edges of H' that are adjacent to v_1 in G. But in the triangle $\{x, t, t_2\}$ of H', exactly one of its edges is adjacent to v_1 in G, a contradiction. This proves that B_2, and similarly B_1, has length ≥ 2, and so proves (4).

(5) *If B is a branch of H of length 1, with ends t_1, t_2, then $M(t_1)$ is strongly anticomplete to $M(t_2)$.*

If there exists $v_1 \in M(t_1)$ adjacent to some $v_2 \in M(t_2)$, let H' be the graph obtained from H by deleting the edge of B, and adding a two-edge path between t_1, t_2, with edges v_1, v_2 (with v_i incident with t_i for $i = 1, 2$, and the middle vertex of this path being a new vertex). Then H' is robust, and $G|E(H')$ is an $L(H')$-trigraph, contrary to the maximality of H. This proves (5).

For each branch B of H with ends t_1, t_2, we define $C(B), A(t_1, B), A(t_2, B)$ as follows. Let $C(B)$ be the union of $S(B)$ and the set of all $v \in Z$ such that there is a path with interior in Z from v to some vertex in $S(B)$. (Thus if B has length 1 then $C(B)$ is empty.) Let $A(t_1, B)$ be the set of all $v \in M(t_1) \cup M(t_1, B)$ with a neighbour in $C(B)$. Define $A(t_2, B)$ similarly.

(6) *For every branch B with ends t_1, t_2, every vertex in $V(G) \setminus C(B)$ with a neighbour in $C(B)$ belongs to $A(t_1, B) \cup A(t_2, B)$.*

For let $v \in V(G) \setminus C(B)$, with a neighbour in $C(B)$. From the definition of $C(B)$, $v \notin S(B) \cup Z$. Let P be a minimal path of G between $S(B)$ and v with interior in Z. By (3),

$$v \in M(t_1) \cup M(t_1, B) \cup M(t_2) \cup M(t_2, B).$$

Hence $v \in A(t_1, B) \cup A(t_2, B)$. This proves (6).

(7) *Let B be a branch with ends t_1, t_2. If $v \in V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$ has a neighbour in $A(t_1, B)$, then there is a branch B' of H incident with t_1 such that $v \in M(t_1) \cup M(B') \cup M(t_1, B')$. In particular, v is either strongly complete or strongly anticomplete to $A(t_1, B)$.*

The second claim follows from the first and (4). To prove the first, let $v \in V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$, and assume it has a neighbour in $A(t_1, B)$. Since $A(t_1, B)$ is nonempty, it follows that t_1, t_2 are nonadjacent in H. If $t_1 \in Y(v)$, then the claim holds, so we may assume that $t_1 \notin Y(v)$. Suppose first that v is adjacent in G to every $e \in D(t_1)$ that is not in B. Since $t_1 \notin Y(v)$, it follows that $Y(v)$ contains all vertices of H that are adjacent to t_1 and not in $V(B)$. There are at least two such vertices, and $|Y(v)| \leq 2$, and so t_1 has degree 3, and its two neighbours not in B are both in $Y(v)$. By (1), there is a branch B' joining these two vertices, and $v \in M(B')$, contrary to (2). Thus there exists $e \in D(t_1)$ not in B, such that no end of e belongs to $Y(v)$. Now v has a neighbour $a \in A(t_1, B)$. By definition of $A(t_1, B)$, a has a neighbour $c \in C(B)$. Also, a is adjacent in G to v, e, c, and v, e are nonadjacent. Moreover, $v, e \notin A(t_1, B) \cup A(t_2, B) \cup C(B)$, and since $c \in C(B)$, it follows from (6) that c is nonadjacent to v, e. But then $\{a, v, e, c\}$ is a claw in G, a contradiction. This proves (7).

(8) *There is no branch B of H with $S(B)$ nonempty, and consequently every branch has length at most 2. In particular, H is not a theta.*

For suppose that B is a branch with $S(B)$ nonempty. Let its ends be t_1, t_2. Since $S(B)$ is nonempty, it follows that B has length ≥ 2. We claim that $(A(t_1, B), C(B), A(t_2, B))$ is a breaker. To show this, in view of (6) and (7) it remains to check that:

- $A(t_1, B), A(t_2, B)$ are nonempty strong cliques
- there is a vertex in $V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$ with a neighbour in $A(t_1, B)$ and an antineighbour in $A(t_2, B)$; there is a vertex in $V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$ with a neighbour in $A(t_2, B)$ and an antineighbour in $A(t_1, B)$; and there is a vertex in $V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$ with an antineighbour in $A(t_2, B)$ and an antineighbour in $A(t_1, B)$
- if $A(t_1, B)$ is strongly complete to $A(t_2, B)$, then there do not exist adjacent $x, y \in V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$ such that x is $A(t_1, B) \cup A(t_2, B)$-complete and y is $A(t_1, B) \cup A(t_2, B)$-anticomplete.

Since B has length > 1, and $S(B) \neq \emptyset$, it follows that $M(t_1, B)$ is nonempty and is a subset of $A(t_1, B)$, and in particular, $A(t_1, B) \neq \emptyset$, and similarly $A(t_2, B) \neq \emptyset$. By (4), $A(t_1, B), A(t_2, B)$ are strong cliques, and so the first statement holds. For the second, let $e \in E(H) \setminus E(B)$ be incident with t_1; then e has a neighbour in $A(t_1, B)$ and an antineighbour in $A(t_2, B)$, namely the first and last edges of B. Moreover, since H is cyclically 3-connected and at least four vertices of H do not belong to B, it follows that some edge f of H has no end in $V(B)$, and therefore is antiadjacent in G to both the first and last edges of B. The second claim follows. Thus, it remains to check the third.

Suppose then that $x, y \in V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$; x is $A(t_1, B) \cup A(t_2, B)$-complete and y is $A(t_1, B) \cup A(t_2, B)$-anticomplete, and x, y are adjacent. By (7), $x \in M(B)$. Since x, y are adjacent, (2) implies that $y \in M(t_1) \cup M(B') \cup M(t_1, B')$ for some branch B' incident with t_1. But then y is complete to $A(t_1, B)$, by (4). Since $A(t_1, B)$ is nonempty, it is not also anticomplete to $A(t_1, B)$, a contradiction. Consequently $(A(t_1, B), C(B), A(t_2, B))$ is a breaker. By 4.4, G is decomposable, a contradiction. This proves (8).

(9) *$Z = \emptyset$.*

For suppose not, and let W be a component of $G|Z$. Since G does not admit a 0-join, there are vertices not in W with neighbours in W; let X be the set of all such vertices. Thus, for each $x \in X$, $x \notin E(H)$ (since it has a neighbour in Z) and $Y(x)$ is nonempty (since W is a component of $G|Z$). Moreover, the set of neighbours of x in $E(H)$ is a strong clique, since G contains no claw; and consequently $|Y(x)| = 1$, say $Y(x) = \{t\}$. If t belongs to the interior of a branch B then $x \in S(B)$, contrary to (8); and so t is a branch-vertex. Suppose that there exists $x_1, x_2 \in X$ with $Y(x_i) = \{t_i\}$ $(i = 1, 2)$, where $t_1 \neq t_2$. There is a minimal path P between x_1, x_2 with interior in W; and by (3) applied to this path, there is a branch B with ends t_1, t_2. By (8), B has length ≤ 2. Let H' be obtained from H by deleting the edges and interior vertices of B, and adding the members of $V(P)$ to H as the edges of a new branch B' between t_1, t_2, in the appropriate order. Then H' is robust, and $G|E(H')$ is an $L(H')$-trigraph, and so by the maximality of H, we deduce that B' has length at most that of B. In particular, B' has length at most 2, and so $|V(P)| \leq 2$. But $x_1, x_2 \in V(P)$, and so x_1, x_2 are adjacent; and moreover, B has length 2. Now we recall that x_1 has a neighbour w say in W. Since $\{x_1, w, x_2, e\}$ is not a claw in G (where e is some edge of H incident with t_1 and not with t_2), it follows that x_2 is strongly adjacent to w. Thus x_1, x_2 are the only edges of H' that are adjacent to w in G. We deduce that when H is replaced by H', and Y' denotes the function analogous to Y for H', then $Y'(w)$ contains the middle vertex of B', contrary to (8) applied to H'. Consequently there is no such x_2; and so there is a branch-vertex t of H such that $Y(x) = \{t\}$ for all $x \in X$. By 5.5, X is a strong clique. By (3) and (4), every vertex of G not in

$W \cup X$ is either strongly complete or strongly anticomplete to X. But then the result follows from 4.2. This proves (9).

(10) *For every branch B with ends t_1, t_2, if $v_i \in M(t_i) \cup M(t_i, B)$ for $i = 1, 2$, and v_1, v_2 are adjacent in G, then B has length 2 and v_1, v_2 are its two edges.*

For let F_1 be the set of vertices in $M(t_1) \cup M(t_1, B)$ with a neighbour in $M(t_2) \cup M(t_2, B)$, and define F_2 similarly. By (4), F_1, F_2 are strong cliques. We claim that every vertex $v \notin F_1 \cup F_2$ is either strongly complete or strongly anticomplete to F_i, for $i = 1, 2$. For let v have a neighbour $f_1 \in F_1$ say. We may assume that $t_1 \notin Y(v)$, for otherwise v is strongly complete to F_1, by (4). By (3) and (9), there is a branch B' with ends t_1, t_3 say, such that $v \in M(t_3) \cup M(t_3, B')$, and in particular, $t_3 \in Y(v) \subseteq V(B') \setminus \{t_1\}$. Since $v \notin F_2$, it follows that $B' \neq B$, and therefore $t_3 \neq t_2$, since H is not a theta. Since v, f_1 are adjacent, (3) implies that $f_1 \notin M(t_1, B)$, and so $f_1 \in M(t_1)$. Let e be an edge of H incident with t_1 and not in B, B', and let $f_2 \in F_2$ be adjacent in G to f_1. Then f_1 is adjacent in G to all of v, f_2, e. Since $Y(v) \subseteq V(B') \setminus \{t_1\}$, it follows that v, e are antiadjacent in G. Similarly, since $f_2 \in M(t_2) \cup M(t_2, B)$, f_2, e are antiadjacent in G. Since $\{f_1, v, f_2, e\}$ is not a claw, it follows that v, f_2 are strongly adjacent in G. By (3), $v \notin M(t_3, B')$, and so $v \in M(t_3)$; and similarly $f_2 \in M(t_2)$; and also by (3), there is a branch B'' of H with ends t_2, t_3. Let H' be the graph obtained from H by adding a new vertex x and three new edges f_1, v, f_2, joining x to t_1, t_2, t_3 respectively. Then H' is robust, and $G|E(H')$ is an $L(H')$-trigraph, contrary to the maximality of H. This proves our claim that every vertex not in $F_1 \cup F_2$ is either strongly complete or strongly anticomplete to F_i, for $i = 1, 2$. Thus (F_1, F_2) is a homogeneous pair, nondominating since H is not a theta and therefore some edge of H is incident with no vertex in B; and so by 4.3 F_1, F_2 both contain at most one element. To deduce the claim, let v_1, v_2 be as in the statement of (10); if B has length 2, then the edges of B belong to $F_1 \cup F_2$ and the claim follows. If B has length 1, then $v_i \in M(t_i)$ for $i = 1, 2$, contrary to (5). This proves (10).

From (10), every vertex of G not in $E(H)$ belongs either to $M(B)$ for some branch B, or to $M(t)$ for some branch-vertex t. If for all pairs v_1, v_2 of vertices in $V(G) \setminus E(H)$, v_1 is adjacent to v_2 if and only if $Y(v_1) \cap Y(v_2) \neq \emptyset$, then G is a weak line trigraph and the theorem holds by 7.1 (for $\alpha(G) \geq 3$ and $|V(G)| \geq 7$ since H is robust). And we have already shown that this statement holds for all v_1, v_2 such that one of $|Y(v_1)|, |Y(v_2)| = 1$, by (4) and (10), and the "only if" implication holds for all v_1, v_2, by (2). From (4), we may therefore assume that there are antiadjacent $v_1, v_2 \in V(G)$, and distinct branch-vertices t_1, t_2, t_3 of H, and branches B_1, B_2 between t_1, t_3 and t_2, t_3 respectively, such that:

- $v_i \in M(B_i)$ $(i = 1, 2)$
- B_1, B_2 both have length 2, and
- every vertex of $V(H)$ adjacent to t_3 in H either belongs to one of B_1, B_2, or has degree 3 in H and is adjacent to all the ends of B_1, B_2.

Now H is not a theta. Let B_3 be the branch of H with ends t_1, t_2, if it exists. Let N be the set of all neighbours of t_3 that do not belong to B_1, B_2, let $V_1 = N \cup \{t_1, t_2, t_3\} \cup V(B_1) \cup V(B_2)$ and let $V_2 = (V(H) \setminus V_1) \cup \{t_1, t_2\}$. Since (V_1, V_2) is a 2-separation of H, we deduce that either $V(H) = V_1$, or the branch B_3 exists and $V(H) = V_1 \cup V(B_3)$. In either case, no branches of H have length > 1 except possibly B_1, B_2 and B_3 if it exists.

(11) *For $u_1, u_2 \in V(G) \setminus E(H)$, either u_1, u_2 belong to distinct sets $M(B_i)$ $(i = 1, 2, 3)$, or u_1, u_2 are adjacent if and only if $Y(u_1) \cap Y(u_2) \neq \emptyset$.*

For we have seen that if u_1, u_2 are adjacent, then $Y(u_1) \cap Y(u_2) \neq \emptyset$; and the converse holds by (4) unless $u_1 \in M(B)$ and $u_2 \in M(B')$ for distinct branches B, B', both of length ≥ 2. But B_1, B_2, B_3 are the only such branches. This proves (11).

(12) *$M(t) = \emptyset$ for all branch-vertices $t \neq t_1, t_2, t_3$ of H.*

For suppose that $x \in M(t)$ where $t \neq t_1, t_2, t_3$. We have seen that t is adjacent in H to all of t_1, t_2, t_3. Let e be the edge of H between t, t_3. Then e is adjacent in G to all of x, v_1, v_2. But v_1, v_2 are antiadjacent, and x is antiadjacent to v_1, v_2 by (2). Hence $\{e, x, v_1, v_2\}$ is a claw, a contradiction. This proves (12).

For $i = 1, 2, 3$, let $E_i = E(B_i) \cup M(B_i)$, setting $E_3 = \emptyset$ if B_3 does not exist. Thus E_1, E_2, E_3 are three strong cliques. For $i = 1, 2, 3$, let

$$F_i = M(t_i) \cup \bigcup (M(B) : B \neq B_1, B_2, B_3 \text{ is a branch of } H \text{ incident with } t_i).$$

From (8), (9), (10), (12) it follows that the six sets $E_1, E_2, E_3, F_1, F_2, F_3$ are pairwise disjoint and have union $V(G)$. From (4) and (11), F_1, F_2, F_3 are strong cliques. By (4) and (11) E_i is strongly complete to F_i and to F_3 for $i = 1, 2$, and E_3 is strongly complete to $F_1 \cup F_2$. By (2), E_1 is strongly anticomplete to F_2, and E_2 is strongly anticomplete to F_1, and E_3 is strongly anticomplete to F_3. Thus G is expressible as a hex-join, a contradiction. This proves 7.5. ■

8 Prisms

We say a trigraph G is a *prism* if it is a line trigraph of a theta graph. If G is a prism, then there are disjoint strong triangles $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, and three paths P_1, P_2, P_3, where each P_i has ends a_i, b_i, such that $V(G) = V(P_1) \cup V(P_2) \cup V(P_3)$ and for $1 \leq i < j \leq 3$, if $u \in V(P_i)$ and $v \in V(P_j)$ are adjacent then $(u, v) = (a_i, a_j)$ or (b_i, b_j). We say the three paths P_1, P_2, P_3 *form* the prism. A prism formed by paths of length $n_1, n_2, n_3 \geq 1$ is called an *(n_1, n_2, n_3)-prism.*

Our objective in this section is to handle the claw-free trigraphs that contain certain prisms. For big enough prisms, this is accomplished by 7.5. More precisely, we have (immediately from 7.5, taking H to be the theta):

8.1 *Let G be a claw-free trigraph, containing an (n_1, n_2, n_3)-prism, where either $n_1, n_2, n_3 \geq 2$, or $n_1, n_2 \geq 3$. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$, or G is decomposable.*

In this section we prove the same thing for some slightly smaller prisms, namely the $(3, 2, 1)$-prism, the $(2, 2, 1)$-prism and the $(3, 1, 1)$-prism. We need first some lemmas about strips. A *strip* in a trigraph G means a triple (A, C, B) of disjoint subsets of $V(G)$, such that

- A, B are nonempty strong cliques
- every vertex of $A \cup B$ belongs to a rung of the strip (a *rung* means a path between A and B with interior in C)
- for every vertex $v \in C$, there is a path from A to v with interior in C, and a path from v to B with interior in C.

Let (A_i, B_i, C_i) be a strip for $i = 1, 2$. We say they are *parallel* if

- A_1, B_1, C_1 are disjoint from A_2, B_2, C_2
- A_1 is strongly complete to A_2 and B_1 is strongly complete to B_2, and
- if $v_1 \in A_1 \cup B_1 \cup C_1$ and $v_2 \in A_2 \cup B_2 \cup C_2$ are adjacent then either $v_i \in A_i$ for $i = 1, 2$, or $v_i \in B_i$ for $i = 1, 2$.

Then $(A_1 \cup A_2, C_1 \cup C_2, B_1 \cup B_2)$ is a strip that we call the *disjoint union* of the first two strips. If a strip is not expressible as the disjoint union of two strips, we say it is *nonseparable*. We need the following lemma.

8.2 *Let G be a claw-free trigraph, and let $(A_1, B_1, C_1), (A_2, B_2, C_2)$ be parallel strips. Suppose that (A_1, C_1, B_1) is nonseparable and C_1 is nonempty. Then C_1 is connected and every vertex of $A_1 \cup B_1$ has a neighbour in C_1.*

Proof. Let C_3 be a component of C_1 and $C_4 = C_1 \setminus C_3$. Let A_3 be the set of members of A_1 with a neighbour in C_3, and $A_4 = A_1 \setminus A_3$, and define B_3, B_4 similarly.

(1) *If $a \in A_3$, then no neighbour of a belongs to $B_4 \cup C_4$.*

For suppose that $x \in B_4 \cup C_4$ is a neighbour of a. By definition of A_3, a has a neighbour $c \in C_3$; and let $a_2 \in A_2$. Since $\{a, a_2, x, c\}$ is not a claw, it follows that x is adjacent to c. Since $x \notin C_3$ and C_3 is a component of C_1, we deduce that $x \notin C_4$; and since x has a neighbour in C_3, we deduce that $x \notin B_4$, a contradiction. This proves (1).

(2) *Let R be a rung of (A_1, C_1, B_1). Then either $V(R) \subseteq A_3 \cup C_3 \cup B_3$, or $V(R) \subseteq A_4 \cup C_4 \cup B_4$.*

For suppose first that some vertex of the interior of R belongs to C_3. Then C_3 contains all the interior of R, since C_3 is a component of C_1, and so the ends of R belong to $A_3 \cup B_3$ and the claim holds. We may therefore assume that C_3 is disjoint from the interior of R. Let a be the end of R in A_1. Let r be the neighbour of a in R. If $a \in A_3$, then by (1), $r \in B_3 \cup C_3$, and since C_3 is disjoint from the interior of R, we deduce that R has length 1 and $r \in B_3$ and the claim holds. Thus we may assume that $a \notin A_3$, and similarly the other end of R is not in B_3; but then $V(R) \subseteq A_4 \cup C_4 \cup B_4$ and the claim holds. This proves (2).

(3) *(A_3, C_3, B_3) is a strip.*

For since C_3 is nonempty, and (A_1, B_1, C_1) is a strip, it follows that there is a path between C_3 and A_1 with interior in C_1 and hence in C_3; and consequently A_3 is nonempty, and similarly B_3 is nonempty. Consequently (A_3, C_3, B_3) is a strip, by (2). This proves (3).

Suppose that $A_4 \cup B_4 \neq \emptyset$. Then by (2), (A_4, C_4, B_4) is a strip, and by (1) the two strips $(A_3, C_3, B_3), (A_4, C_4, B_4)$ are parallel, contrary to hypothesis that (A_1, B_1, C_1) is nonseparable. Thus $A_4 = B_4 = \emptyset$. If there exists $v \in C_4$, then there is a path from v to A_1 with interior in C_1, which is therefore disjoint from C_3; and consequently this path has interior in C_4. Let its end in A_1 be a. By (1), $a \in A_4$, a contradiction since $A_4 = \emptyset$. This proves 8.2. ■

In several applications later in the paper, we shall have two parallel strips, and a path between them. Here is a lemma for use in that situation.

8.3 *Let G be a claw-free trigraph, and for $i = 1, 2$ let R_i be a path in G of length ≥ 1, with ends a_i, b_i. Suppose that a_1-R_1-b_1-b_2-R_2-a_2-a_1 is a hole. Let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ be connected, and for $i = 1, 2$ let there be a vertex in R_i with a neighbour in X. Then there is a path p_1-$\cdots$-p_k with $p_1, \ldots, p_k \in X \setminus (V(R_1) \cup V(R_2))$ such that:*

- *none of $p_1, \ldots, p_k$ belong to $R_1 \cup R_2$, and*
- *for $1 \leq i \leq k$, p_i has a neighbour in $V(R_1)$ if and only if $i = k$, and p_i has a neighbour in R_2 if and only if $i = 1$, and*
- *p_i, p_j are strongly antiadjacent for $1 \leq i, j \leq k$ with $i \leq j - 2$.*

Moreover, either:

1. *p_1 has exactly two neighbours in R_2 and they are strongly adjacent, and the same for p_k in R_1, or*
2. *$k = 1$, and one of R_1, R_2 has length 1, and the other has length 2, and p_1 is complete to $V(R_1) \cup V(R_2)$, or*
3. *$k = 1$ and for $i = 1, 2$ the neighbours of p_1 in R_i are $\{a_i, b_i\}$, and p_1 is strongly adjacent to all of a_1, b_1, a_2, b_2, or*
4. *$k = 1$, and p_1 is adjacent to both $\{a_1, a_2\}$ or to both $\{b_1, b_2\}$, and p_1 has a unique neighbour in one of R_1, R_2.*

Proof. We may assume that X is minimal with the given property, and therefore X is disjoint from $V(R_1) \cup V(R_2)$, and $X = \{p_1, \ldots, p_k\}$ for some path p_1-$\cdots$-p_k satisfying the three bullets above. Let $M = N_G(p_1) \cap V(R_2)$ and $N = N_G(p_k) \cap V(R_1)$. Suppose first that $|N| = 1$. By 5.4, the vertex of N is not an internal vertex of R_1, and so we may assume that $N = \{a_1\}$. By 5.4, p_k is adjacent to a_2, and therefore $k = 1$ and $a_2 \in M$. But then the final statement of the theorem holds.

We may therefore assume that $|M|, |N| \geq 2$. If M consists of two strongly adjacent vertices, and so does N, then the first statement of the theorem holds. So we may assume that there exist $x, y \in N$, antiadjacent. Since $\{p_k, x, y, p_{k-1}\}$ is not a claw, $k = 1$. Since $\{p_1, x, y, z\}$ is not a claw for z in the interior of R_2, it follows that $M = \{a_2, b_2\}$. Since $\{p_1, x, y, a_2\}$ is not a claw, it follows that

$a_1 \in \{x, y\}$ and the same for b_1. If $|N| = 2$ then the third statement of the theorem holds, and so we may assume that N contains some vertex c from the interior of R_1. Since $\{p_1, c, a_2, b_2\}$ is not a claw, R_2 has length 1. Since $\{p_1, c, a_1, b_2\}$ is not a claw, c is adjacent to a_1 and similarly to b_1. But then R_1 has length 2 and the second statement of the theorem holds. This proves 8.3. ■

Next we show, for several different prisms, that if a claw-free trigraph G contains one of these prisms, then either G is decomposable, or belongs to one of our basic classes. These proofs are quite similar, so we have extracted the main argument in the following lemma.

8.4 *Let G be a claw-free trigraph, and let the three paths R_1, R_2, R_3 form a prism in G. Let R_i have ends a_i, b_i for $1 \leq i \leq 3$, where $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are strong triangles. Suppose that R_1 has length > 1. Then one of the following holds (possibly after exchanging R_2, R_3):*

- *R_1 has length 2, R_2 has length 1, and there is a vertex v complete to $V(R_1) \cup V(R_2)$ and strongly anticomplete to $V(R_3)$, or*
- *R_2 has length 1, and either R_3 has length 1 or R_1 has length 2, and there is a vertex v that is complete to $V(R_2)$ and strongly anticomplete to $V(R_3)$, with exactly two neighbours in R_1, namely either the first two or last two vertices of R_1, or*
- *R_2 and R_3 both have length 1, and there is no vertex w that is complete to one of $V(R_2), V(R_3)$ and anticomplete to the other and to $V(R_1)$, or*
- *$G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$, or G is decomposable.*

Proof. For $i = 2, 3$, let $A_i = \{a_i\}$, $B_i = \{b_i\}$ and C_i be the interior of R_i. Then (A_i, C_i, B_i) is a strip with a unique rung R_i. It follows that there is a strip (A_1, C_1, B_1) such that:

- (A_i, C_i, B_i) $(i = 1, 2, 3)$ are three parallel strips,
- R_1 is a rung of (A_1, B_1, C_1), and
- (A_1, B_1, C_1) is nonseparable.

Choose (A_1, B_1, C_1) such that W is maximal, where W denotes the union of the vertex sets of the three strips.

(1) *We may assume that every vertex $v \in V(G) \setminus (A_1 \cup B_1 \cup C_1)$ is strongly anticomplete to C_1.*

For let $v \in V(G) \setminus (A_1 \cup B_1 \cup C_1)$, and suppose it has a neighbour in C_1. Consequently $v \notin W$. Let $N = N_G(v) \cap W, N^* = N^*_G(v) \cap W$. From the maximality of W, it follows that N meets one of $V(R_2), V(R_3)$. Suppose first that $a_2, a_3 \in N$. Since N meets C_1, it follows from 5.3 that $N \cap V(R_i) = \{a_i\}$ for $i = 2, 3$. Let c_2 be the neighbour of $a_2 \in R_2$. By 5.4 (with a_3-a_2-c_2 and A_1-a_2-c_2), it follows that $a_3 \in N^*$ and $A_1 \subseteq N^*$, and similarly $a_2 \in N^*$, and so v can be added to A_1, contrary to the maximality of W. Thus N contains at most one of a_2, a_3, and at most one of b_2, b_3 by symmetry. By 5.3, it follows that N meets exactly one of R_2, R_3, say R_2.

Now $C_1 \cup \{v\}$ is connected, and so by 8.3 there is a path p_1-$\cdots$-p_k of G with $v = p_1$ and with $p_2, \ldots, p_k \in C_1$, satisfying one of the four statements of 8.3. Certainly none of $p_1, \ldots, p_k$ have neighbours in R_3, and so 5.4 implies that that the fourth statement of 8.3 is impossible. Also 5.4 implies the third is impossible, since R_1 has length > 1. If the second statement of 8.3 holds, then the first statement of the theorem holds. Consequently we may assume that the first statement of 8.3 holds.

Since R_1, R_2, R_3 form a prism, there is a theta H say with two branch-vertices t_1, t_2, and three branches B_1, B_2, B_3, where the edges of B_i are the vertices of R_i in order. For $i = 1, 2$, choose a vertex s_i of H, in the interior of B_i, such that the two edges of B_i incident with s_i are the two neighbours of p_k in R_1 (if $i = 1$) and the two neighbours of p_1 in R_2 (if $i = 2$). Let H' be obtained from H by adding a new branch between s_2 and s_1 with edges $p_1, \dots, p_k$ in order. Then $G|E(H')$ is an $L(H')$-trigraph, and so by 7.5, we may assume that H' is not robust. But H' is a subdivision of K_4, and $|V(H')| \geq 6$. If $|V(H')| = 6$ then $k = 1$ and the second statement of the theorem holds. If $|V(H')| \geq 7$ then some branch of H' contains all its vertices except at most three, and so $k = 1$ and again the second statement holds. This proves (1).

(2) *We may assume that every vertex $v \in V(G) \setminus (A_1 \cup B_1 \cup C_1)$ is either strongly complete or strongly anticomplete to A_1.*

For let $v \in V(G) \setminus (A_1 \cup B_1 \cup C_1)$, and suppose it has a neighbour and an antineighbour in A_1. Then $v \notin W$. Let $N = N_G(v) \cap W, N^* = N^*_G(v) \cap W$. By (1), we may assume that $N \cap C_1 = \emptyset$. By 8.2, every vertex in A_1 has a neighbour in C_1. Since N meets A_1, 5.4 (with a_2-A_1-C_1 and a_3-A_1-C_1) implies that $a_2, a_3 \in N^*$. Choose $a'_1 \in A_1$ such that $a'_1 \notin N^*$. For $i = 2, 3$, if C_i is nonempty then 5.4 (with a'_1-a_i-C_i) implies that N^* meets C_i, and if $C_i = \emptyset$ then 5.4 (with a'_1-a_i-b_i) implies that $b_i \in N^*$. By 5.3, $N \cap (B_2 \cup C_2)$ is complete to $N \cap (B_3 \cup C_3)$; and so C_2, C_3 are empty, and $b_2, b_3 \in N^*$. Suppose there is a vertex w that is complete to one of $V(R_2), V(R_3)$ and anticomplete to the other and to $V(R_1)$. Thus $w \notin W$. Let w be complete to $V(R_2)$ say. By 5.4 (with a'_1-a_2-w) it follows that $w \in N^*$; but that contradicts 5.3, since $N \cap (A_1 \cup \{w, b_3\})$ includes a triad. Thus there is no such w; but then the third statement of the theorem holds. This proves (2).

If every vertex in $V(G) \setminus (A_1 \cup B_1 \cup C_1)$ is strongly complete to one of A_1, B_1, then the third statement of the theorem holds. If not, then from (1) and (2), (A_1, C_1, B_1) is a breaker, and so by 4.4 G is decomposable. This proves 8.4. ■

Now we can process the little prisms.

8.5 *Let G be a claw-free trigraph, containing an (n_1, n_2, n_3)-prism, where $n_1 \geq 3$ and $n_2 \geq 2$. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ or G is decomposable.*

Proof. By 8.1 we may assume that $n_2 = 2$ and $n_3 = 1$. Then the result is immediate from 8.4. ■

8.6 *Let G be a claw-free trigraph, containing an (n_1, n_2, n_3)-prism, where $n_1, n_2 \geq 2$. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ or G is decomposable.*

Proof. By 8.5 and 8.1, we may assume that $n_1 = n_2 = 2$ and $n_3 = 1$. Let R_1, R_2, R_3 be three paths of G, forming a prism, with lengths 2, 2, 1. Let W be the union of their vertex sets. Let R_i be a_i-c_i-b_i for $i = 1, 2$, and let R_3 have vertices a_3-b_3, where $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are strong triangles. By 8.4, we may assume there is a vertex $v_1 \in V(G) \setminus W$, complete to $V(R_3)$, strongly anticomplete to $V(R_2)$, and adjacent to c_1 and to at least one of a_1, b_1. By exchanging R_1, R_2, we may also assume there exists $v_2 \in V(G) \setminus W$ complete to $V(R_3)$, strongly anticomplete to $V(R_1)$, and adjacent to c_2 and to at least one of a_2, b_2. Suppose first that v_1 is adjacent to both a_1, b_1. Since $\{v_1, v_2, a_1, b_1\}$ is not a claw, v_1 is antiadjacent to v_2. Since $\{a_3, v_1, v_2, a_2\}$ is not a claw, v_2 is adjacent to a_2, and by symmetry v_2 is adjacent to b_2. But then the subtrigraph induced on these ten vertices is an $icosa(-2)$-trigraph, and the theorem follows from 5.7. We may therefore assume that v_1 is adjacent to exactly one of a_1, b_1, and v_2 to exactly one of a_2, b_2. Since $\{a_3, a_1, v_1, v_2\}$ and $\{b_3, b_1, v_1, v_2\}$ are not claws, v_1, v_2 are strongly adjacent. Then the subtrigraph induced on these ten vertices is an $L(H)$-trigraph, where H is a graph consisting of a cycle of length six and one more vertex with four neighbours in the cycle, not all consecutive. In particular, H is robust, and the result follows from 7.2. This proves 8.6. ■

Let $(A, \emptyset, B)$ be a strip. A *step* (in this strip) means a hole a_1-a_2-b_2-b_1-a_1 where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. We say the strip is *step-connected* if for every partition (X, Y) of A or of B with $X, Y \neq \emptyset$, there is a step meeting both X, Y. We say an (n_1, n_2, n_3)-prism is *long* if $n_1 + n_2 + n_3 \geq 5$.

8.7 *Let G be a claw-free trigraph, containing a long prism. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$, or G is decomposable.*

Proof. Let the paths R_1, R_2, R_3 form a long (n_1, n_2, n_3)-prism in G. By 8.6 we may assume that $n_1 \geq 3$, and $n_2 = n_3 = 1$. Let the paths R_i have ends a_i, b_i as usual, where R_1 has length ≥ 3, and R_2, R_3 have length 1.

(1) *We may assume that, for every such choice of R_1, R_2, R_3, there is no vertex w that is complete to one of $V(R_2), V(R_3)$ and anticomplete to the other and to $V(R_1)$.*

For if not then by 8.4, we may assume that there is a vertex v that is complete to $V(R_2)$ and anticomplete to $V(R_3)$, with exactly two neighbours in R_1, namely either the first two or last two vertices of R_1. From the symmetry we may assume that v is adjacent to a_1 and its neighbour in R_1. But then $G|(V(R_1) \cup V(R_2) \cup V(R_3) \cup \{v\}) \setminus \{a_2\}$ is a $(2,2,1)$-prism (or longer), and the result follows from 8.6. So we may assume that the statement of (1) holds.

Let $A_1 = \{a_1\}, B_1 = \{b_1\}$, and let C_1 be the interior of R_1. Now $(\{a_2, a_3\}, \emptyset, \{b_2, b_3\})$ is a step-connected strip, parallel to (A_1, C_1, B_1); and therefore we may choose a strip $(A_2, \emptyset, B_2)$ such that

- $(A_2, \emptyset, B_2)$ is step-connected, and $a_2, a_3 \in A_2$ and $b_2, b_3 \in B_2$
- the strips (A_1, C_1, B_1), $(A_2, \emptyset, B_2)$ are parallel, and
- $A_2 \cup B_2$ is maximal.

Let $W = V(R_1) \cup A_2 \cup B_2$.

(2) *Every vertex $v \in V(G) \setminus (A_2 \cup B_2)$ is either strongly complete or strongly anticomplete to A_2.*

For let $v \in V(G) \setminus (A_2 \cup B_2)$, and suppose it has a neighbour and an antineighbour in A_2. Thus

$v \notin W$. Let $N = N_G(v) \cap W, N^* = N^*_G(v) \cap W$. Since $(A_2, \emptyset, B_2)$ is step-connected and $|A_2| \geq 2$, there is a step a'_2-a'_3-b'_3-b'_2-a'_2 such that $a'_2 \in N$ and $a'_3 \notin N^*$. 5.4 (with a'_3-a'_2-b'_2) implies that $b'_2 \in N^*$. Suppose that $b'_3 \in N$. Then 5.4 (with a'_3-b'_3-b_1) implies that $b_1 \in N^*$; 5.3 implies that $C_1 \cap N = \emptyset$; 5.4 (with a'_3-a_1-C_1) implies that $a_1 \notin N$; and 5.4 (with B_2-b_1-C_1) implies that $B_2 \subseteq N^*$. If we add v to B_2 then a'_2-a'_3-b'_3-v-a'_2 is a step of the enlarged strip, showing that this new strip is step-connected; but this contradicts the maximality of W. Thus $b'_3 \notin N$. Let R'_2, R'_3 be the rungs a'_2-b'_2 and a'_3-b'_3; then v is complete to $V(R'_2)$, and anticomplete to $V(R'_3)$. By (1) applied to the paths R_1, R'_2, R'_3, v has a neighbour in $V(R_1)$. Let us apply 8.3 to R_1, R'_2. Since $a'_3, b'_3 \notin N$, the third and fourth outcomes of 8.3 contradict 5.4, and so one of the first two outcomes applies. The second is impossible since R_1, R'_2 both do not have length 2, and so v has two adjacent neighbours in both R_1 and R'_2. If the neighbours of v in R_1 both belong to the interior of R_1, then $G|((V(R_1) \cup V(R'_2) \cup V(R'_3) \cup \{v\})$ is an $L(H)$-trigraph where H is a graph consisting of a cycle and one extra vertex with three pairwise nonadjacent neighbours in the cycle; and in particular, H is robust and the result follows from 7.5. So we may assume that v is adjacent to a_1 and to its neighbour in R_1. Hence $G|((V(R_1) \cup V(R'_2) \cup V(R'_3) \cup \{v\}) \setminus \{a'_2\}$ is a $(2, 2, 1)$-prism or longer, and the result follows from 8.6. This proves (2).

Let $c_1 \in C_1$ be a neighbour of a_1. For $u, v \in A_2$, since $\{a_1, u, v, c_1\}$ is not a claw, it follows that A_2 is a strong clique in G, and similarly so is B_2. From (1) and (2), we deduce that (A_2, B_2) is a homogeneous pair, nondominating since $C_1 \neq \emptyset$, and the result follows from 4.3. This proves 8.7. ■

9 Neighbours in holes

Our goal in the next few sections is to handle claw-free trigraphs that contain holes of length ≥ 7. We begin with some definitions. An *n-hole* in a trigraph G means a hole in G of length n. Let C be a n-hole, with vertices c_1-$\cdots$-c_n-c_1 in order; we call this an *n-numbering.* (We shall read these and similar subscripts modulo n, usually without saying so.) Let $v \in V(G) \setminus V(C)$, and let $N = N_G(v) \cap V(C), N^* = N^*_G(v) \cap V(C)$. We say that

- v is a *hat* (relative to C, and to the given n-numbering) if $N^* = \{c_i, c_{i+1}\}$ for some i
- v is a *clone* if one of N, N^* equals $\{c_{i-1}, c_i, c_{i+1}\}$ for some i
- v is a *star* if $n \geq 5$ and one of N, N^* equals $\{c_{i-1}, c_i, c_{i+1}, c_{i+2}\}$ for some i
- v is a *centre* if $N = V(C)$ (and therefore $n \leq 5$)
- c is a *hub* if $n \geq 6$ and $N = N^* = \{c_i, c_{i+1}, c_j, c_{j+1}\}$ for some i, j such that $i-1, i, i+1, j-1, j, j+1$ are all distinct modulo n.

Since N, N^* may be different, it is possible for v to be both a hat and a clone, and various other combinations are also possible. If $N^* = N$, we say that v is a *strong* hat, clone etc.

9.1 *Let G be a claw-free trigraph, and let C, v, N, N^* as above. If $N^* = N$ then either $N = \emptyset$, or v is a hat, clone, star, hub or centre with respect to C. If $N \neq N^*$, then v is either both a hat and a clone, or both a clone and a star, or both a star and a centre, or (if $n = 4$) both a clone and a centre.*

The proof is clear. We also need:

9.2 *Let G be a claw-free trigraph, and let C be a hole in G. Let $v_1, v_2 \in V(G) \setminus V(C)$, and for $i = 1, 2$, let N_i, N_i^* be respectively the sets of neighbours and strong neighbours of v_i in $V(C)$.*

- *If there exist $x \in N_1 \cap N_2$ and $y \in V(C) \setminus (N_1^* \cup N_2^*)$, consecutive in C, then v_1, v_2 are strongly adjacent.*
- *If there exist $x, y \in N_1 \setminus N_2^*$ that are antiadjacent, then v_1, v_2 are strongly antiadjacent.*

Again, the proof is clear.

9.3 *Let G be claw-free, and let C be a hole in G of length ≥ 7, with a hub. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ or G is decomposable.*

Proof. Let w be a hub for C. Let w have neighbours a_1, a_2, b_1, b_2 in C, where a_1 is adjacent to a_2, and b_1 is adjacent to b_2, and a_1, b_1, b_2, a_2 lie in this order in C. Consequently there are two disjoint paths R_1, R_2 in C between $\{a_1, a_2\}$ and $\{b_1, b_2\}$, with $V(C) = V(R_1) \cup V(R_2)$, where R_i is between a_i, b_i for $i = 1, 2$, and R_1, R_2 both have length at least two, and one of them, say R_1, has length at least three. Let $A_1 = \{a_1\}, B_1 = \{b_1\}$, and let C_1 be the interior of R_1. If $u \in V(R_1)$ and $v \in V(R_2)$ are adjacent, then either $u \in \{a_1, b_1\}$ or $v \in \{a_2, b_2\}$, for otherwise $\{u, v, x, y\}$ would be a claw (where x, y are the neighbours of u in R_1); and if say $u = a_1$, then since $\{u, w, x, v\}$ is not a claw (where x is a neighbour of u in $R_1 \setminus \{a_2\}$), it follows that $v \in \{a_2, b_2\}$; and if $u = a_1$ and $v = b_2$, then $\{u, v, a_2, x\}$ is a claw, with x as before. Hence (u, v) is one of $(a_1, a_2), (b_1, b_2)$. Moreover, since $\{w, a_1, a_2, b_1\}$ is not a claw, it follows that a_1 is strongly adjacent to a_2 and similarly b_1 is strongly adjacent to b_2.

We may therefore choose a strip (A_2, C_2, B_2) with the following properties:

- (A_i, C_i, B_i) $(i = 1, 2)$ are parallel strips
- $a_2 \in A_2, b_2 \in B_2$ and R_2 is a rung of (A_2, C_2, B_2)
- (A_2, C_2, B_2) is nonseparable
- w is strongly complete to $A_2 \cup B_2$ and strongly anticomplete to C_2, and
- $W = V(R_1) \cup A_2 \cup C_2 \cup B_2$ is maximal with these properties.

(1) *We may assume that every $v \in V(G) \setminus (A_2 \cup B_2 \cup C_2)$ is strongly anticomplete to C_2.*

For suppose that $v \in V(G) \setminus (A_2 \cup B_2 \cup C_2)$ has a neighbour in C_2. Then $v \notin W$; let $N = N_G(v) \cap W, N^* = N_G^*(v) \cap W$. From the maximality of W, N meets $\{w\} \cup V(R_1)$. Suppose first that $w \in N$. From 5.4 (with a_1-w-b_1), we may assume that $a_1 \in N^*$. From 5.3 (with C_2, w, C_1 and C_2, a_1, b_1) it follows that $N \cap C_1 = \emptyset$, and $b_1 \notin N$. From 5.4 (with A_2-a_1-C_1), it follows that $A_2 \subseteq N^*$. But then v can be added to A_2, contrary to the maximality of W. Thus $w \notin N$. Consequently N meets $V(R_1)$. Now $C_2 \cup \{v\}$ is connected, by 8.2; choose p_1-$\cdots$-p_k as in 8.3 (with R_1, R_2 exchanged, and taking $X = C_2 \cup \{v\}$), where $p_1 = v$, and $p_2, \ldots, p_k \in C_2$. Then none of $p_1, \ldots, p_k$ are adjacent to w. By 9.1 applied to p_k and the hole w-a_2-R_2-b_2-w, it follows that p_k has at least two neighbours in R_2, and similarly p_1 has at least two neighbours in R_1. Thus the fourth outcome of 8.3

is impossible; and since R_1 has length at least two, 5.4 implies the third is impossible. The second is false since R_1, R_2 have length at least two, and so the first holds. If either $k > 1$ or the four vertices of N in the hole C are not consecutive or R_2 has length > 2, then $G|(V(C) \cup \{w, p_1, \ldots, p_k\})$ is an $L(H)$-trigraph, where H is a robust graph, and the result follows from 7.5. If $k = 1$ and the four vertices of N in C are consecutive and R_2 has length 2, we may assume that v is adjacent to a_1, a_2 and their neighbours in C. But then $G|(V(C) \cup \{v, w\} \setminus \{a_2\})$ is a $(2, 2, 1)$-prism or longer, and the result follows from 8.6. This proves (1).

(2) *Every $v \in V(G) \setminus (A_2 \cup B_2 \cup C_2)$ is either strongly complete or strongly anticomplete to A_2.*

For suppose that $v \in V(G) \setminus (A_2 \cup B_2 \cup C_2)$ has a neighbour and an antineighbour in A_2. Then $v \notin W$. By the assumption of (1), $N_G(v) \cap C_2 = \emptyset$. By 8.2, every vertex in A_2 has a neighbour in C_2, and so 5.4 (with C_2-A_2-w; C_2-A_2-a_1; A_2-w-b_1; and A_2-a_1-C_1) implies that $w, a_1, b_1 \in N_G(v)$, and $N_G(v)$ contains the neighbour of a_1 in R_1. But this contradicts 5.3. This proves (2).

From (1) and (2) we deduce that (A_2, C_2, B_2) is a breaker, and the result follows from 4.4. This proves 9.3. ■

9.4 *Let G be a claw-free trigraph, and let C be a hole in G of length ≥ 7. Let a_1, a_2, b_2, b_1 be four consecutive vertices of C, in order, and let $h, w \in V(G) \setminus V(C)$, such that the neighbours of w in C are a_1, a_2, b_2, b_1, and the strong neighbours of h in C are a_2, b_2. Then G is decomposable.*

Proof. By 9.1, w and h are strongly antiadjacent; and by 9.1 again, it follows that h has no neighbours in C except a_2, b_2. By 5.3 (with a_1, a_2, b_1) it follows that a_1, a_2 are strongly adjacent, and similarly so are b_1, b_2. Let R_1 be the path $C \setminus \{a_2, b_2\}$, and let $C_1 = V(C) \setminus \{a_1, a_2, b_1, b_2\}$. Let R_2 be the path a_2-b_2. Thus $(\{a_1\}, C_1, \{b_1\})$ is a strip, and $(\{a_2\}, \{h\}, \{b_2\})$ is another. We claim that these strips are parallel. For suppose that $u \in V(R_1)$ and $v \in \{a_2, b_2, h\}$ are adjacent. Then $v \neq h$, so we may assume that $v = a_2$. Since $\{v, h, w, u\}$ is not a claw, u is adjacent to w, and so $u \in \{a_1, b_1\}$; and $u \neq b_1$ since $\{a_2, h, a_1, b_1\}$ is not a claw. Thus $(u, v) = (a_1, a_2)$. This proves that the two strips are parallel. Hence we may choose a strip (A_2, C_2, B_2) with the following properties:

- (A_2, C_2, B_2) is parallel to $(\{a_1\}, C_1, \{b_1\})$
- $a_2 \in A_2, h \in C_2, b_2 \in B_2$
- (A_2, C_2, B_2) is nonseparable
- w is strongly complete to $A_2 \cup B_2$ and strongly anticomplete to C_2
- $W = V(R_1) \cup A_2 \cup B_2 \cup C_2$ is maximal subject to these conditions.

(1) *We may assume that every $v \in V(G) \setminus (A_2 \cup B_2 \cup C_2)$ is strongly anticomplete to C_2.*

For suppose that $v \in V(G) \setminus (A_2 \cup B_2 \cup C_2)$ has a neighbour in C_2. Then $v \notin W$; let $N = N_G(v) \cap W, N^* N^*_G(v) \cap W$. From the maximality of W, N meets $\{w\} \cup V(R_1)$. Suppose first that $w \in N$. From 5.4 (with a_1-w-b_1), we may assume that $a_1 \in N^*$. From 5.3 (with C_2, w, C_1 and C_2, a_1, b_1) it follows that $N \cap C_1 = \emptyset$, and $b_1 \notin N$. From 5.4 (with A_2-a_1-C_1), it follows that

$A_2 \subseteq N^*$. But then v can be added to A_2, contrary to the maximality of W. Thus $w \notin N$. Consequently N meets $V(R_1)$. Choose $p_1\text{-}\cdots\text{-}p_k$ as in 8.3 (with R_1, R_2 exchanged), where $p_1 = v$, and $p_2, \ldots, p_k \in C_2$. Then none of $p_1, \ldots, p_k$ are adjacent to w. By 9.1 applied to p_1 and the hole $w\text{-}a_1\text{-}R_1\text{-}b_1\text{-}w$, it follows that p_1 is adjacent to more than one vertex of R_1. Let c_1 be the vertex in R_1 consecutive with a_1. Suppose that the fourth outcome of 8.3 holds; then $k = 1$, and $v = p_1 = p_k$ has a unique neighbour in R_2, say a_2, and v is adjacent to a_1. By 5.4 applied to $w\text{-}a_2\text{-}h$, it follows that v is adjacent to h. Then by 5.4 applied to $w\text{-}a_1\text{-}c_1$, v is adjacent to c_1. By 5.3, v has no neighbours in C except c_1, a_1, a_2; but then the subgraph of G induced on $(V(C) \setminus \{a_2\}) \cup \{h, v\}$ is a long prism, and the result follows from 8.7. Thus we may assume that the fourth outcome of 8.3 does not hold. Since R_1 has length > 1, 5.4 implies the third outcome is impossible. The second is false since R_1 has length ≥ 3, and so the first holds. If $k > 1$ then $G|(V(C) \cup \{p_1, \ldots, p_k\})$ is a long prism, and the result follows from 8.7; so we assume that $k = 1$. If the four vertices of N in the hole C are not consecutive, then v is a hub for C and the result follows from 9.3. We may therefore assume that v is adjacent to c_1, a_1, a_2, b_2. But then $G|(V(C) \cup \{v, w\} \setminus \{a_2\})$ is a long prism, and the result follows from 8.7. This proves (1).

The remainder of the proof of 9.4 is identical with the latter part of the proof of 9.3, and we omit it. This proves 9.4. ■

10 Circular interval trigraphs

So far, our method has been to show that claw-free trigraphs containing subtrigraphs of certain types either are line trigraphs, or are decomposable (with a few sporadic exceptions). That is not adequate to handle all claw-free trigraphs with holes of length ≥ 7, because there is another major basic class of them, the long circular interval trigraphs. In this section we prove the following (we recall that $\mathcal{S}_3$ is the class of all long circular interval trigraphs):

10.1 *Let G be a claw-free trigraph with a hole of length ≥ 7. Then either $G \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_3$, or G is decomposable.*

To prove this we need two lemmas. A subset $X \subseteq V(G)$ is said to be *dominating* if every vertex of G either belongs to X or has a neighbour in X; and a subtrigraph H of G is said to be dominating if $V(H)$ is dominating. Let us say a *maximum* hole is a hole in G of maximum length. Dominating holes are convenient because of the following:

10.2 *Let C be a hole in a claw-free trigraph G, and let $v \in V(G) \setminus V(C)$ with a neighbour in C. Then v has two consecutive strong neighbours in C.*

Proof. Let C have vertices $c_1\text{-}\cdots\text{-}c_n\text{-}c_1$ in order, where v is adjacent to c_1 say. Then 5.4 (with $c_n\text{-}c_1\text{-}c_2$) implies that v is strongly adjacent to one of c_2, c_n, say c_2; and 5.4 (with $c_1\text{-}c_2\text{-}c_3$) implies that v is strongly adjacent to one of c_1, c_3. This proves 10.2. ■

10.3 *Let C be a maximum hole (of length n say) in a claw-free trigraph G. Then either G contains an (n_1, n_2, n_3)-prism, for some $n_1, n_2, n_3 \geq 1$ with $n_1 + n_2 = n - 2$, or G is decomposable, or C is dominating.*

Proof. Let Z be the set of all vertices of G that are not in $V(C)$ and have no neighbour in $V(C)$. We may assume that Z is nonempty; let W be a component of $G|Z$. Let X be the set of all vertices not in W but with a neighbour in W. Let $x \in X$; we claim that it has exactly two neighbours in $V(C)$ and they are strongly adjacent and therefore consecutive in C. For if it has two antiadjacent neighbours $u, v \in V(C)$, let $w \in W$ be adjacent to x; then $\{x, u, v, w\}$ is a claw, a contradiction. From 9.1, this proves that x has precisely two neighbours in C and they are consecutive in C. Suppose there exist $x_1, x_2 \in X$ with distinct sets of neighbours in C. Let P be a path between x_1, x_2 with interior in W. If x_1, x_2 have no common neighbour in C, then the subgraph of G induced on $V(C) \cup V(P)$ is an $(n_1, n_2, |E(P)|)$-prism for some $n_1, n_2 \geq 1$ with $n_1 + n_2 = n - 2$, and the theorem holds. If $c \in V(C)$ is adjacent to both x_1, x_2, then the subgraph induced on $V(C) \cup V(P) \setminus \{c\}$ is a hole of length $> n$, a contradiction. We may therefore assume that there are no such x_1, x_2. Let C have vertices c_1-$\cdots$-c_n-c_1 say, where every member of X is adjacent to c_1 and c_2 and to no other vertex of C. By 5.5, X is a strong clique. Let $v \in V(G) \setminus (X \cup W)$; we claim that v is either strongly complete or strongly anticomplete to X. If $v \in V(C)$ this is true, so we assume $v \notin V(C)$. Suppose that v is adjacent to $x_1 \in X$ and antiadjacent to $x_2 \in X$. Let $w \in W$ be adjacent to x_1. Since $v \notin W \cup X$ it follows that v, w are antiadjacent. Since $\{x_1, w, v, c_1\}$ is not a claw, v is adjacent to c_1 and similarly to c_2. Since $\{c_2, c_3, v, x_2\}$ is not a claw, v is adjacent to c_3 and similarly to c_n; but then $\{v, x_1, c_3, c_n\}$ is a claw, a contradiction. This proves that v is either strongly complete or strongly anticomplete to X. By 4.2, G is decomposable. This proves 10.3. ■

Before the second lemma, we need a few definitions. Let C be a hole in a trigraph G, with vertices c_1-c_2-$\cdots$-c_n-c_1 in order. Let $v_1, \dots, v_k \in V(G) \setminus V(C)$, and for $1 \leq i \leq k$ let $N_i \subseteq V(C)$ such that v_i is complete to N_i and anticomplete to $V(C) \setminus N_i$.

- If $k = 2$ and $N_1 = \{c_i, c_{i+1}\}$ and $N_2 = \{c_j, c_{j+1}\}$ for some i, j, and $N_1 \cap N_2 = \emptyset$, and v_1, v_2 are adjacent, we call $\{v_1, v_2\}$ a *hat-diagonal* for C.

- If $n \geq 5$ and $k = 2$ and $N_1 = \{c_i, c_{i+1}\}$ and $N_2 = \{c_{i-1}, c_i, c_{i+1}, c_{i+2}\}$ for some i, we call $\{v_1, v_2\}$ a *coronet* for C.

- If $n \geq 5$ and $k = 2$ and $N_1 = \{c_i, c_{i+1}, c_{i+2}, c_{i+3}\}$ and $N_2 = \{c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4}\}$ for some i, and v_1, v_2 are antiadjacent, we call $\{v_1, v_2\}$ a *crown* for C.

- If $n = 5$ or 6 and $k = 2$ and $N_1 = \{c_i, c_{i+1}, c_{i+2}, c_{i+3}\}$ and $N_2 = \{c_{i+3}, c_{i+4}, c_{i+5}, c_{i+6}\}$ and v_1, v_2 are adjacent, we call $\{v_1, v_2\}$ a *star-diagonal* for C.

- If $n = 6$ and $k = 3$ and $N_1 = \{c_i, c_{i+1}, c_{i+2}, c_{i+3}\}$ and $N_2 = \{c_{i+2}, c_{i+3}, c_{i+4}, c_{i+5}\}$ and $N_3 = c_{i-2}, c_{i-1}, c_i, c_{i+1}\}$ for some i, and $\{v_1, v_2, v_3\}$ is a clique, we call $\{v_1, v_2, v_3\}$ a *star-triangle* for C.

The second lemma we need is the following, the main result of [3].

10.4 *Let G be a claw-free trigraph with a hole. Suppose that every maximum hole is dominating, and has no hub, coronet, crown, hat-diagonal, star-diagonal, star-triangle or centre. Then either G admits a coherent W-join, or G is a long circular interval trigraph.*

Now we are ready to prove the main result of this section.

Proof of 10.1. Let G be a claw-free trigraph with a hole of length at least seven. By 8.7, we may assume that G does not contain a long prism, and that G is not decomposable. By 10.3, every maximum hole is dominating. By 9.3, we may assume that no maximum hole has a hub, and by 9.4, we may assume that no maximum hole has a coronet. If $\{s_1, s_2\}$ is a crown for a maximum hole C, then G contains a long prism (obtained from $G|V(C) \cup \{s_1, s_2\}$ by deleting the middle common neighbour of s_1, s_2 in C), which is impossible. Also no maximum hole has a hat-diagonal, since G has no long prism. By 10.4, we deduce that $G \in \mathcal{S}_3$. This proves 10.1. ■

11 Near-antiprismatic trigraphs

We turn now to a very special type of claw-free trigraph, which nevertheless turns up surprisingly often as an exceptional case.

11.1 *Let G be a claw-free trigraph, and let $a_0, b_0 \in V(G)$ be semiadjacent. Suppose that no vertex is adjacent to both a_0, b_0, and the set of vertices antiadjacent to both a_0, b_0 is a strong clique. Then one of the following holds:*

- *G admits twins or a nondominating or coherent W-join.*
- *The trigraph obtained from G by making a_0, b_0 strongly antiadjacent is a linear interval trigraph, and a_0, b_0 are the first and last vertices of the corresponding linear order of its vertex set (and in particular, $G \in \mathcal{S}_3$).*
- *G is a line trigraph of some graph H, and a_0, b_0 have a common end in H with degree two.*
- *There is a graph H with $E(H) = V(G)$, such that a_0, b_0 have a common end in H with degree two, and there is a cycle of H of length 4 with edges a_0, a, b, b_0 in order, such that every edge of H is incident with some vertex of this cycle, and a, b are antiadjacent in G, and the trigraph obtained from G by making a, b strongly adjacent is a line trigraph of H (and consequently G is expressible as a hex-join).*
- *$G = H$ or $H \setminus \{a_2\}$, where H is the trigraph with vertex set $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2\}$ and adjacency as follows: $\{a_0, a_1, a_2\}, \{b_0, b_1, b_2, b_3\}$, $\{a_2, c_1, c_2\}$ and $\{a_1, b_1, c_2\}$ are strong cliques; b_2, c_2 are semiadjacent; b_2, c_1 are strongly adjacent; b_3, c_1 are semiadjacent; a_0, b_0 are semiadjacent; and all other pairs are strongly antiadjacent. (Moreover if $G = H$ then G is expressible as a hex-join, and if $G = H \setminus \{a_2\}$ then G admits a generalized 2-join).*
- *G is near-antiprismatic.*

In particular, either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$ or G is decomposable.

Proof. We assume that G does not admit a nondominating or coherent W-join or twins. Let A, B and C be the sets of all vertices different from a_0, b_0 that are adjacent to a_0, to b_0 and to neither of a_0, b_0 respectively. Thus $V(G) = A \cup B \cup C \cup \{a_0, b_0\}$. Moreover, a_0 is strongly complete to A since a_0, b_0 are semiadjacent and $F(G)$ is a matching; and therefore $A \cup \{a_0\}$ is a strong clique since $A \cup \{a_0, b_0\}$ includes no claw. Similarly $B \cup \{b_0\}$ is a strong clique, and by hypothesis C is a strong clique.

(1) *We may assume that $A, B \neq \emptyset$. Moreover, if $a \in A$ and $b \in B$ are adjacent, they have the same neighbours in C (and in particular no vertex in C is semiadjacent to either of a, b).*

For suppose that $A = \emptyset$, say. Then (B, C) is a homogeneous pair, nondominating, and so 4.3 implies that $|B|, |C| \leq 1$. But then G is obtained from a linear interval trigraph as in the second outcome of the theorem. This proves the first claim. For the second, note that if $c \in C$ is adjacent to $a \in A$ and antiadjacent to b say, then $\{a, a_0, b, c\}$ is a claw, a contradiction. This proves (1).

(2) *Every vertex in A has at most one neighbour in B, and vice versa.*

For let H be the graph with vertex set $A \cup B$ and in which $a \in A$ and $b \in B$ are adjacent if they are adjacent in G. Let X be any component of H with $|X| > 1$; then by (1), $(X \cap A, X \cap B)$ is a homogeneous pair, coherent since all X-complete vertices belong to C (because $|X| > 1$), and so $|X \cap A|, |X \cap B| \leq 1$. This proves (2).

(3) *Every vertex in $A \cup B$ has a neighbour in C; and in particular, $C \neq \emptyset$, and we may assume that $|C| \geq 2$.*

For let A_0 be the set of vertices in A with no neighbour in C, and define B_0 similarly. By (1), A_0 is strongly anticomplete to $B \setminus B_0$, and B_0 is strongly anticomplete to $A \setminus A_0$. Consequently, $(A_0 \cup \{a_0\}, B_0 \cup \{b_0\})$ is a homogeneous pair, coherent since a_0, b_0 have no common neighbours. Since G admits no coherent W-join, it follows that A_0, B_0 are empty. This proves the first assertion of (3), and in particular $C \neq \emptyset$. Now suppose that $|C| = 1$, say $C = \{c\}$. Thus c is complete to $A \cup B$. If it is strongly complete to $A \cup B$, then (A, B) is a coherent homogeneous pair, and so $|A| = |B| = 1$ since G does not admit twins or a coherent W-join; and then G arises as in the second outcome of the theorem. We assume therefore that c is semiadjacent to some $a \in A$ say. Then c is strongly complete to $(A \setminus \{a\}) \cup B$, since $F(G)$ is a matching; and a is strongly anticomplete to B, by the second assertion of (1). Hence $(A \setminus \{a\}, B)$ is a coherent homogeneous pair, and so $|A| \leq 2$ and $|B| = 1$ since G does not admits twins or a coherent W-join; and then again G arises as in the second outcome of the theorem. This proves (3).

(4) *If every vertex in A is either strongly C-complete or strongly B-anticomplete, and every vertex in B is either strongly C-complete or strongly A-anticomplete, then the theorem holds.*

For then, let A_1 be the set of vertices in A with a neighbour in B, and define B_1 similarly. It follows that (A_1, B_1) is a coherent homogeneous pair, and so $|A_1|, |B_1| \leq 1$ since G does not admit twins or a coherent W-join. Let us say that $c, c' \in C$ are *A-incomparable* if there exists $a \in A$ adjacent to c and antiadjacent to c', and there exists $a' \in A$ adjacent to c' and antiadjacent to c. Let H be the graph with vertex set C, in which c, c' are adjacent if they are A-incomparable, and suppose that some component X of H satisfies $|X| \geq 2$. Let Y be the set of vertices in A with a neighbour in X and an antineighbour in X. Thus $A_1 \cap Y = \emptyset$. We claim that (X, Y) is a homogeneous pair. For if $u \in A \setminus Y$ then u is strongly Y-complete, and either strongly X-complete or strongly X-anticomplete, from the definition of Y. If $u \in B$, then u is strongly Y-anticomplete, since $A_1 \cap Y = \emptyset$. Suppose that $u \in B$ has a neighbour in X and an antineighbour in X; let X_1 be the set of neighbours of u in X, and let X_2 be the set of its antineighbours in X. Thus $|X_1 \cap X_2| \leq 1$. From the definition of H, and since $|X| \geq 2$, there exist distinct $c_1 \in X_1$ and $c_2 \in X_2$ which are A-incomparable; and so there exists $a \in A$ adjacent to c_1 and antiadjacent to c_2. Hence a is antiadjacent to u; but then $\{c_1, a, c_2, u\}$ is a claw, a contradiction. This proves every $u \in B$ is either strongly X-complete or strongly X-anticomplete. Now let $u \in C \setminus X$; then u is strongly X-complete, and we claim that it is either strongly Y-complete or strongly Y-anticomplete. For let X_1 be the set of vertices $x \in X$ such that every vertex of A adjacent to x is strongly adjacent to u, and let X_2 be the set of all $x \in X$ such that every vertex in A adjacent to u is strongly adjacent to x. For all $x \in X$, x, u are not A-incomparable, from the definition of X, and so $x \in X_1 \cup X_2$. Hence $X_1 \cup X_2 = X$. For all $x_1 \in X_1$ and $x_2 \in X_2$, every vertex in Y adjacent to x_1 is strongly adjacent to u and therefore strongly adjacent to x_2; and so x_1, x_2 are not A-incomparable. Consequently one of $X_1, X_2 = \emptyset$. If $X_1 = \emptyset$, then every neighbour of u in A is strongly complete to X, and so u is strongly Y-anticomplete; and if $X_2 = \emptyset$, then every antineighbour of u in A is strongly X-anticomplete, and so u is strongly Y-complete. This completes the proof of our claim that (X, Y) is a homogeneous pair. It is nondominating, because of b_0, and so 4.3 implies that $|X| \leq 1$, a contradiction. Thus there is no such X.

This proves that no two vertices in C are A-incomparable. For distinct $c, c' \in C$, we write $c \geq_A c'$ if every vertex in A adjacent to c' is strongly adjacent to c. We define $c \geq_B c'$ similarly. We write $c \geq c'$ if $c \geq_A c'$ and $c' \geq_B c$. We claim that the relation $\geq$ is a total order of C. To see this we observe:

- For distinct $c, c' \in C$, not both $c \geq c'$ and $c' \geq c$. For if both these hold, then $c \geq_A c'$, $c' \geq_B c$, $c' \geq_A c$, and $c \geq_B c'$; and so c, c' have the same neighbours in $A \cup B$ and in $C \setminus \{c, c'\}$, and no vertex is semiadjacent to either of them, and so they are twins, a contradiction.

- For distinct $c, c' \in C$, either $c \geq c'$ or $c' \geq c$. For if both are false, then we may assume that $c \not\geq_A c'$, and so $c' \geq_A c$ since c, c' are not A-incomparable; choose $a \in A$ adjacent to c' and antiadjacent to c. Since $c' \not\geq c$, it follows that $c \not\geq_B c'$; choose $b \in B$ adjacent to c' and antiadjacent to c. Then $a \notin A_1$ and $b \notin B_1$, and so $\{c', c, a, b\}$ is a claw, a contradiction.

- For distinct $c_1, c_2, c_3 \in C$, not all of $c_1 \geq c_2, c_2 \geq c_3$, and $c_3 \geq c_1$ hold. For if they do all hold, then since $c_1 \geq_A c_2$ and $c_2 \geq_A c_3$, it follows that $c_1 \geq_A c_3$, and similarly $c_3 \geq_B c_1$, and so $c_1 \geq c_3$; yet $c_3 \geq c_1$, contrary to the first observation above.

From these three observations, we see that $\geq$ is a total order of C. But then the second outcome of the theorem holds. This proves (4).

Let $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_n\}$, where for $1 \leq i \leq k$ a_i is adjacent to b_i, and otherwise each a_i is strongly antiadjacent to each b_j. By (4), we may assume that $k > 0$. Define $A' = \{a_{k+1}, \dots, a_m\}$, and $B' = \{b_{k+1}, \dots, b_n\}$. For each $c \in C$, let

$$I_c = \{i : 1 \leq i \leq k \text{ and } c \text{ is adjacent to } a_i, b_i.\}$$

We observe that, by (1), each $c \in C$ is strongly adjacent to a_i, b_i for all $i \in I_c$.

(5) *If $c, c' \in C$, and $i \in I_c \setminus I_{c'}$, then a_i, b_i are the only vertices in $A \cup B$ that are adjacent to c and antiadjacent to c'. In particular, $|I_c \setminus I_{c'}| \leq 1$.*

For suppose that a_j is adjacent to c and antiadjacent to c', say, where $j \neq i$. Then $\{c, c', a_j, b_i\}$ is a claw by (2), a contradiction. This proves (5).

Let j be the maximum cardinality of the sets I_c ($c \in C$). By (5), $|I_c| = j$ or $j-1$ for all $c \in C$. By (3) $j \geq 1$. Let

$$P = \{c \in C : |I_c| = j - 1\}$$

and $Q = C \setminus P$. Let Z be the set of vertices in $A' \cup B'$ with a neighbour in Q. By (5), if $p \in P$ and $q \in Q$, then $I_p \subseteq I_q$, and every vertex in $A' \cup B'$ that is adjacent to q is strongly adjacent to p. In particular, Z is strongly complete to P. By definition, Q is nonempty. Now there are four cases:

- P is empty and $I_{q_1} = I_{q_2}$ for all $q_1, q_2 \in Q$
- There exist $q_1, q_2 \in Q$ with $I_{q_1} \neq I_{q_2}$
- There exist $p_1, p_2 \in P$ with $I_{p_1} \neq I_{p_2}$, and
- P is nonempty, $I_{q_1} = I_{q_2}$ for all $q_1, q_2 \in Q$, and $I_{p_1} = I_{p_2}$ for all $p_1, p_2 \in P$.

We treat these cases separately. The first case is easy; for if P is empty and $I_{q_1} = I_{q_2}$ for all $q_1, q_2 \in Q$, then by (3), $j = k$ and the hypotheses of (4) are satisfied, and so (4) implies that the theorem holds.

(6) *If there exist $q_1, q_2 \in Q$ with $I_{q_1} \neq I_{q_2}$ then the theorem holds.*

For then by (5), no vertex of $A \cup B$ is semiadjacent to either of q_1, q_2, and q_1, q_2 have the same neighbours in $A' \cup B'$. Let X be the set of neighbours of q_1 (and hence of q_2) in $A' \cup B'$. For any third member $q \in Q$, I_q is different from one of I_{q_1}, I_{q_2}, and so by the same argument, X is the set of neighbours of q in $A' \cup B'$. Consequently Q is strongly complete to X and strongly anticomplete to $(A' \cup B') \setminus X$. Hence $X = Z$, and therefore X is strongly complete to P and hence to C.

Choose $q_1, q_2 \in Q$ with $I_{q_1} \neq I_{q_2}$, and let $Y = I_{q_1} \cap I_{q_2}$. Now $I_p = Y$ for every $p \in P$, by (5). Suppose that there exists $q_3 \in Q$ with $Y \not\subseteq I_{q_3}$. (Hence $P = \emptyset$, and therefore $X = A' \cup B'$ by (3).) Let $Y' = I_{q_1} \cup I_{q_2}$. Since $|I_q \cup I_{q'}| \leq j + 1$ for all $q, q' \in Q$, it follows that $|Y'| = j + 1$ and $I_{q_3} \subseteq Y'$; and since no subset $Y'' \subseteq Y'$ with $|Y''| \leq j - 1$ has intersection of cardinality $\geq j - 1$ with each of $I_{q_1}, I_{q_2}, I_{q_3}$, it follows that $I_q \subseteq Y'$ for all $q \in Q$. By (3), $j + 1 = k$. Moreover, there do not exist $q, q' \in Q$ with $I_q = I_{q'}$, since then q, q' would be twins. Consequently, G is near-antiprismatic, and the theorem holds.

We may therefore assume that $Y \subseteq I_q$ for all $q \in Q$. If $p \in P$ has a neighbour $a \in A' \setminus Z$ and $b \in B' \setminus Z$ then $\{p, q_1, a, b\}$ is a claw, a contradiction; so $P = P_1 \cup P_2$ where P_1, P_2 are the sets of vertices in P strongly anticomplete to $B' \setminus Z, A' \setminus Z$ respectively. Since $I_p = Y$ for all $p \in P$, it follows that $(P_1, A' \setminus Z)$ is a homogeneous pair, nondominating because of b_0, and so $|P_1|, |A' \setminus Z| \leq 1$; and similarly $|P_2|, |B' \setminus Z| \leq 1$. Moreover

$$(\{a_i : i \in Y\} \cup (A' \cap Z), \{b_i : i \in Y\} \cup (B' \cap Z))$$

is a coherent homogeneous pair, and so $|Y| \leq 1$ since G does not admit twins or a coherent W-join, that is, $j \leq 2$; and moreover, either $j = 1$ or $Z = \emptyset$. If $Z = \emptyset$, then the third outcome of the theorem holds; and if $j = 1$ then the fourth outcome holds. This proves (6).

(7) *If there exist $p_1, p_2 \in P$ with $I_{p_1} \neq I_{p_2}$, then the theorem holds.*

For let $Y = I_{p_1} \cup I_{p_2}$; then $|Y| = j$. By (5), $I_q = Y$ for all $q \in Q$. Choose $q \in Q$; then by (5), $I_p \subseteq I_q = Y$ for all $p \in P$. By (3), $j = k$, and so Q is strongly complete to $(A \setminus A') \cup (B \setminus B')$.

By (5), no vertex of $A \cup B$ is semiadjacent to either of p_1, p_2, and p_1, p_2 have the same set of neighbours in $A' \cup B'$, say W. Moreover, if $p \in P$ then I_p is different from one of I_{p_1}, I_{p_2}, and so W is the set of neighbours of p in $A' \cup B'$. We deduce that P is strongly complete to W and strongly anticomplete to $(A' \cup B') \setminus W$. But by (3), every vertex in $A' \cup B'$ has a neighbour in C, and so $Z \cup W = A' \cup B'$; and since Z is strongly complete to P, it follows that $Z \subseteq W$, and so $W = A' \cup B'$ and therefore $A' \cup B'$ is strongly complete to P.

We claim there is at most one value of $i \in \{1, \ldots, k\}$ that belongs to all the sets I_p $(p \in P)$; for if i, i' were two such values, then $(\{a_i, a_{i'}\}, \{b_i, b_{i'}\})$ would be a coherent homogeneous pair, a contradiction. Thus there is at most one such i, and therefore we may assume that for $1 \leq i < k$ there exists $p_i \in P$ with $i \notin I_{p_i}$. There is at most one $p \in P$ with $i \notin P_i$, since two such vertices p, p' would have $I_p = I_{p'}$ and therefore would be twins; and so $P = \{p_1, \ldots, p_{k-1}\}$ or $\{p_1, \ldots, p_k\}$, where p_k is the unique vertex $p \in P$ with $k \notin I_p$, if such a vertex exists. Moreover, if for some $i \in \{1, \ldots, k\}$, a_i, b_i are semiadjacent, then $i \in I_p$ for all $p \in P$; for if $i \notin I_p$ for some p, choose $p' \in P$ with $i \in I_{p'}$, and then $\{p', p, a_i, b_i\}$ is a claw, a contradiction. Hence a_i is strongly adjacent to b_i for $1 \leq i < k$, and also for $i = k$ if p_k exists.

If $q \in Q$ has antineighbours $a' \in A'$ and $b' \in B'$, then $\{p_1, q, a', b'\}$ is a claw, a contradiction; so $Q = Q_1 \cup Q_2$, where Q_1, Q_2 are the sets of vertices in Q strongly complete to B', A' respectively. Since (Q_1, A') is a homogeneous pair, nondominating because of b_0, 4.3 implies that $|Q_1|, |A'| \leq 1$, and similarly $|Q_2|, |B'| \leq 1$. If $|Q| \leq 1$ then G is near-antiprismatic; so we may assume that $Q_1 = \{q_1\}$ and $Q_2 = \{q_2\}$, and $Q_1 \cap Q_2 = \emptyset$. In particular, q_1 is not strongly complete to A', and so A' is nonempty; let $A' = \{a'\}$ say, where q_1, a' are antiadjacent. Similarly, $B' = \{b'\}$ where b', q_2 are antiadjacent. But then again, G is near-antiprismatic. This proves (7).

In view of (6),(7), we may henceforth assume that P is nonempty, $I_{q_1} = I_{q_2}$ for all $q_1, q_2 \in Q$, and $I_{p_1} = I_{p_2}$ for all $p_1, p_2 \in P$. Let $I_p = Y$ for all $p \in P$. Then $|Y| = j - 1$, and $(\{a_i : i \in Y\}, \{b_i : i \in Y\})$ is a coherent homogeneous pair, and so $j \leq 2$ since G does not admit twins or a coherent W-join. By (3), $k = j$. If some $q \in Q$ has antineighbours $a' \in A' \cap Z$ and $b' \in B' \cap Z$, then $\{p, q, a', b'\}$ is a claw where $p \in P$, a contradiction. Thus $Q = Q_1 \cup Q_2$, where Q_1, Q_2 are the sets of members of Q which are strongly complete to $B' \cap Z$ and to $A' \cap Z$ respectively. Since $(Q_1, A' \cap Z)$ is a homogeneous pair, nondominating because of b_0, 4.3 implies that $|Q_1|, |A' \cap Z| \leq 1$, and similarly $|Q_2|, |B' \cap Z| \leq 1$. If some $p \in P$ has neighbours $a' \in A' \setminus Z$ and $b' \in B' \setminus Z$ then $\{p, q, a', b'\}$ is a claw, where $q \in Q$, a contradiction. Thus $P = P_1 \cup P_2$, where P_1, P_2 are the sets of members of P that are strongly anticomplete to $B' \setminus Z$ and to $A' \setminus Z$ respectively. Since $(P_1, A' \setminus Z)$ is a nondominating homogeneous pair, 4.3 implies that $|P_1|, |A' \setminus Z| \leq 1$, and similarly $|P_2|, |B' \setminus Z| \leq 1$.

(8) *If $|Q| \geq 2$ then the theorem holds.*

For in this case it follows that $Q_1, Q_2 \neq Q$. Since $Q_1 \cup Q_2 = Q$ and $|Q_1|, |Q_2| \leq 1$, we deduce that $Q = \{q_1, q_2\}$, where $Q_i = \{q_i\}$ for $i = 1, 2$. Since $q_1 \notin Q_2$, there exists $a' \in A' \cap Z$ antiadjacent to q_1. Suppose that there exists $p \in P \setminus P_1$. Since $p \notin P_1$, p has a neighbour $b' \in B' \setminus Z$; but then $\{p, q_1, a', b'\}$ is a claw, a contradiction. This proves that $P_1 = P$, and similarly $P_2 = P$. Hence $|P| = 1$, $P = \{p\}$ say. Since $p \in P_1$, p has no neighbours in $B' \setminus Z$; but every vertex in $B' \setminus Z$ is adjacent to p, by (3), and so $B' \subseteq Z$. Similarly $A' \subseteq Z$, and so G is near-antiprismatic, and the theorem holds. This proves (8).

In view of (8) we may assume that $|Q| = 1$. If $Z = \emptyset$ then the third outcome of the theorem holds, so we may assume that Z is nonempty. If $Y \neq \emptyset$, let $1 \in Y$, say; then $((Z \cap A') \cup \{a_1\}, (Z \cap B') \cup \{b_1\})$ is a coherent W-join, a contradiction. Thus $Y = \emptyset$, and so $k = 1$. If Q is strongly complete to Z, and one of $Z \cap A, Z \cap B$ is empty, then again the third outcome of the theorem holds; while if Q is strongly complete to Z and $Z \cap A, Z \cap B$ are both nonempty, then the fourth outcome holds. Thus we may assume that $Q = \{q\}$ say, and q is antiadjacent to $z \in Z \cap B$. Hence $Z \cap B = \{z\}$. Since z has a neighbour in Q, we deduce that q, z are semiadjacent. Now $q \notin Q_1$, and so $q \in Q_2$ and therefore q is strongly complete to $Z \cap A$. If there exists $p \in P \setminus P_2$, let $a \in A' \setminus Z$ be adjacent to p; then $\{p, a, q, z\}$ is a claw, a contradiction. Thus $P = P_2$, and so $|P| = 1$, say $P = \{p\}$. Since $A' \setminus Z$ is therefore strongly anticomplete to C, (3) implies that $A' \subseteq Z$. If also $B' \subseteq Z$ then G is antiprismatic, so we may assume that $|B' \setminus Z| = 1$, and the vertex in $B' \setminus Z$ is adjacent to p. If it is strongly adjacent to p, then $(B', \{q\})$ is a nondominating homogeneous pair, contrary to 4.3. Thus the vertex in $B' \setminus Z$ is semiadjacent to p. Any two vertices in $Z \cap A$ are twins, and so $|Z \cap A| \leq 1$. If $Z \cap A = \emptyset$, then G admits a generalized 2-join $(\{b_1\}, A \cup \{a_0, b_0\}, B \cup C \setminus \{b_1\})$; and if $Z \cap A \neq \emptyset$, then G admits a hex-join, since $Z \cap A, \{a_0\} \cup A \setminus Z, C, \{b_0\} \cup B$ are four strong cliques with union $V(G)$ and the first is strongly complete to the second and third and strongly anticomplete to the last. Thus the fifth statement of the theorem holds. This proves 11.1. ∎

The previous result has a convenient corollary, the following.

11.2 *Let G be a claw-free trigraph with $\alpha(G) \geq 3$, and let $a_0, b_0 \in V(G)$ be antiadjacent. Suppose that the set of all vertices in $V(G) \setminus \{a_0, b_0\}$ adjacent to a_0 is a strong clique, and they are all strongly adjacent to a_0; and the same for b_0. Suppose that no vertex is adjacent to both a_0, b_0, and the set of vertices antiadjacent to both a_0, b_0 is a strong clique. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$ or G is decomposable.*

Proof. Let G' be the trigraph obtained from G by making a_0, b_0 semiadjacent, and leaving the adjacency of all other pairs unchanged. Then G' is claw-free, from the hypothesis, and therefore satisfies the hypotheses of 11.1. Hence either $G' \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$ or G' is decomposable. Certainly if G' is decomposable then so is G, so we assume that G' is not decomposable. It is easy to see that if $G' \in \mathcal{S}_3$ then the same holds for G, and if $G' \in \mathcal{S}_6$ then $G \in \mathcal{S}_6 \cup \mathcal{S}_7$. Suppose then that $G' \in \mathcal{S}_0$, and let G' be a line trigraph of some graph H. Since a_0, b_0 are semiadjacent in G', there is a vertex of degree two in H incident with them both in H. It follows that G is also a line trigraph, for G is an $L(H')$-trigraph where H' is obtained from H by "splitting" v into two vertices both of degree one. This proves 11.2. ∎

12 The icosahedron minus a triangle

Now we begin the next part of the paper. The objective of the next several sections is to prove 17.2, that every claw-free trigraph with a hole of length ≥ 6 either belongs to one of our basic classes or is decomposable. We begin by outlining the plan of the proof, as follows.

- We can assume G is a claw-free trigraph with a maximum 6-hole, and with no long prism. Consequently we may assume that every 6-hole is dominating, by 10.3.
- (In 13.6) If some 6-hole has a hub, and either a clone or a semiadjacent pair of consecutive vertices, then either G belongs to one of the basic classes or G is decomposable.
- (In 13.7) If some 6-hole has both a star-diagonal and a clone then either G belongs to one of the basic classes or G is decomposable.
- (In 14.3) Every 5-hole is dominating (or else either G is decomposable, or it belongs to one of our basic classes). Consequently, no 6-hole has a coronet.
- (In 15.1) If some 6-hole has both a hub and a hat, then either G is in one of the basic classes or it is decomposable.
- (In 15.2) If some 6-hole has both a star-diagonal and a hat, then either G is in one of the basic classes or G is decomposable.
- (In 16.3) If no 6-hole has a hub, but some 6-hole has both a star-triangle and either a hat or clone, then G is decomposable.
- (In 16.4) If no 6-hole has a hub or star-diagonal, but some 6-hole has a crown, then G is decomposable.
- (In 17.1) If no 6-hole has a hub, star-diagonal, star-triangle or crown, then either G is a circular interval trigraph or G is decomposable.
- To complete the proof of 17.2, we may therefore assume that some 6-hole has either a hub, a star-diagonal, or a star-triangle, and has no hat or clone. We deduce that G is either decomposable or antiprismatic.

The first step is to handle $icosa(-3)$, and that is the goal of this section. We recall that $icosa(-3)$ is the graph obtained from $icosa(0)$ by deleting three pairwise adjacent vertices. Thus it has nine vertices $c_1, \dots, c_6, b_1, b_3, b_5$, where $\{b_1, b_3, b_5\}$ is a triangle, c_1-$\cdots$-c_6-c_1 is a 6-numbering, and for $i = 1, 3, 5$, b_i is adjacent to c_{i-1}, c_i, c_{i+1} and antiadjacent to the other three of $c_1, \dots, c_6$.

12.1 *Let G be claw-free, and with no long prism or hole of length > 6, containing an $icosa(-3)$-trigraph. Then G is decomposable.*

Proof. Let $c_1, \dots, c_6, b_1, b_3, b_5 \in V(G)$ such that the subtrigraph induced on these nine vertices is an $icosa(-3)$-trigraph, labelled as above. By 10.3 we may assume that $\{c_1, \dots, c_6\}$ is dominating. For $i = 1, 3, 5$, let B_i be the set of all $v \in V(G)$ such that v is adjacent to b_1, b_3, b_5, c_i. Let $W = \{c_1, \dots, c_6\} \cup B_1 \cup B_3 \cup B_5$.

(1) *For $i = 1, 3, 5$, if $v \in B_i$ then $c_{i-1}, c_i, c_{i+1} \in N^*(v)$, and $c_{i+2}, c_{i+3}, c_{i+4} \notin N(v)$, and in particular, B_1, B_3, B_5 are pairwise disjoint. Moreover, $B_1 \cup B_3 \cup B_5$ is a strong clique.*

For let $v \in B_1$. By 5.3 it follows that $c_3, c_5 \notin N(v)$, so the sets B_1, B_3, B_5 are pairwise disjoint.

By 5.4 (with c_3-c_4-c_5), $c_4 \notin N(v)$. By 5.4 (with c_4-b_5-c_6 and c_4-b_3-c_2), $c_2, c_6 \in N^*(v)$; and by 5.4 (with c_1-c_2-c_3), $c_1 \in N^*(v)$. This proves the first two claims. For the final claim, suppose that $u, v \in B_1 \cup B_3 \cup B_5$ are antiadjacent. From the symmetry we may assume that $u, v \notin B_1$; and then $\{b_1, u, v, c_1\}$ is a claw, a contradiction. This proves that $B_1 \cup B_3 \cup B_5$ is a strong clique, and therefore proves (1).

For $i = 1, 3, 5$, let C_i be the set of all $v \in V(G) \setminus (B_1 \cup B_3 \cup B_5)$ such that $B_i \subseteq N^*(v)$ and $B_{i-2}, B_{i+2} \not\subseteq N^*(v)$. For $i = 2, 4, 6$, let C_i be the set of all $v \in V(G) \setminus (B_1 \cup B_3 \cup B_5)$ such that $B_{i-1}, B_{i+1} \subseteq N^*(v)$ and $B_{i+3} \not\subseteq N^*(v)$.

(2) *We may assume that the nine sets $C_1, \dots, C_6, B_1, B_3, B_5$ are pairwise disjoint and have union $V(G)$.*

For we have seen that B_1, B_3, B_5 are pairwise disjoint, and therefore the nine sets are pairwise disjoint. We must show they have union $V(G)$. Let $v \in V(G)$. Since each $c_i \in C_i$ and $b_i \in B_i$, we may assume that $v \neq b_1, b_3, b_5, c_1, \dots, c_6$. If $N^*(v)$ includes either one or two of B_1, B_3, B_5, then v belongs to one of $C_1, \dots, C_6$; so we may assume that $N^*(v)$ includes none or all of B_1, B_3, B_5. Suppose first that $B_1, B_3, B_5 \subseteq N^*(v)$. If $c_1 \in N(v)$ then $v \in B_1$, so we may assume that $c_1 \notin N(v)$, and similarly $c_3, c_5 \notin N(v)$. By 5.4 (with c_1-c_2-c_3), $c_2 \notin N(v)$, and similarly $c_6 \notin N(v)$, contrary to 5.4 (with c_2-b_1-c_6}). Second, suppose that $B_1, B_3, B_5 \not\subseteq N^*(v)$. If $N(v)$ contains at least two of c_2, c_4, c_6, say c_2, c_4, then 5.4 (with c_1-c_2-B_3}) implies that $c_1 \in N(v)$ and similarly $c_5 \in N(v)$; 5.4 (with c_3-c_4-B_5) implies that $c_3 \in N(v)$; and then $\{v, c_1, c_3, c_5\}$ is a claw, a contradiction. Thus $N(v)$ contains at most one of c_2, c_4, c_6 and we may assume that $c_2, c_4 \notin N(v)$. By 5.4 (with c_2-c_3-c_4), $c_3 \notin N(v)$; by 5.4 (with c_3-B_3-B_5), $B_3 \cap N(v) = \emptyset$; and by 5.4 (with B_1-B_5-c_4 and B_5-B_1-c_2), $N(v)$ is disjoint from B_1, B_5. If $c_1 \in N(v)$ then 5.4 (with c_6-c_1-c_2) implies that $c_6 \in N(v)$; and if $c_6 \in N(v)$ then 5.4 (with B_5-c_6-c_1) implies that $c_1 \in N(v)$. Since $\{c_1, \dots, c_6\}$ is dominating, it follows that $c_5, c_6, c_1 \in N(v)$. But then the subtrigraph induced on $\{c_1, \dots, c_6, b_1, b_3, b_5, v\}$ is an *icosa*(-2)-trigraph, and the result follows from 5.7. This proves (2).

We remind the reader that $v \in N(v)$.

(3) *Let $1 \leq i \leq 6$ and let $v \in C_i$.*

- *If i is odd then $N(v) \cap W = B_i \cup \{c_{i-1}, c_i, c_{i+1}\} \cup X$, where X is a subset of one of $B_{i-2} \cup \{c_{i-2}\}, B_{i+2} \cup \{c_{i+2}\}$.*
- *If i is even then $N(v) \cap W = B_{i-1} \cup B_{i+1} \cup \{c_{i-1}, c_i, c_{i+1}\} \cup X$, where X is one of $\emptyset, \{c_{i-2}\}, \{c_{i+2}\}$.*

For let $v \in C_1$. Thus $B_1 \subseteq N^*(v)$, and $B_3, B_5 \not\subseteq N^*(v)$. By 5.4 (with B_5-B_1-c_2 and B_5-B_1-c_1 if $v \neq c_1$), $c_2, c_1 \in N^*(v)$ and similarly $c_6 \in N^*(v)$. By 5.3, $c_4 \notin N(v)$, and not both $c_3, c_5 \in N(v)$; we assume $c_3 \notin N(v)$. By 5.4 (with c_3-B_3-B_5), $N(v) \cap B_3 = \emptyset$. This proves the first claim. For the second, let $v \in C_2$. Thus $B_1, B_3 \subseteq N^*(v)$ and $B_5 \not\subseteq N^*(v)$. By 5.4 (with B_5-B_1-c_2 if $v \neq c_2$, and B_5-B_1-c_1), $c_1, c_2 \in N^*(v)$, and similarly $c_3 \in N^*(v)$. By 5.3, $B_5 \cup \{c_5\}$ is disjoint from $N(v)$, and not both $c_4, c_6 \in N^*(v)$. This proves the second claim and hence proves (3).

(4) *For $i = 1, 3, 5$, $B_i \cup C_i$ is a strong clique, and for $i = 2, 4, 6$, C_i is a strong clique.*

For first suppose that $u, v \in B_1 \cup C_1$ are antiadjacent. By (1), at least one of $u, v \in C_1$, say $v \in C_1$. By (3) we may assume that $N(v) \cap (B_3 \cup \{c_3\}) = \emptyset$. If also $u \in C_1$, choose $x \in B_3$ antiadjacent to u; then $\{b_1, x, u, v\}$ is a claw, a contradiction. So $u \in B_1$; but then $\{c_2, c_3, u, v\}$ is a claw, a contradiction. This proves the first claim. For the second, suppose that $u, v \in C_2$ are antiadjacent. Then $\{b_3, b_5, u, v\}$ is a claw, a contradiction. This proves (4).

(5) *For $i = 1, 3, 5$, $B_i \cup C_i$ is strongly complete to $C_{i-1} \cup C_{i+1}$.*

For let $u \in B_1 \cup C_1$ and $v \in C_2$ say, and suppose that u, v are antiadjacent. Since B_1 is strongly complete to C_2 from the definition of C_2, it follows that $u \in C_1$. Choose $x \in B_5$ antiadjacent to u. Then $\{b_1, x, u, v\}$ is a claw, a contradiction. This proves (5).

(6) *For $i = 1, 3, 5$, $B_i \cup C_i$ is strongly anticomplete to C_{i+3}.*

For let $u \in B_1 \cup C_1$ and $v \in C_4$ say, and suppose that u, v are adjacent. From (3), $u \notin B_1$, so $u \in C_1$. Choose $x \in B_5$ antiadjacent to u. Since $\{v, u, x, c_3\}$ is not a claw, it follows that $c_3 \in N(u)$, and similarly $c_5 \in N(u)$, contrary to (3). This proves (6).

From (2),(4),(5),(6), it follows that G is expressible as a hex-join and therefore decomposable. This proves 12.1. ■

13 6-holes with clones

Let c_1-$\cdots$-c_n-c_1 be an n-numbering in a trigraph G, and let $v \in V(G)$. We saw in 9.1 that if $v \neq c_1, \ldots, c_n$, and is neither strongly complete nor strongly anticomplete to $\{c_1, \ldots, c_n\}$, and not a hub, then there is an "interval" $c_i, c_{i+1}, .., c_j$ of C with

$$\emptyset \neq \{c_i, c_{i+1}, \ldots, c_j\} \neq \{c_1, \ldots, c_n\},$$

such that v is adjacent to the vertices in this interval and antiadjacent to the other vertices of $c_1, \ldots, c_n$. In this case we say that v is in *position* $(i+j)/2$ relative to c_1-$\cdots$-c_n-c_1. (Possibly there are two such intervals, if v is semiadjacent to one of $c_1, \ldots, c_n$, and then v has two positions relative to c_1-$\cdots$-c_n-c_1.) It is helpful also to say that for $1 \leq i \leq n$, c_i is in position i relative to c_1-$\cdots$-c_n-c_1.

Let c_1-$\cdots$-c_6-c_1 be a 6-numbering of a 6-hole C. If v is a hub relative to C, we say that v is *in hub-position* i if v is adjacent to $c_{i-2}, c_{i-1}, c_{i+1}, c_{i+2}$. (Thus hub-position i is the same as hub-position $i+3$.)

13.1 *Let G be a claw-free trigraph. Let C be a 6-hole in G with vertices c_1-$\cdots$-c_6-c_1 in order, and let w be a hub in hub-position i. Let $v \in V(G) \setminus (V(C) \cup \{w\})$. Then w, v are strongly adjacent if and only if either:*

- *v is a hub in hub-position i, or*
- *v is a hat in position $i + 1\frac{1}{2}$ or in position $i - 1\frac{1}{2}$, or*

- *v is a clone in position $i+1, i+2, i-2$ or $i-1$, or*
- *v is a star in position $i+\frac{1}{2}, i+2\frac{1}{2}, i-\frac{1}{2}$ or $i-2\frac{1}{2}$*

and strongly antiadjacent otherwise.

Proof. In each case listed, if v, w are antiadjacent there is a claw; and in the cases not listed, if v, w are adjacent there is a claw. We leave the details to the reader. ■

This has the following consequence.

13.2 *Let G be a claw-free trigraph, and let C be a 6-hole in G with vertices c_1-$\cdots$-c_6-c_1 in order. If there are two hubs in the same hub-position, then G admits twins.*

Proof. By 13.1, any two hubs in the same hub-position are strongly adjacent, and every other vertex is either strongly adjacent to them both, or strongly antiadjacent to them both. Thus they are twins. This proves 13.2. ■

Two n-numberings are *proximate* if they differ in exactly one place (and therefore they number n-holes with $n-1$ vertices in common; the exceptional vertex of each is a clone with respect to the other). Note that we regard c_1-$\cdots$-c_n-c_1 and c_2-c_3-$\cdots$-c_n-c_1-c_2 as different numberings; the choice of initial vertex is important. A nonempty set $\mathcal{C}$ of n-numberings is *connected by proximity* if the graph with vertex set $\mathcal{C}$, in which two n-numberings are adjacent if they are proximate, is connected. The *proximity distance* between two n-numberings is the length of the shortest path between them in this graph, if such a path exists, and is undefined otherwise. A *proximity component of order n* means a set $\mathcal{C}$ of n-numberings that is connected by proximity and maximal with this property.

13.3 *Let G be a claw-free trigraph, and let $\mathcal{C}$ be a proximity component of order 6. Let $v \in V(G)$ be a hub in hub-position i for some member of $\mathcal{C}$. Then v is a hub in hub-position i for every member of $\mathcal{C}$.*

Proof. It suffices to show that if c_1-$\cdots$-c_6-c_1 and c'_1-$\cdots$-c'_6-c'_1 are proximate, and v is a hub in hub-position i for the first 6-numbering, then v is a hub in hub-position i for the second. We may assume that $i = 1$. From the symmetry we may assume that $c_j = c'_j$ for $j = 3, 4, 5$; and since v is adjacent to c_3, c_5 and not to c_4, it follows from 9.1 that v is a hub in hub-position 1 relative to c'_1-$\cdots$-c'_6-c'_1. This proves 13.3. ■

If $\mathcal{C}$ is a proximity component of order n, we denote the union of the vertex sets of its members by $V(\mathcal{C})$; and for $1 \le i \le n$, the set of vertices that are the ith term of some member of $\mathcal{C}$ is denoted by $A_i(\mathcal{C})$, or just A_i when there is no ambiguity. If these n sets are pairwise disjoint, we say that $\mathcal{C}$ is *pure*.

13.4 *Let G be a claw-free trigraph containing no long prism, with a maximum hole of length six, in which every maximum hole is dominating. Let $\mathcal{C}$ be a pure proximity component of order 6. Then*

- *For $1 \le i \le 6$, A_i is a strong clique, and A_i is strongly anticomplete to A_{i+3}*
- *If $v \in V(G)$ and $v \notin A_1 \cup \cdots \cup A_6$, then for $1 \le i \le 6$, v is either strongly complete or strongly anticomplete to A_i; and v is strongly complete to either two or four of the sets $A_1, \ldots, A_6$.*

- *For $1 \le i \le 6$, every $v \in A_i$ is either strongly complete to A_{i+1} or strongly anticomplete to A_{i+2}.*
- *For $1 \le i \le 6$, either A_i is strongly complete to A_{i-1} or A_i is strongly anticomplete to A_{i+2}.*
- *For $1 \le i \le 6$, A_i is strongly complete to one of A_{i-1}, A_{i+1}.*

Proof. For each vertex $v \in V(G)$, let $P(v)$ be the set of all k such that v is in position k relative to some member of $\mathcal{C}$ (and therefore v is not a hub relative to this 6-numbering). Since v may be semiadjacent to a vertex of the 6-numbering, it may have two distinct positions relative to the same 6-numbering, and therefore the same 6-numbering may contribute two different terms to $P(v)$, differing by $\frac{1}{2}$.

(1) *For every vertex $v \in V(G)$, if k is an integer, then $k \in P(v)$ if and only if $v \in A_k$. Moreover, $|P(v)| \le 3$, and the members of $P(v)$ are consecutive multiples of $\frac{1}{2}$ modulo 6.*

For suppose first that $v \in A_k$. Then since the sets $A_1, \dots, A_6$ are pairwise disjoint, v is the kth term of every member of $\mathcal{C}$ that contains it, and there is such a member since $v \in A_k$; and so $k \in P(v)$. For the converse, suppose that k is an integer and $k \in P(v)$. We may assume that $k = 1$. Choose a 6-numbering $c_1\text{-}\cdots\text{-}c_6\text{-}c_1 \in \mathcal{C}$ such that v is in position 1 relative to this 6-numbering. Hence either $v = c_1$ or v is a clone in position 1. In either case the 6-numbering $v\text{-}c_2\text{-}\cdots\text{-}c_6\text{-}v$ also belongs to $\mathcal{C}$, because of the maximality of $\mathcal{C}$, and so $v \in A_1$. This proves the first claim. For the second claim, we may assume that $P(v)$ is nonempty, and so by 13.3, v is not a hub with respect to any member of $\mathcal{C}$. Since every member of $\mathcal{C}$ is dominating (by hypothesis) and has no centre, it follows that v has (at least) one position with respect to every member of $\mathcal{C}$. But for any two proximate 6-numberings, v has a position with respect to each of them such that these two positions differ by at most $\frac{1}{2}$; and so the members of $P(v)$ are consecutive multiples of $\frac{1}{2}$ (modulo 6). Since $P(v)$ contains at most one integer, as we have seen, it follows that $|P(v)| \le 3$. This proves (1).

To prove the first statement of the theorem, we may assume that $i = 1$. Let $u, v \in A_1$, and let $c_1\text{-}\cdots\text{-}c_6\text{-}c_1 \in \mathcal{C}$ with $c_1 = u$. Since $v \in A_1$, it follows that $1 \in P(v)$; and so $P(v) \subseteq \{\frac{1}{2}, 1, 1\frac{1}{2}\}$ by (1). In particular, v is in position $\frac{1}{2}, 1$ or $1\frac{1}{2}$ relative to $c_1\text{-}\cdots\text{-}c_6\text{-}c_1$; and in each case, it is strongly adjacent to u. Hence A_1 is a strong clique. Now let $u \in A_1$ and $v \in A_4$. As before, $P(u) \subseteq \{\frac{1}{2}, 1, 1\frac{1}{2}\}$. Choose $c_1\text{-}\cdots\text{-}c_6\text{-}c_1 \in \mathcal{C}$ with $c_4 = v$; then u is in position $\frac{1}{2}, 1$ or $1\frac{1}{2}$ relative to $c_1\text{-}\cdots\text{-}c_6\text{-}c_1$, and in each case u, v are strongly antiadjacent. This proves the first statement.

For the second statement, let $v \in V(G)$ with $v \notin A_1 \cup \cdots \cup A_6$. By 13.3 we may assume that v is not a hub relative to any member of $\mathcal{C}$. By (1), $P(v)$ contains no integer, and so $P(v) = \{i + \frac{1}{2}\}$ for some integer i. Thus v is in position $i + \frac{1}{2}$ relative to every member of $\mathcal{C}$, and it is either a hat or a star. Since it is not a clone (because $P(v)$ contains no integer), it follows that v is not semiadjacent to any member of $A_1 \cup \cdots \cup A_6$. If v is sometimes a hat and sometimes a star, then there are two proximate members of $\mathcal{C}$ such that v is a hat relative to one and a star relative to the other, which is impossible. Hence either it is a hat in position $i + \frac{1}{2}$ relative to all members of $\mathcal{C}$, or it is a star in the same position for them all, and in either case the claim follows. This proves the second statement.

For the third statement, we may assume that $i = 1$; let $v \in A_1$, and suppose it has a neighbour $a_3 \in A_3$ and an antineighbour $a_2 \in A_2$. Choose $c_1\text{-}c_2\text{-}\cdots\text{-}c_6\text{-}c_1 \in \mathcal{C}$ so that $a_2 = c_2$. Since $v \in A_1$ it follows that $P(v) \subseteq \{\frac{1}{2}, 1, 1\frac{1}{2}\}$, and since v is antiadjacent to c_2, we deduce that v is in position $\frac{1}{2}$

relative to c_1-c_2-$\cdots$-c_6-c_1. Hence v is either a hat or a star. If v is a star, then v is semiadjacent to c_2, and therefore v is also a clone in position 6, a contradiction, since $6 \notin P(v)$. Thus v is a hat. Since $a_3 \in A_3$, it follows that $P(a_3) \subseteq \{2\frac{1}{2}, 3, 3\frac{1}{2}\}$. If a_3 is adjacent to both c_2, c_4 then $\{a_3, c_2, c_4, v\}$ is a claw; and so a_3 is a hat in position $2\frac{1}{2}$ or $3\frac{1}{2}$ relative to c_1-c_2-$\cdots$-c_6-c_1. But then $G|\{c_1, \ldots, c_6, v, a_3\}$ is a long prism, a contradiction. This proves the third statement.

For the fourth statement, let us first prove the following.

(2) *If $1 \leq i \leq 6$, then every vertex in A_i is either strongly complete to A_{i-1} or strongly anticomplete to A_{i+2}.*

For we may assume that $i = 2$. Let $v \in A_2$, and suppose that v has a neighbour $a_4 \in A_4$ and an antineighbour $a_1 \in A_1$. Choose c_1-c_2-$\cdots$-c_6-c_1 and c_1'-c_2'-$\cdots$-c_6'-c_1' in $\mathcal{C}$, with $c_1 = a_1$ and $c_4' = a_4$, and choose these two 6-numberings so that their proximity distance (k say) is as small as possible. Since $v \in A_2$, it follows that $P(v) \subseteq \{1\frac{1}{2}, 2, 2\frac{1}{2}\}$. Since v is antiadjacent to c_1 we deduce that, relative to c_1-c_2-$\cdots$-c_6-c_1, either v is a hat in position $2\frac{1}{2}$, or $v = c_2$ and c_2 is semiadjacent to c_1; and in either case v is strongly antiadjacent to c_4 (since $3 \notin P(v)$). Similarly, relative to c_1'-c_2'-$\cdots$-c_6'-c_1', either v is a star in position $2\frac{1}{2}$, or $v = c_2'$ and c_2' is semiadjacent to c_4; and in either case v is strongly adjacent to c_1'. In particular, $c_1 \neq c_1'$, and $c_4 \neq c_4'$. It follows that the two 6-numberings are not proximate, and so $k > 1$. Consequently there is a third 6-numbering c_1''-c_2''-$\cdots$-c_6''-c_1'' in $\mathcal{C}$, proximate to c_1'-c_2'-$\cdots$-c_6'-c_1', and with proximity distance to c_1-c_2-$\cdots$-c_6-c_1 less than k. From the minimality of k, it follows that c_4'' is strongly antiadjacent to v, and therefore $c_4'' \neq a_4$; and so $c_i'' = c_i'$ for all $i \in \{1, \ldots, 6\}$ with $i \neq 4$. Consequently c_1'-v-c_3'-c_4''-c_5'-c_6'-c_1' is a 6-numbering, and therefore belongs to $\mathcal{C}$. Since c_1 is antiadjacent to v, and $P(c_1) \subseteq \{\frac{1}{2}, 1, 1\frac{1}{2}\}$, it follows that relative to this last 6-numbering, c_1 is in position $\frac{1}{2}$ and is a hat. Consequently c_1 is strongly antiadjacent to c_3', c_4', c_5', and is strongly adjacent to c_6'.

Suppose that c_1 is in position 1 relative to c_1'-c_2'-$\cdots$-c_6'-c_1'. Then c_1-c_2'-c_3'-$\cdots$-c_6'-c_1 belongs to $\mathcal{C}$, and yet v is in position 3 relative to it, contradicting that $P(v) \subseteq \{1\frac{1}{2}, 2, 2\frac{1}{2}\}$. So c_1 is not in position 1 relative to c_1'-c_2'-$\cdots$-c_6'-c_1'. Since $P(c_1) \subseteq \{\frac{1}{2}, 1, 1\frac{1}{2}\}$ and c_1 is strongly antiadjacent to c_3' and strongly adjacent to c_6', we deduce that c_1 is in position $\frac{1}{2}$ relative to c_1'-c_2'-$\cdots$-c_6'-c_1'. Since c_1 is strongly antiadjacent to c_5', it follows that c_1 is antiadjacent to c_2'.

Since $\{c_2, c_2', c_4', c_1\}$ is not a claw, it follows that c_2, c_4' are strongly antiadjacent and therefore $v \neq c_2$; and since A_4 is a strong clique, c_4' is strongly adjacent to c_4. Since $\{c_4', v, c_4, c_6\}$ is not a claw, c_4' is strongly antiadjacent to c_6. Thus if c_4' is in position $4\frac{1}{2}$ relative to c_1-$\cdots$-c_6-c_1, then it is a hat and therefore antiadjacent to c_3; but then $G|\{c_1, \ldots, c_6, v, c_4'\}$ is a long prism, a contradiction. If c_4' is in position 4 relative to c_1-c_2-$\cdots$-c_6-c_1, then v is in position 3 relative to c_1-c_2-c_3-c_4'-c_5-c_6-c_1, contradicting that $P(v) = \{1\frac{1}{2}, 2, 2\frac{1}{2}\}$. Thus, c_4' is in position $3\frac{1}{2}$ relative to c_1-c_2-$\cdots$-c_6-c_1, and therefore is a hat, since c_2, c_4' are strongly antiadjacent. Then c_1-c_2-v-c_4'-c_4-c_5-c_6-c_1 is a 7-hole, a contradiction. Thus there is no such vertex v. This proves (2).

To complete the proof of the fourth statement of the theorem, again we may assume that $i = 2$. Suppose that $v, v' \in A_2$, and v has a neighbour $a_4 \in A_4$, and v' has an antineighbour $a_1 \in A_1$. By (2), v', a_4 are antiadjacent, and v, a_1 are adjacent. But then $\{v, v', a_1, a_4\}$ is a claw, a contradiction. This proves the fourth statement of the theorem.

For the fifth statement, let us first prove the following:

(3) *For $1 \le i \le 6$, every vertex in A_i is either strongly A_{i+1}-complete or strongly A_{i-1}-complete.*

For we may assume that $i = 2$. Let $a_2 \in A_2$, and assume it has antineighbours $a_1 \in A_1$ and $a_3 \in A_3$. Since a_1 is not strongly complete to A_2, it is therefore strongly anticomplete to A_3 by the third statement of the theorem; and in particular, a_1, a_3 are strongly antiadjacent. Choose $x, y \in A_2$ adjacent to a_1, a_3 respectively. Since $\{x, a_1, a_2, a_3\}$ is not a claw, it follows that x is not adjacent to a_3, and similarly y is not adjacent to a_1. Thus a_1-x-y-a_3 is a path. Choose c_1-c_2-$\cdots$-c_6-$c_1 \in \mathcal{C}$ with $a_2 = c_2$. Now $P(a_1) \subseteq \{\frac{1}{2}, 1, 1\frac{1}{2}\}$, and a_1 is antiadjacent to a_2; and so relative to c_1-c_2-$\cdots$-c_6-c_1, a_1 is a hat in position $\frac{1}{2}$. Similarly, a_3 is a hat in position $3\frac{1}{2}$. Now x is strongly anticomplete to A_4, A_5, A_6, by respectively the third, first and fourth statements of the theorem, since x is not strongly complete to A_3. Similarly y is strongly anticomplete to $A_4 \cup A_5 \cup A_6$. It follows that a_1-x-y-a_3-c_4-c_5-c_6-a_1 is a 7-hole in G, a contradiction. This proves (3).

Now to prove the fifth statement of the theorem, we may assume that $i = 2$. Suppose that $a_1 \in A_1$ and $a_3 \in A_3$ both have antineighbours in A_2. By (3) they have no common antineighbour, and so there is a path a_1-x-y-a_3 where $x, y \in A_2$. Choose $c_i \in A_i$ for $i = 4, 5, 6$, such that c_4-c_5-c_6 is a path. By (3) and the first, third and fourth statements of the theorem, a_1-x-y-a_3-c_4-c_5-c_6-a_1 is a 7-hole in G, a contradiction. This proves the fifth statement, and therefore proves 13.4. ■

We have two applications for the previous theorem, but first we need another lemma.

13.5 *Let G be a claw-free trigraph containing no long prism, with a maximum hole of length six, in which every maximum hole is dominating. Let $\mathcal{C}$ be a pure proximity component of order 6, such that there is a hub for some member of $\mathcal{C}$. Suppose that for some $i \in \{1, \dots, 6\}$, $A_i(\mathcal{C})$ is not strongly complete to $A_{i+1}(\mathcal{C})$. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$, or G is decomposable.*

Proof. Let $A_i = A_i(\mathcal{C})$ for $1 \le i \le 6$, and let $W = A_1 \cup \cdots \cup A_6$. For $i = 0, \dots, 5$, let $H_{i+\frac{1}{2}}$ and $S_{i+\frac{1}{2}}$ be respectively the set of all hats and stars in $V(G) \setminus W$ in position $i + \frac{1}{2}$ relative to $\mathcal{C}$. For $i = 1, 2, 3$, let W_i be the set of all vertices in $V(G) \setminus W$ that are strongly complete to $A_{i+1}, A_{i+2}, A_{i-1}, A_{i-2}$ (and therefore strongly anticomplete to A_i, A_{i+3}). Then since every 6-hole is dominating, from 9.2, 9.1 and 13.4 we have:

- $S_{\frac{1}{2}}, \dots, S_{5\frac{1}{2}}, H_{\frac{1}{2}}, \dots, H_{5\frac{1}{2}}, W_1, W_2, W_3$ are pairwise disjoint strong cliques with union $V(G) \setminus W$
- for $1 \le i \le 6$, $H_{i+\frac{1}{2}}$ is strongly complete to $A_i \cup A_{i+1}$, and strongly anticomplete to A_j for $j \ne i, i+1$
- for $1 \le i \le 6$, $S_{i+\frac{1}{2}}$ is strongly complete to $A_{i-1}, A_i, A_{i+1}, A_{i+2}$ and strongly anticomplete to A_{i+3}, A_{i+4}
- for $i, j \in \{1, \dots, 6\}$, $H_{i+\frac{1}{2}}$ is strongly complete to $S_{j+\frac{1}{2}}$ if $i - j \in \{1, -1\}$, and strongly anticomplete otherwise
- $H_{\frac{1}{2}}, \dots, H_{5\frac{1}{2}}$ are pairwise strongly anticomplete (since G has no 7-hole or long prism)
- for $1 \le i, j \le 6$, W_i is strongly complete to $H_{j+1\frac{1}{2}}$ if $j \in \{i+1, i-2\}$ and strongly anticomplete otherwise

- for $1 \le i,j \le 6$, W_i is strongly anticomplete to $S_{j+1\frac{1}{2}}$ if $j \in \{i+1, i-2\}$ and strongly complete otherwise.

Thus the only adjacencies that are not yet determined are between some pairs of A_i's and between some pairs of $S_{j+\frac{1}{2}}$'s.

From the symmetry we may assume that A_1 is not strongly complete to A_2, and so 13.4 implies that A_1 is strongly complete to A_6 and strongly anticomplete to A_5, and similarly A_2 is strongly complete to A_3 and strongly anticomplete to A_4. Moreover $S_{1\frac{1}{2}}$ is strongly anticomplete to $S_{4\frac{1}{2}}$ since $S_{1\frac{1}{2}} \cup S_{4\frac{1}{2}} \cup A_1 \cup A_2$ includes no claw.

(1) *$S_{\frac{1}{2}}, S_{2\frac{1}{2}}, H_{1\frac{1}{2}}, W_3$ are all empty.*

For there exist $c_1 \in A_1$ and $c_2 \in A_2$, antiadjacent. If there exists $v \in S_{\frac{1}{2}}$, choose $c_5 \in A_5$; then by 13.4, c_5 is antiadjacent to c_1, c_2, and so $\{v, c_1, c_2, c_5\}$ is a claw, a contradiction. Thus $S_{\frac{1}{2}} = \emptyset$, and similarly $S_{2\frac{1}{2}}, W_3$ are empty. If there exists $v \in H_{1\frac{1}{2}}$, then choose $a_1\text{-}\cdots\text{-}a_6\text{-}c_1$ in $\mathcal{C}$; by 13.4, c_1 is antiadjacent to a_3 since c_1 is not strongly complete to A_2, and similarly c_2 is antiadjacent to a_6, and so

$$c_1\text{-}v\text{-}c_2\text{-}a_3\text{-}a_4\text{-}a_5\text{-}a_6\text{-}c_1$$

is a 7-hole, a contradiction. This proves (1).

(2) *If $W_1 \neq \emptyset$, then A_5 is strongly complete to A_6, A_1 is strongly anticomplete to A_3, and A_6 is strongly anticomplete to A_4.*

For A_5 is strongly complete to A_6 since $W_1 \cup A_5 \cup A_6 \cup A_3$ includes no claw; A_1 is strongly anticomplete to A_3 since $A_3 \cup A_1 \cup A_4 \cup W_1$ includes no claw; and A_6 is strongly anticomplete to A_4 since $A_6 \cup A_4 \cup A_1 \cup W_1$ includes no claw. This proves (2).

(3) *If A_4 is not strongly complete to A_5 then either $G \in \mathcal{S}_0$ or G is decomposable.*

For then it follows as in (1) that $S_{3\frac{1}{2}}, S_{5\frac{1}{2}}, H_{4\frac{1}{2}}$ are all empty. Moreover, 13.4 implies that for $i \in \{2,3,5,6\}$, A_i is strongly complete to A_{i+1}, and for $i \in \{2,5\}$, A_i is strongly anticomplete to A_{i+2}. Since one of W_1, W_2 is nonempty, from the symmetry we may assume that $W_1 \neq \emptyset$. By (2), A_5 is strongly complete to A_6, A_1 is strongly anticomplete to A_3, and A_6 is strongly anticomplete to A_4. If also $W_2 \neq \emptyset$, then similarly A_3 is strongly complete to A_4, A_6 is strongly anticomplete to A_2, and A_3 is strongly anticomplete to A_5; and so A_i is strongly anticomplete to A_{i+2} for $i = 1, \ldots, 6$, and 9.2 implies that $G \in \mathcal{S}_0$. We may therefore assume that $W_2 = \emptyset$. But then

$$(A_2 \cup H_{2\frac{1}{2}} \cup S_{1\frac{1}{2}}, A_1 \cup H_{\frac{1}{2}}, A_6)$$

is a breaker, and 4.4 implies that G is decomposable. This proves (3).

In view of (3) we assume henceforth that A_4 is strongly complete to A_5.

(4) *If $H_{5\frac{1}{2}} \neq \emptyset$ then:*

- *A_4 is strongly anticomplete to A_6,*
- *A_5 is strongly complete to A_6,*
- *$S_{3\frac{1}{2}}$ is strongly complete to $S_{5\frac{1}{2}}$,*
- *$S_{5\frac{1}{2}}$ is strongly complete to $S_{1\frac{1}{2}}$, and*
- *$S_{4\frac{1}{2}}$ is strongly anticomplete to $S_{5\frac{1}{2}}$.*

For let $h \in H_{5\frac{1}{2}}$. Then A_4 is strongly anticomplete to A_6 since $A_6 \cup A_4 \cup A_1 \cup \{h\}$ includes no claw; A_5 is strongly complete to A_6 since G contains no 7-hole; $S_{3\frac{1}{2}}$ is strongly complete to $S_{5\frac{1}{2}}$ since $A_5 \cup S_{3\frac{1}{2}} \cup S_{5\frac{1}{2}} \cup \{h\}$ includes no claw; $S_{5\frac{1}{2}}$ is strongly complete to $S_{1\frac{1}{2}}$ since $A_6 \cup S_{5\frac{1}{2}} \cup S_{1\frac{1}{2}} \cup \{h\}$ includes no claw; and $S_{4\frac{1}{2}}$ is strongly anticomplete to $S_{5\frac{1}{2}}$ since $S_{4\frac{1}{2}} \cup S_{5\frac{1}{2}} \cup A_3 \cup \{h\}$ includes no claw. This proves (4).

(5) *If $H_{5\frac{1}{2}}$ and $S_{5\frac{1}{2}}$ are both nonempty then either $G \in \mathcal{S}_0$ or G is decomposable.*

For let $h \in H_{5\frac{1}{2}}$ and $s \in S_{5\frac{1}{2}}$. By 9.2, h, s are strongly antiadjacent. Since $S_{3\frac{1}{2}} \cup S_{4\frac{1}{2}} \cup A_2 \cup \{s\}$ includes no claw, it follows that $S_{3\frac{1}{2}}$ is strongly anticomplete to $S_{4\frac{1}{2}}$. Since $A_6 \cup A_2 \cup \{h, s\}$ includes no claw, A_6 is strongly anticomplete to A_2, and similarly A_5 is strongly anticomplete to A_3. If A_1 is strongly anticomplete to A_3 and $S_{1\frac{1}{2}}$ is strongly complete to $S_{3\frac{1}{2}}$ then $G \in \mathcal{S}_0$; so we may assume that not both these hold. If $W_1 \neq \emptyset$ then (2) implies that A_1 is strongly anticomplete to A_3, and since $\{W_1, S_{1\frac{1}{2}}, S_{3\frac{1}{2}}, h\}$ includes no claw, it folows that $S_{1\frac{1}{2}}$ is strongly complete to $S_{3\frac{1}{2}}$, a contradiction. Thus $W_1 = \emptyset$, and so there exists $w \in W_2$. By (2), A_3 is strongly complete to A_4. If $S_{3\frac{1}{2}} = \emptyset$, then

$$(A_6 \cup S_{5\frac{1}{2}}, A_5 \cup H_{5\frac{1}{2}} \cup H_{4\frac{1}{2}}, A_4 \cup S_{4\frac{1}{2}})$$

is a breaker, and the result follows from 4.4. So we may assume that $S_{3\frac{1}{2}} \neq \emptyset$. If also $H_{3\frac{1}{2}} \neq \emptyset$, then from the symmetry between W_1, W_2 we deduce that $W_2 = \emptyset$, a contradiction. Consequently $H_{3\frac{1}{2}} = \emptyset$. Suppose that there exists $s_{4\frac{1}{2}} \in S_{4\frac{1}{2}}$. We recall that either A_1 is not strongly anticomplete to A_3 or $S_{1\frac{1}{2}}$ is not strongly complete to $S_{3\frac{1}{2}}$. But if $a_1 \in A_1$ is adjacent to $a_3 \in A_3$, then $\{a_3, a_1, s_{4\frac{1}{2}}, s_{3\frac{1}{2}}\}$ is a claw (where $s_{3\frac{1}{2}} \in S_{3\frac{1}{2}}$), and if $s_{1\frac{1}{2}} \in S_{1\frac{1}{2}}$ is antiadjacent to $s_{3\frac{1}{2}} \in S_{3\frac{1}{2}}$ then $\{a_3, s_{1\frac{1}{2}}, s_{3\frac{1}{2}}, s_{4\frac{1}{2}}\}$ is a claw (where $a_3 \in A_3$), in either case a contradiction. Thus $S_{4\frac{1}{2}} = \emptyset$. But then

$$(A_6 \cup H_{\frac{1}{2}}, H_{5\frac{1}{2}}, A_5 \cup H_{4\frac{1}{2}})$$

is a breaker, and again the result follows from 4.4. This proves (5).

(6) *If $H_{5\frac{1}{2}}$ is nonempty then either $G \in \mathcal{S}_0$ or G is decomposable.*

For suppose that $H_{5\frac{1}{2}} \neq \emptyset$. By (5) we may assume that $S_{5\frac{1}{2}} = \emptyset$. By (4), A_4 is strongly anticomplete to A_6, and A_5 is strongly complete to A_6. Suppose that $W_2 = \emptyset$; then $W_1 \neq \emptyset$. By (2),

A_3 is strongly anticomplete to A_1. Since $W_1 \cup S_{1\frac{1}{2}} \cup S_{3\frac{1}{2}} \cup H_{5\frac{1}{2}}$ includes no claw, $S_{1\frac{1}{2}}$ is strongly complete to $S_{3\frac{1}{2}}$. But then

$$(A_5 \cup S_{4\frac{1}{2}}, A_4 \cup H_{3\frac{1}{2}} \cup H_{4\frac{1}{2}}, A_3 \cup S_{3\frac{1}{2}})$$

is a breaker, and the result follows from 4.4. Thus we may assume that $W_2 \neq \emptyset$. By (2), A_3 is strongly complete to A_4, and A_3 is strongly anticomplete to A_5, and A_2 is strongly anticomplete to A_6. Suppose that $S_{3\frac{1}{2}} = \emptyset$. If also $W_1 = \emptyset$ then

$$(A_1 \cup S_{1\frac{1}{2}}, A_2 \cup H_{2\frac{1}{2}}, A_3)$$

is a breaker and the result follows from 4.4. On the other hand, if $S_{3\frac{1}{2}} = \emptyset$ and $W_1 \neq \emptyset$, then from (2) A_1 is strongly anticomplete to A_3 and therefore $G \in \mathcal{S}_0$. We may therefore assume that $S_{3\frac{1}{2}} \neq \emptyset$. By (5) we may assume that $H_{3\frac{1}{2}} = \emptyset$. Since $S_{4\frac{1}{2}} \cup S_{3\frac{1}{2}} \cup W_2 \cup H_{5\frac{1}{2}}$ includes no claw, it follows that $S_{4\frac{1}{2}}$ is strongly anticomplete to $S_{3\frac{1}{2}}$. If $W_1 \neq \emptyset$, then by (2), A_1 is strongly anticomplete to A_3, and $S_{1\frac{1}{2}}$ is strongly complete to $S_{3\frac{1}{2}}$ since $W_1 \cup S_{1\frac{1}{2}} \cup S_{3\frac{1}{2}} \cup H_{5\frac{1}{2}}$ includes no claw; and then $G \in \mathcal{S}_0$. So we may assume that $W_1 = \emptyset$. If $S_{4\frac{1}{2}} = \emptyset$ then $(A_5, H_{5\frac{1}{2}}, A_6)$ is a breaker, and the result follows from 4.4; so we may assume that $S_{4\frac{1}{2}} \neq \emptyset$. Since $A_3 \cup S_{1\frac{1}{2}} \cup S_{3\frac{1}{2}} \cup S_{4\frac{1}{2}}$ includes no claw, $S_{1\frac{1}{2}}$ is strongly complete to $S_{3\frac{1}{2}}$; and since $A_3 \cup A_1 \cup S_{3\frac{1}{2}} \cup S_{4\frac{1}{2}}$ includes no claw, A_1 is strongly anticomplete to A_3. But then again $G \in \mathcal{S}_0$. This proves (6).

In view of (6) and the symmetry between $H_{5\frac{1}{2}}, H_{3\frac{1}{2}}$, we henceforth assume that $H_{5\frac{1}{2}} = H_{3\frac{1}{2}} = \emptyset$.

(7) *If $H_{4\frac{1}{2}}, S_{4\frac{1}{2}}$ are both nonempty, then either $G \in \mathcal{S}_0$ or G is decomposable.*

For since $S_{3\frac{1}{2}} \cup S_{4\frac{1}{2}} \cup A_2 \cup H_{4\frac{1}{2}}$ includes no claw, $S_{3\frac{1}{2}}$ is strongly anticomplete to $S_{4\frac{1}{2}}$; and similarly $S_{4\frac{1}{2}}$ is strongly anticomplete to $S_{5\frac{1}{2}}$. Since $A_3 \cup S_{1\frac{1}{2}} \cup S_{3\frac{1}{2}} \cup S_{4\frac{1}{2}}$ includes no claw, $S_{1\frac{1}{2}}$ is strongly complete to $S_{3\frac{1}{2}}$, and similarly $S_{1\frac{1}{2}}$ is strongly complete to $S_{5\frac{1}{2}}$. Since $S_{4\frac{1}{2}} \neq \emptyset$ it follows that A_3 is strongly complete to A_4, and A_5 is strongly complete to A_6. Since $A_5 \cup A_3 \cup H_{4\frac{1}{2}} \cup A_6$ includes no claw, A_5 is strongly anticomplete to A_3, and similarly A_4 is strongly anticomplete to A_6. Since $A_5 \cup S_{3\frac{1}{2}} \cup S_{5\frac{1}{2}} \cup S_{4\frac{1}{2}}$ includes no claw, $S_{3\frac{1}{2}}$ is strongly complete to $S_{5\frac{1}{2}}$. If also A_2 is strongly anticomplete to A_6 and A_1 is strongly anticomplete to A_3 then $G \in \mathcal{S}_0$; so we may assume that A_1 is not strongly anticomplete to A_3, from the symmetry. By (2), $W_1 = \emptyset$, and so $W_2 \neq \emptyset$, and by (2), A_2 is strongly anticomplete to A_6. Since $A_3 \cup S_{3\frac{1}{2}} \cup A_1 \cup S_{4\frac{1}{2}}$ includes no claw, it follows that $S_{3\frac{1}{2}} = \emptyset$. But then

$$(A_1 \cup S_{1\frac{1}{2}}, A_2 \cup H_{2\frac{1}{2}}, A_3)$$

is a breaker, and the result follows from 4.4. This proves (7).

(8) *If either $S_{1\frac{1}{2}} \neq \emptyset$ or some vertex in $S_{3\frac{1}{2}} \cup S_{5\frac{1}{2}}$ is strongly complete to $S_{4\frac{1}{2}}$, or $S_{4\frac{1}{2}} = \emptyset$, then either $G \in \mathcal{S}_0$ or G is decomposable.*

For let $S'_{i+\frac{1}{2}}$ be the set of all vertices in $S_{i+\frac{1}{2}}$ that are strongly complete to $S_{4\frac{1}{2}}$, for $i = 3, 5$.

(Thus if $S_{4\frac{1}{2}} = \emptyset$ then $S'_{i+\frac{1}{2}} = S_{i+\frac{1}{2}}$.) For $i = 3, 5$, let $S''_{i+\frac{1}{2}} = S_{i+\frac{1}{2}} \setminus S'_{i+\frac{1}{2}}$. Since $A_3 \cup S''_{3\frac{1}{2}} \cup S_{1\frac{1}{2}} \cup S_{4\frac{1}{2}}$ includes no claw, it follows that $S''_{3\frac{1}{2}}$ is strongly complete to $S_{1\frac{1}{2}}$, and similarly $S''_{5\frac{1}{2}}$ is strongly complete to $S_{1\frac{1}{2}}$. Moreover, since $S'_{3\frac{1}{2}} \cup S''_{5\frac{1}{2}} \cup S_{4\frac{1}{2}} \cup A_2$ includes no claw, it follows that $S'_{3\frac{1}{2}}$ is strongly anticomplete to $S''_{5\frac{1}{2}}$, and similarly $S'_{5\frac{1}{2}}$ is strongly anticomplete to $S''_{3\frac{1}{2}}$. But

$$A_1 \cup A_6 \cup H_{\frac{1}{2}} \cup S''_{5\frac{1}{2}} \cup W_2, A_2 \cup A_3 \cup H_{2\frac{1}{2}} \cup S''_{3\frac{1}{2}} \cup W_1, A_4 \cup A_5 \cup H_{4\frac{1}{2}} \cup S_{4\frac{1}{2}}$$

are strong cliques; also, $S'_{3\frac{1}{2}}, S'_{5\frac{1}{2}}, S_{1\frac{1}{2}}$ are strong cliques; these six cliques are pairwise disjoint and have union $V(G)$; and for $i = 1, 2, 3$, the ith clique of the first three is strongly anticomplete to the ith clique of the second three, and strongly complete to the other two of the second three. Since the first three cliques are certainly nonempty, we may assume that the second three are all empty, for otherwise G admits a hex-join. This proves the first two assertions of the claim. In particular, $S_{1\frac{1}{2}} = \emptyset$. For the third assertion, suppose that also $S_{4\frac{1}{2}} = \emptyset$. Then since $S'_{3\frac{1}{2}}, S'_{5\frac{1}{2}} = \emptyset$, it follows that $S_{3\frac{1}{2}}, S_{5\frac{1}{2}} = \emptyset$. From the symmetry we may assume that $W_1 \neq \emptyset$, and so from (2), A_5 is strongly complete to A_6, A_1 is strongly anticomplete to A_3, and A_6 is strongly anticomplete to A_4. If also $W_2 \neq \emptyset$, then similarly A_3 is strongly complete to A_4, A_6 is strongly anticomplete to A_2, and A_5 is strongly anticomplete to A_3, and therefore $G \in \mathcal{S}_0$. We may therefore assume that $W_2 = \emptyset$. But then $(A_6, A_1 \cup H_{\frac{1}{2}}, A_2)$ is a breaker, and the result follows from 4.4. This proves (8).

Thus we may assume that $S_{1\frac{1}{2}} = \emptyset$ and $S_{4\frac{1}{2}} \neq \emptyset$; and no vertex in $S_{3\frac{1}{2}} \cup S_{5\frac{1}{2}}$ is strongly complete to $S_{4\frac{1}{2}}$. Consequently $H_{4\frac{1}{2}} = \emptyset$, from (7). Suppose that A_1 is not strongly anticomplete to A_3. From (2), $W_1 = \emptyset$; and so $W_2 \neq \emptyset$, and therefore from (2), A_3 is strongly complete to A_4, A_2 is strongly anticomplete to A_6, and A_3 is strongly anticomplete to A_5. Since $A_3 \cup H_{2\frac{1}{2}} \cup A_1 \cup S_{4\frac{1}{2}}$ includes no claw, $H_{2\frac{1}{2}} = \emptyset$. If there exists $s_{3\frac{1}{2}} \in S_{3\frac{1}{2}}$, choose $s_{4\frac{1}{2}} \in S_{4\frac{1}{2}}$ antiadjacent to $s_{3\frac{1}{2}}$ (this exists, by (8)); and choose $a_1 \in A_1$ and $a_3 \in A_3$, adjacent. Then $\{a_3, a_1, s_{3\frac{1}{2}}, s_{4\frac{1}{2}}\}$ is a claw, a contradiction. Hence $S_{3\frac{1}{2}} = \emptyset$. Consequently (A_1, A_2, A_3) is a breaker, and the result follows from 4.4. Thus we may assume that A_1 is strongly anticomplete to A_3, and similarly A_2 is strongly anticomplete to A_6. Hence (A_1, A_2) is a nondominating homogeneous pair of cliques. By 4.3, we may assume that $|A_1| = |A_2| = 1$; let $A_i = \{a_i\}$ for $i = 1, 2$. It follows that a_1, a_2 are semiadjacent. The set of vertices antiadjacent to both a_1, a_2 is $A_4 \cup A_5 \cup S_{4\frac{1}{2}}$, and this is a strong clique. No vertex is adjacent to both a_1, a_2. Thus the hypotheses of 11.1 are satisfied, and therefore 11.1 implies that $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$ and the theorem holds. This proves 13.5. ■

Our first application of 13.4 is the following.

13.6 *Let G be a claw-free trigraph containing no long prism, in which every maximum hole is dominating; and let C_0 be a maximum hole of length six. Suppose that there is a hub for C_0, and either some vertex of $V(G) \setminus V(C_0)$ is a clone with respect to C_0, or some two consecutive vertices of C_0 are semiadjacent. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$, or G is decomposable.*

Proof. Let C_0 have vertices a_1-$\cdots$-a_6-a_1 in order, and let w be a hub, adjacent to a_1, a_2, a_4, a_5 say. Let $\mathcal{C}$ be the proximity component containing C_0, and let $A_i = A_i(\mathcal{C})$ for $1 \leq i \leq 6$. By 13.3, w is a hub in hub-position 3 relative to every member of $\mathcal{C}$. Consequently, w is strongly complete to

$A_1 \cup A_2 \cup A_4 \cup A_5$, and strongly anticomplete to $A_3 \cup A_6$, and in particular, A_3, A_6 are disjoint from A_1, A_2, A_4, A_5. We observe first that:

(1) *Let $1 \leq i \leq 6$, and let $v \in A_i$. Let c_1-$\cdots$-c_6-$c_1 \in \mathcal{C}$. If $i = 3, 6$, then $N(v)$ contains c_i and at least one of c_{i-1}, c_{i+1}, and none of $c_{i+2}, c_{i+3}, c_{i+4}$. (Consequently, A_3 is strongly anticomplete to $A_5 \cup A_6 \cup A_1$.) If $i = 1, 2$, then $N(v)$ contains both of c_1, c_2, and at most one of c_4, c_5 (and symmetrically if $i = 4, 5$).*

For v belongs to some member of $\mathcal{C}$, and the claim holds for that member. Consequently it suffices to show that if c_1-$\cdots$-c_6-c_1 and c'_1-$\cdots$-c'_6-c'_1 are proximate members of $\mathcal{C}$, and the claim holds for c_1-$\cdots$-c_6-c_1 then it holds for c'_1-$\cdots$-c'_6-c'_1. Let these two 6-numberings differ in their jth entry. Assume first that $i \in \{3, 6\}$, say $i = 3$. Thus $N(v)$ contains at least two of c_2, c_3, c_4 and none of w, c_5, c_6, c_1. Hence if $j \in \{5, 6, 1\}$ then $N(v)$ contains at least two of c'_2, c'_3, c'_4, and if $j \in \{2, 3, 4\}$ then $N(v)$ contains none of c'_5, c'_6, c'_1; and in either case, since $w \notin N(v)$, it follows from 13.1 that $N(v)$ contains c'_3 and at least one of c'_2, c'_4, and contains none of c'_5, c'_6, c'_1 as required. Now assume that $i \in \{1, 2\}$, and consequently $c_1, c_2, w \in N(v)$, and not both $c_4, c_5 \in N(v)$. Thus if $j \in \{3, 4, 5, 6\}$ then $c'_1, c'_2 \in N(v)$, and if $j \in \{6, 1, 2, 3\}$ then not both $c'_4, c'_5 \in N(v)$. Since $w \in N(v)$ and v is not a hub relative to c'_1-$\cdots$-c'_6-c'_1 (by 13.3), it follows in either case from 13.1 applied to c'_1-$\cdots$-c'_6-c'_1 that v is $\{c'_1, c'_2\}$-complete and not $\{c'_4, c'_5\}$-complete, as required. This proves (1).

(2) *$\mathcal{C}$ is pure.*

We must show that $A_1, \ldots, A_6$ are pairwise disjoint. The members of A_1, A_2, A_4, A_5 are adjacent to w, and those of A_3, A_6 are not. Also, by (1), members of $A_1 \cup A_2$ are $\{a_1, a_2\}$-complete and not $\{a_4, a_5\}$-complete; and members $A_4 \cup A_5$ are $\{a_4, a_5\}$-complete and not $\{a_1, a_2\}$-complete. Thus the three sets $A_3 \cup A_6, A_1 \cup A_2, A_4 \cup A_5$ are pairwise disjoint. To prove the claim, it remains to show that the intersections $A_3 \cap A_6, A_1 \cap A_2, A_4 \cap A_5$ are all empty. Now members of A_3 are adjacent to a_3 and not to a_6 by (1), and vice versa for A_6, and so $A_3 \cap A_6 = \emptyset$. Suppose that $v \in A_1 \cap A_2$ say. Since $v \in A_1$, there exists c_1-$\cdots$-c_6-$c_1 \in \mathcal{C}$ with $c_1 = v$; and since $c_6 \in A_6$, it follows that v has a neighbour x say in A_6. Similarly v has a neighbour y in A_3; and since A_3, A_6 are anticomplete by (1), it follows that $\{v, w, x, y\}$ is a claw, a contradiction. Thus $A_1 \cap A_2 = \emptyset$ and similarly $A_4 \cap A_5 = \emptyset$. This proves (2).

We deduce that the five statements of 13.4 hold. In particular, each A_i is a strong clique, and A_i is strongly anticomplete to A_{i+3}, and every vertex not in $A_1 \cup \cdots \cup A_6$ is strongly complete or strongly anticomplete to each A_i. By (2) and 13.5, we may assume that for $i = 1, \ldots, 6$, A_i is strongly complete to A_{i+1}. (In particular, every two consecutive vertices of C_0 are strongly adjacent, and so by hypothesis, some member of $V(G) \setminus V(C_0)$ is a clone relative to C_0). Now if $c_6 \in A_6$ and $c_2 \in A_2$ are adjacent, choose $c_3 \in A_3$; then $\{c_2, c_3, c_6, w\}$ is a claw, a contradiction Consequently A_6 is strongly anticomplete to A_2, and similarly A_i is strongly anticomplete to A_{i+2} for $i = 1, 3, 4, 6$. It follows that A_3 is a homogeneous set, and (A_2, A_4) is a homogeneous pair, nondominating since $A_6 \neq \emptyset$; and so we may assume that A_2, A_3, A_4 all have cardinality one, for otherwise G is decomposable by 4.3. Similarly we may assume (for a contradiction) that A_5, A_6, A_1 all have cardinality one, contradicting the hypothesis that some member of $V(G) \setminus V(C_0)$ is a clone relative to C_0. This proves 13.6. ■

Let c_1-$\cdots$-c_6-c_1 be a 6-hole. We recall that if b_1, b_2 are adjacent stars in positions $i+\frac{1}{2}, i+3\frac{1}{2}$ for some $i \in \{1,\dots,6\}$, we call $\{b_1, b_2\}$ a *star-diagonal*. The trigraph induced on these eight vertices is also an induced subtrigraph of the icosahedron, obtained by deleting two vertices at distance two and both their common neighbours. If v is a star relative to a hole C, we say v is a *strong* star if v is not semiadjacent to any vertex of C. The next result is our second application of 13.4.

13.7 *Let G be a claw-free trigraph containing no long prism, and such that every maximum hole is dominating; and let C_0 be a maximum hole of length six, with a star-diagonal. Then no two consecutive vertices of C_0 are semiadjacent. Moreover, if some vertex of $V(G) \setminus V(C_0)$ is a clone with respect to C_0, then either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$, or G is decomposable.*

Proof. Let C_0 have vertices a_1-$\cdots$-a_6-a_1, and let b_1, b_2 be adjacent stars in positions $1\frac{1}{2}, -1\frac{1}{2}$ respectively, say. Since $\{b_1, b_2, a_1, a_2\}$ is not a claw, a_1 is strongly adjacent to a_2; and since $\{b_1, a_2, a_3, a_6\}$ is not a claw, a_2 is strongly adjacent to a_3. Similarly every two consecutive vertices of C_0 are strongly adjacent. This proves the first assertion.

We oberve also that b_1, b_2 are strong stars relative to C_0. For b_1 is strongly adjacent to a_1 since $\{a_6, a_1, b_1, a_5\}$ is not a claw; and b_1 is strongly adjacent to a_3 since $\{b_2, b_1, a_3, a_5\}$ is not a claw; and b_1 is strongly antiadjacent to a_4 since $\{b_1, a_2, a_4, a_6\}$ is not a claw. Similarly b_1 is strongly adjacent to each of a_6, a_1, a_2, a_3 are strongly antiadjacent to a_4, a_5, and so b_1 is a strong star; and similarly so is b_2.

We may assume that some vertex of $V(G) \setminus V(C_0)$ is a clone with respect to C_0. By 12.1, we may assume that G does not contain an *icosa*(-3)-trigraph. By 13.6, we may assume that no vertex is a hub for C_0. Let $\mathcal{C}$ be the proximity component containing a_1-$\cdots$-a_6-a_1, and let $A_i = A_i(\mathcal{C})$ for $1 \le i \le 6$.

(1) *For every c_1-$\cdots$-c_6-$c_1 \in \mathcal{C}$, b_1, b_2 are strong stars in positions $1\frac{1}{2}, -1\frac{1}{2}$ respectively.*

For let c_1-$\cdots$-c_6-c_1 and c'_1-$\cdots$-c'_6-c'_1 be proximate members of $\mathcal{C}$, differing only in their jth term say; it suffices to show that if the claim holds for c_1-$\cdots$-c_6-c_1 then it holds for c'_1-$\cdots$-c'_6-c'_1. Let $N = N_G(c'_j)$ and $N^* = N^*_G(c'_j)$. Thus $c_{j-1}, c_j, c_{j+1} \in N$, and $c_{j+2}, c_{j+3}, c_{j+4} \notin N^*$. From the symmetry we may assume that $j \in \{2,3\}$. Suppose first that $j = 2$. Then we must prove that $b_1 \in N^*$ and $b_2 \notin N$. Now 5.4 (with b_1-c_3-c_4) implies that $b_1 \in N^*$; and 5.4 (with c_4-b_2-c_6) implies that $b_2 \notin N$. Next, suppose that $j = 3$; we must prove that $b_1, b_2 \in N^*$. If $b_1, b_2 \notin N$, then $G|\{c_1,\dots,c_6,b_1,b_2,c'_3\}$ is an *icosa*(-3)-trigraph, a contradiction. Thus N contains at least one of b_1, b_2, and from the symmetry we may assume it contains b_1. By 5.4 (with c_1-b_1-b_2), it follows that $b_2 \in N^*$, and similarly $b_1 \in N$. This proves (1).

(2) *Let $1 \le i \le 6$ and let $v \in A_i$. Let c_1-$\cdots$-c_6-$c_1 \in \mathcal{C}$. If $i = 3, 6$, then $c_{i+3} \notin N_G(v)$, and $c_{i-1}, c_i, c_{i+1} \in N^*_G(v)$. If $i = 1, 2$, then $N^*(v)$ contains both of c_1, c_2, and $N_G(v)$ contains at most one of c_4, c_5 (and symmetrically if $i = 4, 5$). In particular, A_3 is strongly anticomplete to A_6.*

For v belongs to some member of $\mathcal{C}$, and the claim is true for that member. Consequently, it suffices to show that if c_1-$\cdots$-c_6-c_1 and c'_1-$\cdots$-c'_6-c'_1 are proximate members of $\mathcal{C}$, and the claim holds for c_1-$\cdots$-c_6-c_1, then it holds for c'_1-$\cdots$-c'_6-c'_1. Let these two 6-numberings differ in their jth entry. Assume first that $i \in \{3,6\}$, say $i = 3$. Thus $N^*_G(v)$ contains b_1, b_2, c_2, c_3, c_4 and $c_6 \notin N_G(v)$.

If $j \neq 6$, then $c'_6 = c_6 \notin N_G(v)$, and by 5.4 (with c'_2-b_1-c'_6, c'_3-b_1-c'_6, and c'_4-b_2-c'_6), it follows that $c'_2, c'_3, c'_4 \in N^*_G(v)$. If $j = 6$, then $c'_2, c'_4 \in N^*_G(v)$, and so $c'_6 \notin N_G(v)$ by 5.3. Thus in either case the claim holds. Now assume that $i = 1$. Thus $b_1 \in N^*_G(v)$ and $b_2 \notin N_G(v)$. By 5.4 (with b_2-b_1-c'_1), $c'_1 \in N^*_G(v)$ and similarly $c'_2 \in N^*_G(v)$. Since v is not a hub relative to c'_1-$\cdots$-c'_6-c'_1 by 13.3, it follows that $N_G(v)$ contains at most one of c'_4, c'_5. This proves (2).

(3) *$\mathcal{C}$ is pure, and A_i is strongly complete to A_{i+1} for $1 \leq i \leq 6$.*

We must show that $A_1, \ldots, A_6$ are pairwise disjoint. By (1), the members of $A_3 \cup A_6$ are strongly adjacent to both b_1, b_2; the members of $A_1 \cup A_2$ are strongly adjacent to b_1 and strongly antiadjacent b_2; and the members of $A_4 \cup A_5$ are strongly adjacent to b_2 and strongly antiadjacent to b_1. Consequently the three sets $A_3 \cup A_6$, $A_1 \cup A_2$, $A_4 \cup A_5$ are pairwise disjoint. By (2), the members of $A_3 \setminus \{a_3\}$ are strongly adjacent to a_3, and the members of A_6 are strongly antiadjacent to a_3, and so $A_3 \cap A_6 = \emptyset$. Suppose that $v \in A_1 \cap A_2$ say. Since $v \in A_1$, there exists c_1-$\cdots$-c_6-$c_1 \in \mathcal{C}$ with $c_1 = v$; and since $c_3 \in A_3$, it follows that v has an antineighbour x say in A_3. Similarly v has an antineighbour y in A_6; and since A_3, A_6 are anticomplete by (1), it follows that $\{b_1, v, x, y\}$ is a claw, a contradiction. Thus $A_1 \cap A_2 = \emptyset$ and similarly $A_4 \cap A_5 = \emptyset$. Thus $\mathcal{C}$ is pure, and the final claim follows from (2). This proves (3).

We deduce that the five statements of 13.4 hold. In particular, each A_i is a strong clique, and A_i is strongly anticomplete to A_{i+3}, and every vertex not in $A_1 \cup \cdots \cup A_6$ is strongly complete or strongly anticomplete to each A_i.

(4) *We may assume (possibly after renumbering $A_1, \ldots, A_6$) that there is a vertex $h \in V(G) \setminus (A_1 \cup \cdots \cup A_6 \cup \{b_1, b_2\})$, such that h is strongly $A_1 \cup A_2$-complete and strongly anticomplete to A_3, A_4, A_5, A_6.*

For since some vertex is a clone relative to C_0, at least one of the sets $A_1, \ldots, A_6$ has at least two members, and therefore from the symmetry we may assume that not all of A_1, A_3, A_5 have cardinality 1. Now A_1, A_3, A_5 are strong cliques, all nonempty, and their union is not equal to $V(G)$; so by 4.5, we may assume that some $h \in V(G) \setminus (A_1 \cup A_3 \cup A_5)$ does not have the property that it is strongly complete to two of A_1, A_3, A_5 and strongly anticomplete to the third. Consequently $h \notin A_1 \cup \cdots \cup A_6 \cup \{b_1, b_2\}$, and therefore h is strongly complete or strongly anticomplete to each A_i, and is complete to exactly two of $A_1, \ldots, A_6$, necessarily consecutive. If say h is complete to A_2, A_3, then by 9.1 h is adjacent to b_1 and antiadjacent to b_2; and then $\{b_1, b_2, h, a_1\}$ is a claw, a contradiction. Thus h is complete to either A_1, A_2 or to A_4, A_5, and anticomplete to the other four sets. This proves (4).

(5) *A_2 is strongly anticomplete to $A_4 \cup A_6$, and A_1 is strongly anticomplete to $A_3 \cup A_5$.*

For let $x \in A_2$ and $y \in A_4$, and let h be as in (4). Then h is adjacent to x and antiadjacent to y; and h is antiadjacent to b_1 by 9.2. Since $\{x, b_1, y, h\}$ is not a claw, it follows that x, y are strongly antiadjacent. Thus A_2, A_4 are strongly anticomplete, and similarly so are A_1, A_5. Now let $x \in A_2, y \in A_6$. Let z be a neighbour of x in A_3 (this exists, since x belongs to a member of $\mathcal{C}$). Since $\{x, y, z, h\}$ is not a claw, x, y are strongly antiadjacent, and so A_2, A_6 are strongly anticomplete. Similarly A_1, A_3 are strongly anticomplete. This proves (5).

To complete the proof, we recall that one of $A_1, \ldots, A_6$ has cardinality > 1 since there is a clone relative to a_1-$\cdots$-a_6-a_1. But (A_3, A_5) is a homogeneous pair, nondominating since $A_1 \neq \emptyset$; and similarly (A_4, A_6) is a nondominating homogeneous pair, and A_1, A_2 are homogeneous sets. Hence 4.3 implies that G is decomposable. This proves 13.7. ■

14 Nondominating 5-holes

Let us say that a triple (A, C, B) is a *generalized breaker* in a trigraph G if it satisfies:

- A, B, C are disjoint nonempty subsets of $V(G)$, and A, B are strong cliques
- every vertex in $V(G) \setminus (A \cup B \cup C)$ is either strongly A-complete or strongly A-anticomplete, and either strongly B-complete or strongly B-anticomplete, and strongly C-anticomplete,
- there is a vertex in $V(G) \setminus (A \cup B \cup C)$ with a neighbour in A and an antineighbour in B; there is a vertex in $V(G) \setminus (A \cup B \cup C)$ with a neighbour in B and an antineighbour in A; and there is a vertex in $V(G) \setminus (A \cup B \cup C)$ with an antineighbour in A and an antineighbour in B.

Thus, this is the same as the definition of a breaker, except that the final condition has been removed. There is an analogue of 4.4 for generalized breakers, the following.

14.1 *Let G be a claw-free trigraph. If there is a generalized breaker in G, then either G is decomposable, or $G \in \mathcal{S}_2 \cup \mathcal{S}_5$.*

Proof. We assume G is not decomposable. Let (D_3, D_5, D_4) be a generalized breaker; let $V_1 = D_3 \cup D_4 \cup D_5$, let D_2 be the set of vertices in $V(G) \setminus V_1$ that are $D_3 \cup D_4$-complete, and let $V_2 = V(G) \setminus (D_2 \cup V_1)$. Let A be the set of vertices in V_2 that are D_3-complete, and B the set that are D_4-complete. Let D_1 be the set of all vertices in $V_2 \setminus (A \cup B)$ with a neighbour in D_2. By hypothesis, D_3, D_4, A, B are nonempty, and as in the proof of 4.4, it follows that $A \cup D_2$ and $B \cup D_2$ are strong cliques. By 4.4, $D_1 \neq \emptyset$ and D_3 is strongly complete to D_4. Since $D_3 \cup D_4$ is not an internal clique cutset (because G is not decomposable), it follows that $|D_5| = 1$, $D_5 = \{d_5\}$ say. We may assume that d_5 has a neighbour $d_3 \in D_3$. Let $d_4 \in D_4$ and $a \in A$; then since $\{d_3, d_4, d_5, a\}$ is not a claw, it follows that d_5, d_4 are strongly adjacent, and therefore that d_5 is strongly complete to D_4. Similarly d_5 is strongly complete to D_3. Hence D_3 is a homogeneous set, so $D_3 = \{d_3\}$ and similarly $D_4 = \{d_4\}$, and d_5 is strongly adjacent to d_3, d_4. Every vertex in D_1 has a neighbour in D_2, and since $D_2 \cup A \cup D_1 \cup D_4$ includes no claw, $A \cup D_1$ is a strong clique and similarly $B \cup D_1$ is a strong clique. Let $V_3 = V_2 \setminus (A \cup B \cup D_1)$.

(1) *A is not strongly complete to B.*

For suppose it is. Since $D_2 \cup A \cup B$ is not an internal clique cutset and $D_1 \neq \emptyset$, it follows that $V_3 = \emptyset$. But then $(D_3 \cup D_4, A \cup B)$ is a coherent homogeneous pair, a contradiction. This proves (1).

Let C be the set of vertices in V_3 with a neighbour in D_1, and $Z = V_3 \setminus C$. For each $c \in C$, let N_c, M_c be the sets of neighbours and antineighbours of c in $A \cup B$ respectively.

(2) *For each $c \in C$, M_c is a strong clique, and $M_c \cap A$ is strongly anticomplete to $N_c \cap B$, and $M_c \cap B$ is strongly anticomplete to $N_c \cap A$. Moreover, C is strongly complete to D_1; and $|D_1| = |D_2| = 1$.*

For let $c \in C$, and let $d_1 \in D_1$ be adjacent to c. If $a \in M_c \cap A$ and $b \in M_c \cap B$, then since $\{d_1, a, b, c\}$ is not a claw, it follows that a, b are strongly adjacent; and so M_c is a strong clique. If $a \in M_c \cap A$ and $b \in N_c \cap B$, then since $\{b, a, c, d_4\}$ is not a claw, a, b are strongly antiadjacent. Hence $M_c \cap A$ is strongly anticomplete to $N_c \cap B$, and similarly $M_c \cap B$ is strongly anticomplete to $N_c \cap A$. Now choose $a \in A$ and $b \in b$, antiadjacent (this is possible by (1)). Then one of $a, b \notin M_c$, say a. For all $d_1 \in D_1$, since $\{a, d_3, d_1, c\}$ is not a claw, it follows that c is strongly adjacent to d_1. Hence C is strongly complete to D_1, and so (D_1, D_2) is a nondominating homogeneous pair. Consequently $|D_1| = |D_2| = 1$. This proves (2).

Let $D_1 = \{d_1\}$ and $D_2 = \{d_2\}$.

(3) *Z is strongly anticomplete to $A \cup B$, and $|Z| \leq 1$, and $M_c \neq \emptyset$ for each $c \in C$.*

For since $\{a, z, d_3, d_1\}$ is not a claw, it follows that each z in Z is strongly antiadjacent to each $a \in A$, and so Z is strongly anticomplete to A and similarly to B. Since C is not an internal clique cutset, it follows that $|Z| \leq 1$. If $c \in C$ and $M_c = \emptyset$, then c is strongly anticomplete to Z (since $\{c, z, a, b\}$ is not a claw, where $a \in A, b \in B$ are antiadjacent and $z \in Z$), and so $(\{d_2\}, \{c, d_1\})$ is a nondominating homogeneous pair, a contradiction. This proves (3).

Let A_0 be the set of vertices in A with no neighbour in B, and define $B_0 \subseteq B$ similarly.

(4) *Every vertex in A has at most one neighbour in B and vice versa. Moreover, if $c \in C$ then $M_c \cap N_c \subseteq A_0 \cup B_0$.*

For suppose that H is a component with $|V(H)| \geq 2$ of the bipartite graph with vertex set $A \cup B$ in which $a \in A$ and $b \in B$ are adjacent if and only if they are adjacent in G. For each $c \in C$, c is either strongly complete or strongly anticomplete to $V(H)$ by (2), and so $(A \cap V(H), B \cap V(H))$ is a nondominating homogeneous pair. Hence $|V(H)| = 2$, and the claim follows.

Let $A \setminus A_0 = \{a_1, \ldots, a_n\}$, and for $1 \leq i \leq n$ let b_i be the neighbour of $a_i \in B$. Thus $B \setminus B_0 = \{b_1, \ldots, b_n\}$. Let P, Q be the set of all $c \in C$ with $M_c \subseteq A_0$ and $M_c \subseteq B_0$ respectively, and for $1 \leq i \leq n$ let C_i be the set of vertices $c \in C$ with $M_c = \{a_i, b_i\}$.

(5) *The sets $P, Q, C_1, \ldots, C_n$ are pairwise disjoint and have union C. Moreover, if $c \in C$ has a neighbour in Z, then either*

- *$n = 0$ and M_c is one of A_0, B_0, or*
- *$n = 1$ and one of A_0, B_0 is empty, say B_0, and M_c is one of $A_0, \{a_1, b_1\}$, or*
- *$n = 2$ and $A_0, B_0 = \emptyset$ and $M_c = \{a_i, b_i\}$ for some $i \in \{1, 2\}$;*

and in each case $M_c \cap N_c = \emptyset$.

For let $c \in C$. Since $M_c \neq \emptyset$, c belongs to at most one of the sets $P, Q, C_1, \ldots, C_n$. If $M_c \cap A$, $M_c \cap B$ are both nonempty, then since M_c is a strong clique it follows that $M_c = \{a_i, b_i\}$ for some $i \in \{1, \ldots, n\}$, and so $c \in C_i$. We may assume then that $M_c \subseteq A$. For $1 \leq i \leq n$, since $b_i \in N_c$, (2) implies that $a_i \notin M_c$, and so $M_c \subseteq A_0$ and $c \in P$. This proves the first claim.

For the second, suppose that c is adjacent to $z \in Z$. Since $\{c, z\} \cup N_c$ includes no claw, N_c is a strong clique. Suppose first that $M_c \subseteq A_0$. Then $b_1, \ldots, b_n \notin M_c$, and so by (2), $a_1, \ldots, a_n \in N_c \setminus M_c$. Then since there exists $b \in B$, it follows that $b \in N_c$ and so b is strongly adjacent to all members of $A \cap N_c$. Hence $|A \cap N_c| \leq 1$, and $N_c \cap A_0 = \emptyset$; and so $n \leq 1$, and $M_c \cap N_c = \emptyset$, and $M_c = A_0$, and if $n = 1$ then b is the unique member of B and so $B_0 = \emptyset$. Similarly if $M_c \subseteq B_0$ then the claim holds. Suppose then that $M_c = \{a_1, b_1\}$ say. By (4), $M_c \cap N_c \subseteq M_c \cap (A_0 \cup B_0) = \emptyset$. If $a \in A \setminus \{a_1\}$ and $b \in B \setminus \{b_1\}$, then $a, b \in N_c$, and so they are strongly adjacent; and therefore $n = 2$, and $A_0 = B_0 = \emptyset$, and $a = a_2, b = b_2$, and $N_c = \{a_2, b_2\}$, and the claim holds. We may assume then that there does not exist $b \in B \setminus \{b_1\}$, and so $n = 1$ and $B_0 = \emptyset$, and again the claim holds. This proves (5).

(6) *If* $Z \neq \emptyset$ *then* $G \in \mathcal{S}_2$.

For suppose that $Z \neq \emptyset$, and let C_0 be the set of all $c \in C$ that are strongly anticomplete to Z. Since G admits no 0-join, it follows that $C_0 \neq C$. Let $Z = \{z\}$. Let N be the union of all the sets N_c ($c \in C \setminus C_0$). If $c_0 \in C_0$ and $m \in N_c$ for some $c \in C \setminus C_0$, then since $\{c, c_0, m, z\}$ is not a claw, it follows that m, c_0 are strongly adjacent; and so C_0 is strongly complete to N. Suppose first that $N_c = N$ for all $c \in C \setminus C_0$. Then N is a strong clique, and hence $N \cup C_0$ is a strong clique, and therefore is an internal clique cutset (since $C \setminus C_0, Z \neq \emptyset$), a contradiction. This proves that there exist $c_1, c_2 \in C \setminus C_0$ with $N_{c_1} \neq N_{c_2}$, and therefore with $M_{c_1} \cap M_{c_2} = \emptyset$, by (5). Hence $N = A \cup B$, and since $M_{c_0} \neq \emptyset$ and $M_{c_0} \cap N = \emptyset$ for every $c_0 \in C_0$, it follows that $C_0 = \emptyset$. We claim that z is strongly complete to C; for let $c_3 \in C$. Then M_{c_3} is different from one of M_{c_1}, M_{c_2}, say M_{c_1}, and so there exists $v \in M_{c_3} \setminus M_{c_1}$, by (5). Since $\{c_1, c_3, z, v\}$ is not a claw, it follows that z, c_3 are strongly adjacent. This proves that Z is strongly complete to C. Hence each set C_i is a homogeneous set, and so each $|C_i| \leq 1$. Moreover, (P, A_0) and (Q, B_0) are nondominating homogeneous pairs, and so P, Q, A_0, B_0 each have cardinality at most one.

By (5) there are now three cases, $n = 2$, $n = 1$ and $n = 0$. Suppose first that $n = 2$. By (5), $C_1 \cup C_2 = C$, and so $|C| = 2$; and then $G \in \mathcal{S}_2$. Next, suppose that $n = 1$. Then by (5), one of A_0, B_0 is empty, say B_0, and $C = C_1 \cup P$. Since there exists $c \in C$ with $N_c \neq \{a_1, b_1\}$, it follows that $P, A_0 \neq \emptyset$, and so $|P| = |A_0| = 1$. But then again $G \in \mathcal{S}_2$. Finally, suppose that $n = 0$. Thus $C = P \cup Q$, and so P, Q, A_0, B_0 all have cardinality one; and again $G \in \mathcal{S}_2$. This proves (6).

Henceforth we therefore may assume that $Z = \emptyset$. Consequently each C_i is a homogeneous set, and so $|C_i| \leq 1$ for $1 \leq i \leq n$. Now again (P, A_0) is a nondominating homogeneous pair, and so $|P|, |A_0| \leq 1$, and similarly $|Q|, |B_0| \leq 1$. We claim there is at most one value of $i \in \{1, \ldots, n\}$ with $C_i = \emptyset$; for if there were two, say $i = 1, 2$, then $(\{a_1, a_2\}, \{b_1, b_2\})$ would be a nondominating homogeneous pair, contrary to 4.3. Thus we may assume that $C_1, \ldots, C_{n-1}$ are all nonempty. For $1 \leq i \leq n$, since $\{d_1, a_i, b_i\} \cup C_i$ includes no claw, it follows that either a_i, b_i are strongly adjacent or $C_i = \emptyset$ (and hence $i = n$). Moreover, if C is strongly complete to B then G is the hex-join of $G|\{d_3\}$ and $G \setminus \{d_3\}$, which is impossible; so C is not strongly complete to B, and similarly not to A. But then $G \in \mathcal{S}_5$. This proves 14.1. ■

Before the main result of this section, we prove another lemma.

14.2 *Let H be a graph with seven vertices $v_1, \dots, v_7$, where v_1-$\cdots$-v_5-v_1 is a cycle of length 5, v_6 has three neighbours in this cycle, and v_7 has two. Then some subgraph of H is a theta with seven vertices.*

Proof. By deleting one (appropriately chosen) edge incident with v_6, we obtain a subgraph consisting of the cycle v_1-$\cdots$-v_5-v_1, a vertex with two consecutive neighbours (say v_1, v_2) in this cycle, and a second vertex with two nonconsecutive neighbours in the cycle. Delete the edge v_1v_2 from this subgraph; the result is a 7-vertex theta. This proves 14.2. ■

The main result of this section is the following, which will have a number of consequences.

14.3 *Let G be a claw-free trigraph, containing no hole of length > 6 or long prism. If some 5-hole in G is not dominating, then either G is decomposable or $G \in \mathcal{S}_0 \cup \mathcal{S}_2 \cup \mathcal{S}_4 \cup \mathcal{S}_5$.*

Proof. We assume that G is not decomposable. Let C_0 be a 5-hole, and let c_1-$\cdots$-c_5-c_1 be a 5-numbering of it. Let Z be the set of all vertices that are strongly $V(C_0)$-anticomplete, and assume that Z is nonempty. Let $z \in Z$, and let Y be the set of vertices in $V(G) \setminus Z$ that have a neighbour in the component of Z containing z.

(1) *Z is strongly stable, and Y is a strong clique, and Y is the set of neighbours of z. Moreover, every member of Y is a strong hat relative to c_1-$\cdots$-c_5-c_1.*

For let Z_0 be the component of Z containing z, and let $y \in Y$. Then y has a neighbour in Z_0, say z_0, and has a neighbour in $\{c_1, \dots, c_5\}$ from the maximality of Z_0. For any two of its neighbours $x_1, x_2 \in \{c_1, \dots, c_5\}$, $\{y, z_0, x_1, x_2\}$ is not a claw, and so x_1, x_2 are strongly adjacent. Hence y is a strong hat. To see that Y is a clique, let $y_1, y_2 \in Y$, and suppose that they are antiadjacent. Both y_1, y_2 are strong hats relative to c_1-$\cdots$-c_5-c_1, and are not in the same position, since they are antiadjacent and G is claw-free; let P be a path between y_1, y_2 with interior in Z_0. If y_1, y_2 share a neighbour in $\{c_1, \dots, c_5\}$, say c_5, then $G|(\{c_1, \dots, c_4\} \cup V(P))$ is a hole of length > 6, a contradiction. If y_1, y_2 share no neighbour in $\{c_1, \dots, c_5\}$, then $G|(\{c_1, \dots, c_5\} \cup V(P))$ is a long prism, a contradiction. Consequently Y is a strong clique. Since Y is not an internal clique cutset, it follows that $|Z_0| = 1$, and therefore $Z_0 = \{z\}$. In particular, Y is the set of neighbours of z, and z has no neighbours in Z. Since the latter holds for all choices of z, it follows that Z is strongly stable. This proves (1).

For $1 \le i \le 5$, let Y_i be the set of all members of Y that are strong hats in position $i + 2\frac{1}{2}$ relative to c_1-$\cdots$-c_5-c_1. Thus $Y = Y_1 \cup \cdots \cup Y_5$.

(2) *Let $v \in V(G) \setminus (Y \cup \{z\})$. Then for $1 \le i \le 5$, v is either strongly complete or strongly anticomplete to Y_i. Moreover, if v is a hat relative to c_1-$\cdots$-c_5-c_1, then v is complete to Y_i if and only if v is in position $i + 2\frac{1}{2}$.*

For suppose that v has a neighbour y_1 and an antineighbour y_2, both in Y_i. Since $v \notin Y \cup \{z\}$, it follows that v is antiadjacent to z. Now y_1, y_2 are hats in position $i + 2\frac{1}{2}$. By 5.4 applied to c_{i+2}-y_1-z, it follows that v is adjacent to c_{i+2} and similarly to c_{i+3}. By 5.4 applied to y_2-c_{i+2}-c_{i+1}, we deduce that v is adjacent to c_{i+1} and similarly to c_{i-1}. But then $\{v, y_1, c_{i+1}, c_{i-1}\}$ is a claw, a contradiction. This proves the first claim of (2). For the second claim, suppose that v is a hat, in position $j + 2\frac{1}{2}$ say. Since $v \notin Y$, it follows that v, z are antiadjacent. If $j = i$ then v is Y_i-complete by 5.5. If $j \neq i$, choose $a \in \{c_{i+2}, c_{i-2}\}$ antiadjacent to v; then for $y \in Y_i$, $\{y, z, a, v\}$ is not a claw, and so v is antiadjacent to y, and hence to Y_i. This proves (2).

(3) *We may assume that $Y_i \neq \emptyset$ for at least three values of $i \in \{1, \dots, 5\}$. Also, every hat antiadjacent to z is strongly antiadjacent to every other hat except those in the same position relative to c_1-$\cdots$-c_5-c_1.*

For if all the sets Y_i are empty except possibly for say Y_1, then G is decomposable, by (2) and 4.2 applied to $Y_1, \{z\}$, a contradiction. If exactly two of the sets are nonempty, say Y_i, Y_j, then $(Y_i, \{z\}, Y_j)$ is a generalized breaker by (2), and the result follows from 14.1. This proves the first assertion of (3). For the second, let h be a hat antiadjacent to z, and let h' be some other hat in a different position. Suppose that h, h' are adjacent. By (2), h, h' are strongly antiadjacent to z. Choose three hats adjacent to z, all in different positions, say y_1, y_2, y_3. Then, since (1) implies that y_1, y_2, y_3 are pairwise strongly adjacent, it follows that $G|\{c_1, \dots, c_5, y_1, y_2, y_3, h, h'\}$ is a line trigraph of a graph satisfying the hypotheses of 14.2; and so by 14.2, G contains a long prism, a contradiction. This proves (3).

(4) $|Z| = 1$.

For choose $y_1, y_2, y_3 \in Y$, all hats in different positions relative to c_1-$\cdots$-c_5-c_1. Suppose that $z' \in Z$ is different from z; then similarly there are vertices y'_1, y'_2, y'_3, all hats in different positions, and all adjacent to z'. If say y'_1 is adjacent to z, then $\{y'_1, z, z', a\}$ is a claw, where $a \in \{c_1, \dots, c_5\}$ is adjacent to y'_1. Thus y'_1, y'_2, y'_3 are antiadjacent to z, and yet they are adjacent to each other by (1), contrary to (3). This proves (4).

Let $\mathcal{C}$ be the proximity component containing c_1-$\cdots$-c_5-c_1, and for $1 \leq i \leq 5$ let $A_i = A_i(\mathcal{C})$.

(5) *z has no neighbours in $A_1 \cup \cdots \cup A_5$. Moreover, for $1 \leq i \leq 5$ and each $y \in Y_i$, if a_1-$\cdots$-a_5-a_1 belongs to $\mathcal{C}$ then y is a strong hat in position $i + 2\frac{1}{2}$ relative to a_1-$\cdots$-a_5-a_1.*

For let a_1-$\cdots$-a_5-a_1 and a'_1-$\cdots$-a'_5-a'_1 be proximate, with $a'_j \neq a_j$ say. Suppose first that z is strongly antiadjacent to $a_1, \dots, a_5$; then since $\{a'_j, a_{j-1}, a_{j+1}, z\}$ is not a claw, it follows that z is strongly antiadjacent to a'_j. Consequently z has no neighbours in $A_1 \cup \cdots \cup A_5$. Now, with a_1-$\cdots$-a_5-a_1 and a'_1-$\cdots$-a'_5-a'_1 as before, suppose that $y \in Y$ is a strong hat in position $i + 2\frac{1}{2}$ relative to a_1-$\cdots$-a_5-a_1. If $j = i + 2$, then by 9.2, a'_j is strongly adjacent to y and therefore y is a strong hat in position $i + 2\frac{1}{2}$ relative to a'_1-$\cdots$-a'_5-a'_1. If $j = i$, then by 9.2, a'_j is strongly antiadjacent to y, and again y is a strong hat in position $i + 2\frac{1}{2}$ relative to a'_1-$\cdots$-a'_5-a'_1. Thus from the symmetry we may assume that $j = i - 1$. Since $\{y, a'_j, z, a_{i+2}\}$ is not a claw, it follows that y, a'_j are strongly antiadjacent, and again the claim holds. This proves (5).

From (3) we may assume that there exist $y_3 \in Y_3$, and $y_5 \in Y_5$.

(6) *$A_1, \dots, A_5$ are pairwise disjoint; A_4 is strongly anticomplete to A_1, A_2; A_1 is strongly anticomplete to A_3; A_2 is strongly anticomplete to A_5; and $A_1 \cup A_5, A_2 \cup A_3, A_4$ are strong cliques.*

For by (5), y_3 is strongly complete to $A_5 \cup A_1$ and strongly anticomplete to $A_2 \cup A_3 \cup A_4$, and y_5 is strongly complete to $A_2 \cup A_3$ and strongly anticomplete to $A_1 \cup A_4 \cup A_5$. Consequently $A_5 \cup A_1, A_2 \cup A_3, A_4$ are pairwise disjoint. Let H be the bipartite subgraph of G with vertex set $A_1 \cup A_2$ and edges all pairs (x, y) with $x \in A_1$ and $y \in A_2$ such that x, y are adjacent in G. Since $\mathcal{C}$ is a proximity component, it follows that H is connected. Let $a_4 \in A_4$, and assume that a_4 has a neighbour in $A_1 \cup A_2$. Since it also has an antineighbour in $A_1 \cup A_2$ (because a_4 belongs to some member of $\mathcal{C}$), it follows that for some edge of H, a_4 is adjacent to one of its ends and antiadjacent to the other; say $a_1 \in A_1$ and $a_2 \in A_2$ are adjacent, and a_4 is adjacent to a_1 and antiadjacent to a_2. But then $\{a_1, a_2, a_4, y_3\}$ is a claw, a contradiction. This proves that a_4 is strongly $A_1 \cup A_2$-anticomplete, and so A_4 is strongly $A_1 \cup A_2$-anticomplete. Since no vertex of A_3 is strongly A_4-anticomplete, it follows that $A_2 \cap A_3 = \emptyset$, and similarly $A_1 \cap A_5 = \emptyset$. Thus $A_1, \dots, A_5$ are pairwise disjoint. Let $a_1 \in A_1$ and $a_3 \in A_3$, and let $a_4 \in A_4$ be adjacent to a_3. Since $\{a_3, a_1, y_5, a_4\}$ is not a claw, it follows that a_1, a_3 are strongly antiadjacent. So A_1 is strongly anticomplete to A_3, and similarly A_2 is strongly anticomplete to A_5. Next, let $u, v \in A_1 \cup A_5$; since $\{y_3, z, u, v\}$ is not a claw it follows that u, v are strongly adjacent. Consequently $A_1 \cup A_5$ and similarly $A_2 \cup A_3$ are strong cliques. Finally, suppose that $u, v \in A_4$ are antiadjacent. Choose $a_1\text{-}\cdots\text{-}a_5\text{-}a_1 \in \mathcal{C}$ with $a_4 = u$. Since A_4 is strongly anticomplete to $A_1 \cup A_2$, it follows that v is strongly antiadjacent to a_1, a_2, a_4, and therefore also to a_3, a_5, since there is no claw. But then by (5), with v, z exchanged, it follows that v has no neighbour in any member of $\mathcal{C}$, a contradiction. Thus A_4 is a strong clique. This proves (6).

Let $W = A_1 \cup \cdots \cup A_5$.

(7) *For every vertex $v \in V(G) \setminus W$, let N, N^* be the sets of neighbours and strong neighbours of v in W, respectively. Then either*

- *$N = N^* = \emptyset$ and $v = z$, or*
- *for some $i \in \{1, \dots, 5\}$, $N = N^* = A_{i+2} \cup A_{i-2}$ (let H_i be the set of all such v), or*
- *for some $i \in \{1, \dots, 5\}$, $N = N^* = W \setminus A_i$ (let S_i be the set of all such v), or*
- *N^* contains at least four of $a_1, \dots, a_5$ for every $a_1\text{-}\cdots\text{-}a_5\text{-}a_1 \in \mathcal{C}$, and N contains all five vertices for some choice of $a_1\text{-}\cdots\text{-}a_5\text{-}a_1$ (let T be the set of all such v).*

For we may assume that $v \neq z$. From the maximality of $\mathcal{C}$, it follows that for every $a_1\text{-}\cdots\text{-}a_5\text{-}a_1 \in \mathcal{C}$, either N, N^* both contain exactly two of $a_1, \dots, a_5$, or N^* contains at least four of $a_1, \dots, a_5$; and since $\mathcal{C}$ is connected by proximity, the claim follows. This proves (7).

(8) *The sets H_i and S_i are strong cliques, for $1 \leq i \leq 5$, and so is T. For $1 \leq i, j \leq 5$, H_i is strongly complete to S_j if $j = i + 1$ or $i - 1$, and otherwise H_i is strongly anticomplete to S_j. Also, T is strongly anticomplete to H_i for $1 \leq i \leq 5$.*

For H_i and S_i are strong cliques by 5.5, and the adjacency between the sets H_i and the sets S_j is forced by 9.2. Let $t \in T$; if t is adjacent to some $h \in H_i$, then $\{t, h, a_{i+1}, a_{i-1}\}$ is a claw (where a_1-$\cdots$-a_5-$a_1 \in \mathcal{C}$ is chosen so that t is adjacent to all of $a_1, \ldots, a_5$), a contradiction. Thus T is strongly anticomplete to all the sets H_i. Let $t_1, t_2 \in T$. Since they are both adjacent to at least four of $c_1, \ldots, c_5$, they have at least three common neighbours in $\{c_1, \ldots, c_5\}$; and consequently one of these common neighbours, say a, is adjacent to one of y_3, y_5, say to y_3. Since $\{a, y_3, t_1, t_2\}$ is not a claw, it follows that t_1, t_2 are strongly adjacent, and so T is a strong clique. This proves (8).

(9) *For $1 \le i \le 5$, if $H_i \neq \emptyset$, then T is strongly complete to A_{i-1} and to A_{i+1}.*

For let $t \in T$ and $h \in H_i$. By (8), t, h are strongly antiadjacent. Let a_1-$\cdots$-a_5-$a_1 \in \mathcal{C}$. Since t, h are strongly antiadjacent and t has at least four strong neighbours in the hole a_1-$\cdots$-a_5-a_1, 9.2 implies that t, a_{i-1} are strongly adjacent. This proves (9).

(10) *For $1 \le i \le 5$, if T is not strongly complete to A_i, then $i \in \{3, 5\}$, $|A_i| = |T| = 1$ and the vertices in A_i and in T are semiadjacent, Y_{i-2}, Y_{i+2}, Y_i are nonempty, and $H_{i-1}, H_{i+1} = \emptyset$.*

For by (9), T is strongly complete to $A_1 \cup A_2 \cup A_4$, and so $i \in \{3, 5\}$. By (9), $H_4 = \emptyset$. From the symmetry we may assume that $i = 3$. By (9), $H_2 = \emptyset$ (and so $H_{i-1}, H_{i+1} = \emptyset$ as claimed). By (3), there exists $y_1 \in Y_1$, and so T is strongly complete to A_5 by (9). By (6) with y_5, y_1 exchanged, A_3 is strongly complete to A_4 and strongly anticomplete to A_5. Let $v \in V(G) \setminus (T \cup A_3)$; we claim that v is either strongly T-complete or strongly T-anticomplete. If $v \in W$ then v is strongly T-complete, and if $v \in H_i$ for some i then v is strongly T-anticomplete by (8). So we may assume that $v \in S_1 \cup \cdots \cup S_5$. If $v \in S_1$ then v is strongly T-complete, since for $t \in T$, $\{c_5, v, t, y_3\}$ is not a claw. Similarly v is strongly T-complete if $v \in S_5$. If $v \in S_2$ then v is strongly T-complete, since for $t \in T$, $\{a_3, v, t, y_5\}$ is not a claw, where $a_3 \in A_3$ is adjacent to t; and similarly v is strongly T-complete if $v \in S_4$. If $v \in S_3$ then v is strongly T-complete, since for $t \in T$, $\{c_4, v, t, y_1\}$ is not a claw. This proves the claim. But every such v is also strongly complete or strongly anticomplete to A_3, and so (A_3, T) is a homogeneous pair, nondominating since $Z \neq \emptyset$; and therefore 4.3 implies that $|A_3| = |T| = 1$, and therefore the members of A_3, T are semiadjacent. This proves (10).

(11) *The following hold:*

- *For $1 \le i \le 5$, S_i is strongly complete to S_{i+2}*
- *For $1 \le i \le 5$, if $H_i \neq \emptyset$ then S_i is strongly anticomplete to S_{i+1}, S_{i-1}*
- *T is strongly complete to $S_1, \ldots, S_5$.*

For suppose first that $s \in S_i$ and $s' \in S_{i+2}$ are antiadjacent. If there exists $h \in H_i$, then $\{c_{i-2}, s, h, s'\}$ is a claw, a contradiction. Thus $H_i = \emptyset$, and similarly $H_{i+2} = \emptyset$. By (3), there exists $y_{i+1} \in Y_{i+1}$; but then $\{y_{i+1}, s, s', z\}$ is a claw, again a contradiction. This proves the first claim. For the second, if $h \in H_i$ and $s \in S_i$ and $s' \in S_{i+1}$ are adjacent, then $\{s', h, s, c_i\}$ is a claw, a contradiction. This proves the second claim. For the third, suppose that $t \in T$ and $s_j \in S_j$ are antiadjacent, for some j with $1 \le j \le 5$. Now one of H_j, H_{j+2}, H_{j-2} is nonempty, and both t, s_j are anticomplete to these three sets; so there is a hat h antiadjacent to both t, s_j. But one of $c_1, \ldots, c_5$ is adjacent to all of t, s_j, h, and hence these four vertices form a claw, a contradiction. This proves (11).

(12) *We may assume that $A_1, \ldots, A_5$ all have cardinality 1; and for $1 \le i \le 5$, A_i is strongly complete to A_{i+1} and strongly anticomplete to A_{i+2}.*

For by (10), T is strongly complete to $A_1 \cup A_2$, and so (A_1, A_2) is a homogeneous pair, nondominating since $Z \neq \emptyset$; and hence $|A_1| = |A_2| = 1$. Suppose that there exists $y_1 \in Y_1$. Then from the symmetry between A_2 and A_4 (fixing A_3), it follows that $|A_4| = |A_5| = 1$ and A_3 is strongly anticomplete to A_5 and so the third claim holds. If $|A_3| > 1$ then by (10) all members of A_3 are twins, a contradiction, and so $|A_3| = 1$, and the first claim holds. Moreover, T is complete to W; and since $T \cup W$ includes no claw, the second claim holds. Hence (12) holds if $Y_1 \neq \emptyset$.

Thus we may assume that Y_1 is empty, and similarly $Y_2 = \emptyset$. By (10), T is strongly complete to W; and by (3) there exists $y_4 \in Y_4$. Since $\{y_4, z\} \cup A_1 \cup A_2$ includes no claw, A_1 is strongly complete to A_2. Suppose that A_4 is not strongly complete to A_5, and choose $a_4 \in A_4$ and $a_5 \in A_5$, antiadjacent. If there exists $t \in S_1 \cup S_3 \cup T$ then $\{t, a_4, a_5, c_2\}$ is a claw, and so $T, S_1, S_3 = \emptyset$. Suppose that there exists $h \in H_2$, necessarily antiadjacent to z; then by (2) it is strongly antiadjacent to y_3, y_4. Let $a_3 \in A_3$ be adjacent to a_4. Since $\{a_3, a_4, a_5, y_5\}$ is not a claw, a_3 is antiadjacent to a_5; but then c_2-a_3-a_4-h-a_5-y_3-y_4-c_2 is a 7-hole, a contradiction. Thus H_2 is empty. Suppose that also A_4 is not strongly complete to A_3; then similarly $S_2, S_5, H_1 = \emptyset$. But then (A_3, A_4, A_5) is a breaker, contrary to 4.4. Thus A_4 is strongly complete to A_3. Let A'_5 be the set of vertices in A_5 with an antineighbour in A_4, and let A''_5 be the set of vertices in A_5 with a neighbour in A_3. If there exists $a'_5 \in A'_5 \cap A''_5$, then $\{a'_3, a'_4, a'_5, y_5\}$ is a claw, where $a'_3 \in A_3$ is a neighbour of a'_5 and $a'_4 \in A_4$ is an antineighbour of a'_5. Also, both $(A_5 \setminus A''_5, A_4)$ and $(A_5 \setminus A'_5, A_3)$ are nondominating homogeneous pairs, and hence by 4.3, $|A_3| = |A_4| = 1$ and $|A_5 \setminus A''_5|, |A_5 \setminus A'_5| \le 1$. Suppose that $|A_5| > 1$; then $A_5 = \{a'_5, a''_5\}$, where $A'_5 = \{a'_5\}$ and $A''_5 = \{a''_5\}$. By (11), S_2 is strongly complete to S_4; and $S_5 = \emptyset$ since $\{c_3, y_5, a''_5\} \cup S_5$ includes no claw. But then $(A_3, A_4 \cup H_1, A_5 \cup S_2)$ is a breaker, contrary to 4.4. Thus $|A_5| = 1$; but then $G \in \mathcal{S}_0$.

This proves that the claim holds if A_4 is not strongly complete to A_5, so we may assume that A_4 is strongly complete to A_5 and similarly to A_3. Hence (A_3, A_5) is a nondominating homogeneous pair, and so A_3, A_5 both have cardinality 1; and all members of A_4 are twins, so $|A_4| = 1$. Let $A_i = \{a_i\}$ for $1 \le i \le 5$. Since T is a homogeneous set it follows that $|T| \le 1$. Suppose that a_3, a_5 are semiadjacent. Since $\{a_3, a_5, y_5\} \cup S_5$ includes no claw, it follows that $S_5 = \emptyset$, and similarly $S_3 = \emptyset$. Since $\{a_3, a_5, y_5\} \cup H_1$ includes no claw, we deduce that $H_1 = \emptyset$, and similarly $H_2 = \emptyset$. Since $(\{a_3\} \cup S_1, \{a_5\} \cup S_2)$ is a nondominating homogeneous pair, it follows that $S_1 = S_2 = \emptyset$. If $T = \emptyset$ then $(\{a_3\}, \{a_4\}, \{a_5\})$ is a breaker, a contradiction; so $T \neq \emptyset$ and so $|T| = 1$, $T = \{t\}$ say. Since $\{a_1, t, y_4\} \cup H_3$ includes no claw, $H_3 = Y_3$, and similarly $H_5 = Y_5$; and since $\{a_1, t, y_3\} \cup H_4$, includes no claw, $H_4 = Y_4$. Since Y_3, Y_4, Y_5 are all homogeneous sets, they each have cardinality one. But then $G \in \mathcal{S}_4$. Thus we may assume that a_3 is strongly antiadjacent to a_5. This proves (12).

(13) *Let $1 \le i \le 5$. Then S_i is strongly anticomplete to S_{i+1}.*

For suppose that S_i, S_{i+1} are not strongly anticomplete. By (11), H_i, H_{i+1} are both empty, and since H_3, H_5 are nonempty, it follows that $i = 1$, and Y_4 is nonempty. By (10), T is strongly complete to W. Choose $s_1 \in S_1$ and $s_2 \in S_2$, adjacent. If there exists $s_3 \in S_3$, then by (11) s_3 is adjacent to s_1 and antiadjacent to s_2 (since $H_3 \neq \emptyset$), and so $\{s_1, s_3, s_2, y_5\}$ is a claw, a contradiction. Thus S_3 is empty, and similarly S_5 is empty. But then $(S_2 \cup A_5, S_1 \cup A_3)$ is a nondominating homogeneous pair, and $|S_2 \cup A_5| \geq 2$, contrary to 4.3. This proves (13).

Now (11) and (13) imply that $|S_i| \leq 1$ for each i; by (12), $|A_i| = 1$ for each i; and by (8) and (3), $|H_i \setminus Y_i|, |Y_i| \leq 1$ for each i. From (8), (10) and (11) it follows that $|T| \leq 1$. If $T = \emptyset$ then $G \in \mathcal{S}_0$, so we may assume that $T = \{t\}$ say. If T is strongly complete to W and $H_j = Y_j$ for $1 \leq j \leq 5$, then $G \in \mathcal{S}_4$. Thus we may assume that for some $j \in \{1, \dots, 5\}$, either there exists $h \in H_j \setminus Y_j$ or t is semiadjacent to a_j. Suppose that there exists $h' \in H_{j-1}$. Then by (10), t is strongly adjacent to a_j, so $h \in H_j \setminus Y_j$ and $\{c_{j+2}, h, h', t\}$ is a claw by (3), a contradiction. Thus H_{j-1} and similarly H_{j+1} are empty. Since Y_3, Y_5 are nonempty, it follows that $j \in \{3, 5\}$ and from the symmetry we may assume that $j = 3$. Thus H_2, H_4 are empty, and therefore there exists $y_1 \in Y_1$. Moreover j is unique, and so $H_i = Y_i$ for $i = 1, 5$. Suppose that there exists $s \in S_2$. If $h \in H_3 \setminus Y_3$ then $\{s, h, t, y_1\}$ is a claw, while if t is semiadjacent to a_3 then $\{s, t, a_3, y_3\}$ is a claw, in either case a contradiction; so $S_2 = \emptyset$, and similarly $S_4 = \emptyset$. If $S_3 \neq \emptyset$, then $(S_3 \cup T, A_3)$ is a nondominating homogeneous pair, contrary to 4.3; and so $S_3 = \emptyset$. But then $G \in \mathcal{S}_2$. (To see this, let $v_1, \dots, v_{13}$ in the definition of $\mathcal{S}_2$ be

$$c_5, c_1, c_2, y_5, y_1, c_4, h, z, t, c_3, s_1, s_5, y_3$$

respectively, where $v_{11} = s_1$ is the unique member of S_1 if $S_1 \neq \emptyset$ and v_{11} is undefined otherwise, and similarly either $v_{12} \in S_5$ or is undefined, and either $v_7 \in H_3 \setminus Y_3$ or is undefined.) This proves 14.3. ■

15 6-holes with hubs and hats

In this section we handle 6-holes that have both a hub and a hat.

15.1 *Let G be a claw-free trigraph, containing no long prism and no hole of length > 6, and such that every hole of length 5 or 6 is dominating. If there is a 6-hole in G relative to which some vertex is a hub and some vertex is a hat, then either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$, or G is decomposable.*

Proof. For a contradiction, we assume that G is not decomposable. Let C_0 be the 6-hole, and let its vertices be $a_2^1, a_3^1, a_3^2, a_1^2, a_1^3, a_2^3$ in order. Define $A_j^i = \{a_j^i\}$ for $1 \leq i, j \leq 3$ with $i \neq j$. For $1 \leq i \leq 3$ let A_i^i be the set of all hubs that are antiadjacent to a_k^j, a_j^k, where $\{i, j, k\} = \{1, 2, 3\}$. By hypothesis, at least one of the sets A_i^i is nonempty. By 13.2, $|A_i^i| \leq 1$ for $1 \leq i \leq 3$, since G is not decomposable; if A_i^i is nonempty, let a_i^i be its unique member. Let W be the union of the nine sets A_j^i.

For $1 \leq i \leq 3$, define $A^i = A_1^i \cup A_2^i \cup A_3^i$, and for $1 \leq j \leq 3$ define $A_j = A_j^1 \cup A_j^2 \cup A_j^3$. For $1 \leq i \leq 3$, let H^i, H_i, S^i, S_i be four subsets of $V(G) \setminus W$, defined as follows. For $v \in V(G) \setminus W$, let N, N^* denote the set of neighbours and strong neighbours, respectively, of v in W; then

- $v \in H^i$ if $N = N^* = A^i$
- $v \in H_i$ if $N = N^* = A_i$
- $v \in S^i$ if $N = N^* = W \setminus A^i$

- $v \in S_i$ if $N = N^* = W \setminus A_i$.

(1) *The twelve sets H^i, H_i, S^i, S_i ($1 \le i \le 3$) are pairwise disjoint strong cliques, and they have union $V(G) \setminus W$, and at least one of $H^1, H^2, H^3, H_1, H_2, H_3$ is nonempty.*

For clearly they are pairwise disjoint, and they are all strong cliques by 5.5. Let $v \in V(G) \setminus W$, and let N, N^* be as before. If v is a hub relative to C_0, then v belongs to one of the sets A_i^i, and therefore belongs to W, a contradiction. Since C_0 is dominating, it follows from 9.1 that $2 \le |N^*| \le |N| \le 4$ and the members of N are consecutive in C_0. If $|N| = 3$ or $|N^*| = 3$ then v is a clone relative to C_0, which we may assume is false by 13.6 since G is not decomposable. Thus either $|N| = 4$ or $|N^*| = 2$; and since $|N| - |N^*| \le 1$, it follows that $|N| = |N^*|$. Hence v belongs to one of the twelve sets. Thus the twelve sets have union $V(G) \setminus W$. The final assertion follows since by hypothesis there is a hat relative to C_0. This proves (1).

(2) *The sets $A_1, A_2, A_3, A^1, A^2, A^3$ are strong cliques. Moreover, if $A_1^1 \ne \emptyset$ and $x, y \in W$ are adjacent, then either $\{x, y\}$ is a subset of one of these cliques, or $x, y \in A_2^1 \cup A_1^3$, or $x, y \in A_3^1 \cup A_1^2$. The analogous statements hold for A_2^2, A_3^3.*

The first claim follows from 13.6 and 9.1. For the second, let $x \in A_j^i$ and $y \in A_{j'}^{i'}$ say. We may assume that none of the six cliques includes $\{x, y\}$, and so $i \ne i'$ and $j \ne j'$. If $i = j$, then x is a hub and 9.1 implies that $y \in A^i \cup A_i$, a contradiction. Thus $i \ne j$ and similarly $i' \ne j'$, and so $x, y \in V(C_0)$. If x, y are opposite vertices of C_0 (that is, if $i = j'$ and $j = i'$), then there is a claw with members x, y and the two vertices of C_0 consecutive with x, a contradiction. In particular, at least one of x, y is adjacent to a_1^1; so from the symmetry we may assume that $i = 1, j = 2$, and so (i', j') is one of $(2, 3), (3, 1)$. In the first case $\{x, y, a_1^1, a_2^3\}$ is a claw, a contradiction, and in the second case the claim holds. This proves (2).

(3) *The six sets $H^1, H^2, H^3, H_1, H_2, H_3$ are pairwise strongly anticomplete. Moreover, for $1 \le i, j \le 3$, H^i is strongly anticomplete to S_j; and H^i is strongly complete to S^j if $j \ne i$, and strongly anticomplete to S^i. Analogous statements hold for H_i.*

For the members of distinct sets $H^1, H^2, H^3, H_1, H_2, H_3$ are hats in different positions relative to C_0; if some two are adjacent, then either G contains a hole of length > 6 or a long prism, in either case a contradiction. This proves the first assertion. The second follows from 9.2. This proves (3).

(4) *For $1 \le i \le 3$ one of H^i, S_i is empty, and one of H_i, S^i is empty.*

For suppose that $h^1 \in H^1$ and $s_1 \in S_1$ say. Then s_1-a_3^2-a_1^2-a_1^3-a_2^3-s_1 is a 5-hole that does not dominate h^1, a contradiction.

(5) *For $1 \le i \le 3$, S^i is strongly anticomplete to S_i.*

For suppose that $s^1 \in S^1$ and $s_1 \in S_1$ are adjacent, say. By (4), $H^1, H_1 = \emptyset$, and so from the symmetry we may assume that there exists $h^2 \in H^2$. Then $\{s^1, s_1, h^2, a_1^3\}$ is a claw, a contradiction. This proves (5).

(6) *For $1 \leq i \leq 3$, if $S^i \neq \emptyset$ and $H_1 \cup H_2 \cup H_3 \neq \emptyset$ then $A_i^i = \emptyset$.*

For suppose that, say, $s^1 \in S^1$ and $h \in H_1 \cup H_2 \cup H_3$, and $A_1^1 = \{a_1^1\}$. By (4), $h \notin H_1$, and so we may assume that $h \in H_2$. But then s^1-a_1^3-a_1^1-a_3^1-a_3^2-s^1 is a 5-hole not dominating h, a contradiction. This proves (6).

(7) *If $H_1 \cup H_2 \cup H_3 \neq \emptyset$ then S^1, S^2, S^3 are pairwise strongly complete.*

For suppose that $s^1 \in S^1$ is antiadjacent to $s^2 \in S^2$ say, and let $h \in H_1 \cup H_2 \cup H_3$. By (4), $h \in H_3$. By (6), $A_1^1 = A_2^2 = \emptyset$, and so $A_3^3 = \{a_3^3\}$. But then $\{a_3^3, s^1, s^2, h\}$ is a claw, a contradiction. This proves (7).

(8) *We may assume that $S^1 \cup S^2 \cup S^3$ is not strongly anticomplete to $S_1 \cup S_2 \cup S_3$.*

For suppose it is. If also S^1, S^2, S^3 are pairwise strongly complete and S_1, S_2, S_3 are pairwise strongly complete then G is a line trigraph by (1),(3), so we may assume that, say, S^1, S^2 are not strongly complete. By (7), $H_1, H_2, H_3 = \emptyset$. Suppose that there exists $s_j \in S_j$ for some j with $1 \leq j \leq 3$. Choose $s^1 \in S^1$ and $s^2 \in S^2$, antiadjacent. One of a_1^3, a_2^3 is adjacent to s_j, say x; and then $\{x, s_j, s^1, s^2\}$ is a claw, a contradiction. Thus $S_1, S_2, S_3 = \emptyset$. Now each of the three strong cliques S^1, S^2, S^3 is strongly complete to two of the three strong cliques $A^1 \cup H^1, A^2 \cup H^2, A^3 \cup H^3$ and strongly anticomplete to the third, and so G is the hex-join of $G|(W \cup H^1 \cup H^2 \cup H^3)$ and $G|(S^1 \cup S^2 \cup S^3)$, a contradiction. This proves (8).

(9) *For $1 \leq i \leq 3$, at least one of H^i, H_i is empty.*

For suppose that $h^1 \in H^1$ and $h_1 \in H_1$ say. By (4), $S_1 = S^1 = \emptyset$. By (7), S^2 is strongly complete to S^3, and S_2 is strongly complete to S_3. By (5), S^i is strongly anticomplete to S_i for $i = 2, 3$. By (8) we may assume from the symmetry that there exist $s^3 \in S^3$ and $s_2 \in S_2$, adjacent. From (6), $A_2^2 = A_3^3 = \emptyset$, and so $A_1^1 = \{a_1^1\}$. By (4), $H_3 = H^2 = \emptyset$. Then $(A_3^1 \cup S^3, A_1^2 \cup S_2)$ is a homogeneous pair by (2), nondominating since A_2^3 is nonempty, a contradiction. This proves (9).

(10) *At least one of $H^1 \cup H^2 \cup H^3$, $H_1 \cup H_2 \cup H_3$ is empty.*

For suppose they are both nonempty; then by (9), we may assume from the symmetry that there exist $h_1 \in H_1$ and $h^2 \in H^2$. By (4), $S^1, S_2 = \emptyset$, and by (9), $H^1, H_2 = \emptyset$. By (7), S_1 is strongly complete to S_3 and S^2 is strongly complete to S^3. By (5), S^3 is strongly anticomplete to S_3. Suppose first that $S^2 = \emptyset$. From (8), there exist $s^3 \in S^3$ and $s_1 \in S_1$, adjacent. From (6), $A_1^1, A_3^3 = \emptyset$. Then by (2) and (3), $(A_2^1 \cup S_1, A_3^2 \cup S^3)$ is a homogeneous pair, nondominating since $A_1^3 \neq \emptyset$, a contradiction. Hence $S^2 \neq \emptyset$, and similarly $S_1 \neq \emptyset$. From (6), $A_1^1 = A_2^2 = \emptyset$, and therefore $A_3^3 = \{a_3^3\}$. By (6) again, $S^3 = S_3 = \emptyset$. But now $(A_1^3, H_1 \cup H^2 \cup A_1^2, A_3^2)$ is a breaker, contrary to 4.4. This proves (10).

(11) *Exactly one of $H^1, H^2, H^3, H_1, H_2, H_3$ is nonempty.*

For by hypothesis, at least one is nonempty, say H_1. By (10), $H^1, H^2, H^3 = \emptyset$. Suppose that $H_2 \neq \emptyset$. By (4), $S^1, S^2 = \emptyset$, and by (8), S^3 is nonempty. From (4), $H_3 = \emptyset$, and from (6), $A_3^3 = \emptyset$. Since $\{a_2^1, a_1^3, a_3^1, h_2\}$ is not a claw, a_2^1 is strongly antiadjacent to a_1^3, and similarly a_2^3 is strongly antiadjacent to a_1^2. Since one of $A_1^1, A_2^2 \neq \emptyset$, (2) implies that $(H_1 \cup A_1^3, H_2 \cup A_2^3)$ is a homogeneous pair, nondominating since $A_3^1 \neq \emptyset$, a contradiction. This proves (11).

In view of (11) we assume henceforth that H_3 is nonempty, and therefore H^1, H^2, H^3, H_1, H_2 are empty. Choose $h_3 \in H_3$. By (4), $S^3 = \emptyset$.

(12) *Either S^1 is nonempty or a_1^3 is semiadjacent to a_3^2; and either S^2 is nonempty, or a_2^3 is semiadjacent to a_3^1. Consequently $A_1^1 = A_2^2 = \emptyset$, and $A_3^3 = \{a_3^3\}$.*

For suppose that $S^2 = \emptyset$, say. From (8), $S^1 \neq \emptyset$. From (6), $A_1^1 = \emptyset$. Since $(H_3 \cup A_3^1, A_2^1)$ is not a homogeneous pair, (nondominating since $A_1^2 \neq \emptyset$), it follows that some vertex of C_0 is semiadjacent to one of a_3^1, a_2^1. By (2), one of the pairs $a_1^2 a_3^1, a_2^3 a_3^1, a_3^2 a_2^1, a_1^3 a_2^1$ is semiadjacent. The first is impossible since $\{a_3^1, a_1^2, a_2^1, h_3\}$ is not a claw; the second is the desired result; the third is impossible since $\{a_3^2, a_2^1, a_1^2, h_3\}$ is not a claw. Suppose that the fourth holds, that is, a_1^3 is semiadjacent to a_2^1. By (2) $A_2^2 = A_3^3 = \emptyset$, and so $A_1^1 \neq \emptyset$, a contradiction. This proves that either S^2 is nonempty, or a_2^3 is semiadjacent to a_3^1. Similarly either S^1 is nonempty or a_1^3 is semiadjacent to a_3^2. If $S^1 \neq \emptyset$ then (6) implies that $A_1^1 = \emptyset$, and if a_1^3 is semiadjacent to a_3^2 then (2) implies that $A_1^1 = \emptyset$; so in either case $A_1^1 = \emptyset$ and similarly $A_2^2 = \emptyset$, and so $A_3^3 = \{a_3^3\}$. This proves (12).

(13) *S_3 is strongly complete to $S^1 \cup S^2$.*

For suppose not; then from the symmetry we may assume that there exist $s_3 \in S_3$ and $s^2 \in S^2$, antiadjacent. If $S^1 \neq \emptyset$, choose $s^1 \in S^1$, and otherwise let $s^1 = a_1^3$; then in either case, s^1 is adjacent to a_3^2 by (12). By (7) if $s^1 \in S^1$, and by definition otherwise, s^1, s^2 are adjacent. If s_3, s^1 are antiadjacent, then s_3-a_2^1-s^2-s^1-a_1^2-s_3 is a 5-hole, not dominating H_3, a contradiction. If s_3, s^1 are adjacent, then $\{s^1, s_3, s^2, a_3^2\}$ is a claw, a contradiction. This proves (13).

Let S_1' be the set of vertices in S_1 with an antineighbour in S^2, and let S_2' be the set of vertices in S_2 with an antineighbour in S^1.

(14) *$S_1' \cup S_2'$ is strongly anticomplete to S_3, S_1' is strongly complete to S_2, and S_2' is strongly complete to S_1.*

For suppose that some vertex $s_1 \in S_1'$ say has a neighbour $s_3 \in S_3$. Let $s^2 \in S^2$ be an antineighbour of s_1. Then by (13), $\{s_3, s_1, s^2, a_1^2\}$ is a claw, a contradiction. Thus S_3 is strongly anticomplete to S_1', and similarly to S_2'. Now suppose that some $s_1 \in S_1'$ has an antineighbour $s_2 \in S_2$. Let $s^2 \in S^2$ be an antineighbour of s_1; then by (5), $\{a_3^3, s_2, s_1, s^2\}$ is a claw, a contradiction. Hence S_1' is strongly complete to S_2. Similarly S_2' is strongly complete to S_1. This proves (14).

But now the following six sets are strong cliques: $S_1 \setminus S_1'$; $S_2 \setminus S_2'$; S_3; $S^1 \cup A_1$; $S^2 \cup A_2$; $H_3 \cup A_3 \cup S_1' \cup S_2'$. Every vertex belongs to exactly one of these cliques; and each of the first three cliques is strongly complete to two of the last three, and strongly anticomplete to the other, in the manner required for a hex-join. Consequently G is expressible as a hex-join, a contradiction. This proves 15.1. ■

There is an (easy) analogue of 15.1 for 6-holes with a star-diagonal and a hat, the following.

15.2 *Let G be a claw-free trigraph, containing no long prism and no hole of length > 6, and such that every hole of length 5 or 6 is dominating. If there is a 6-hole in G with a star-diagonal, relative to which some vertex is either a hat or a clone, then either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$, or G is decomposable.*

Proof. Let C_0 be the 6-hole, with vertices $c_1, \ldots, c_6$-c_1. Let b_1, b_2 be adjacent stars, in positions $1\frac{1}{2}, -1\frac{1}{2}$ respectively. Let h be either a hat or clone relative to C_0. If it is a clone, the result follows from 13.7. We assume then that h is a strong hat. From the symmetry we may assume that it is in position $\frac{1}{2}$ or $1\frac{1}{2}$. If it is in position $\frac{1}{2}$, then by 9.2 h is adjacent to b_1 and antiadjacent to b_2, and then $\{b_1, h, b_2, c_3\}$ is a claw, a contradiction. If it is in position $1\frac{1}{2}$, then it is strongly antiadjacent to b_1 by 9.2, and then b_1-c_4-c_5-c_6-c_1-b_1 is a nondominating 5-hole, a contradiction. This proves 15.2. ■

16 Star-triangles.

We recall that, if c_1-$\cdots$-c_6-c_1 is a 6-hole, and there are three pairwise adjacent stars in positions $1\frac{1}{2}, 3\frac{1}{2}, 5\frac{1}{2}$ respectively, we call the set of these three stars a *star-triangle* for the 6-hole. Our next goal is to prove an analogue of 13.7 for star-triangles. We need the following lemma.

16.1 *Let G be claw-free, and let B_1, B_2, B_3 be strong cliques in G. Let $B = B_1 \cup B_2 \cup B_3$. Suppose that:*

- *$B \neq V(G)$,*
- *there are two triads $T_1, T_2 \subseteq B$ with $|T_1 \cap T_2| = 2$, and*
- *there is no triad T in G with $|T \cap B| = 2$.*

Then either

- *there exists $V \subseteq B$ with $T_1, T_2 \subseteq V$ such that V is a union of triads, and G is a hex-join of $G|V$ and $G|(V(G) \setminus V)$, where $(V \cap B_1, V \cap B_2, V \cap B_3)$ is the corresponding partition of V into strong cliques, or*
- *there is a homogeneous set with at least two members, included in one of B_1, B_2, B_3, such that all its members are in triads, or*
- *there is a nondominating homogeneous pair (V_1, V_2) with $\max(|V_1|, |V_2|) \geq 2$, such that V_1 is a subset of one of B_1, B_2, B_3 and V_2 is a subset of another.*

In particular, G is decomposable.

Proof. Since $|T_1 \cap T_2| = 2$, it follows that there are distinct $u_1, \ldots, u_t \in B$ with $t \geq 4$, such that $T_1 \cup T_2 = \{u_1, u_2, u_3, u_4\}$, and for $3 \leq s \leq t$, $\{u_1, \ldots, u_s\}$ is expressible as a union of triads. Choose such a sequence with t maximum, and let $U = \{u_1, \ldots, u_t\}$. Since every triad included in B contains only one vertex of B_1, B_2, B_3, each vertex of such a triad belongs to only one of B_1, B_2, B_3; and hence $U \cap B_1, U \cap B_2, U \cap B_3$ are disjoint.

(1) *Every vertex not in U is strongly complete to two of $U \cap B_1, U \cap B_2, U \cap B_3$ and strongly anticomplete to the third.*

For let $v \in V(G) \setminus U$. We claim that there is no triad T with $T \setminus U = \{v\}$. For if $v \in B$, this holds from the maximality of t (for otherwise we could set $u_{t+1} = v$), and if $v \notin B$ it follows from a hypothesis of the theorem. On the other hand, v is not complete to any triad, since G is claw-free; and so for every triad $T \subseteq U$, v is strongly adjacent to two members of T and strongly antiadjacent to the third. In particular, since $\{u_1, u_2, u_3\}$ is a triad, we may assume that v is strongly antiadjacent to u_1 and strongly adjacent to u_2, u_3, and $u_i \in B_i$ for $i = 1, 2, 3$. We claim that for $1 \le s \le t$, v is strongly adjacent to u_s if $s \in B_2 \cup B_3$, and strongly antiadjacent to u_s if $u_s \in B_1$; and we prove this by induction on s. The claim holds when $s \le 3$, so let $4 \le s \le t$; we shall prove that the claim holds for s assuming that it holds for $s-1$. There is a triad T with $u_s \in T \subseteq \{u_1, \ldots, u_s\}$; let $T = \{t_1, t_2, t_3\}$ say, where $t_i \in B_i$ for $i = 1, 2, 3$. As we saw, v is strongly adjacent to exactly two of t_1, t_2, t_3 and strongly antiadjacent to the third. If $u = t_1$, then $t_2, t_3 \in \{u_1, \ldots, u_{s-1}\}$, and from the inductive hypothesis v is strongly adjacent to them both, and therefore strongly antiadjacent to $t_1 = u$. If $u = t_2$, then $t_1, t_3 \in \{u_1, \ldots, u_{s-1}\}$, and from the inductive hypothesis v is strongly adjacent to t_3 and strongly antiadjacent to t_1; and therefore strongly adjacent to $t_2 = u$. Similarly if $u = t_3$ then v is strongly adjacent to u. This completes the inductive proof, and therefore proves (1).

Let $X_i = U \cap B_i$ for $i = 1, 2, 3$, and let Y_i be the set of vertices in $V(G) \setminus U$ that are strongly complete to $U \setminus X_i$ and strongly anticomplete to X_i. By hypothesis, $U \ne V(G)$ since $B \ne V(G)$. As in the proof of 4.5, if Y_1, Y_2, Y_3 are strong cliques then the result holds, so we assume that Y_3 is not a strong clique say. Hence X_3 is strongly anticomplete to $X_1 \cup X_2$; and so X_3 is a homogeneous set, and (X_1, X_2) is a homogeneous pair, and since one of X_1, X_2, X_3 has at least two members (because $t \ge 4$), again the result holds. This proves 16.1. ■

16.2 *Let G be a claw-free trigraph, and let $A = \{a_1, a_2, a_3\}$ be a dominating triangle. Suppose that there are distinct vertices $u_1, u_2, u_3, u_4 \in V(G) \setminus A$ such that:*

- *$u_1, \ldots, u_4$ each have at least two neighbours in A, and at least one antineighbour in A, and*
- *at most one pair of $u_1, \ldots, u_4$ are strongly adjacent.*

Then G is decomposable.

Proof. For $i = 1, 2, 3$, let B_i be the set of all vertices in $V(G) \setminus A$ that are antiadjacent to a_i and adjacent to the other two members of A. From 5.5 it follows that B_1, B_2, B_3 are strong cliques. Let $B = B_1 \cup B_2 \cup B_3$. Thus $u_1, \ldots, u_4 \in B$, and from the hypothesis, there are two triads included in B that have two vertices in common, and so the first two hypotheses of 16.1 hold. For the third, let $v \in V(G) \setminus B$, and suppose that there is a triad $\{v, b_1, b_2\}$, where $b_1 \in B_1$ and $b_2 \in B_2$. By 5.4 (with b_1-a_3-b_2) it follows that v is strongly antiadjacent to a_3. Since $v \notin B_3$, it is strongly antiadjacent to at least one of a_1, a_2, and from the symmetry we may assume that v is strongly antiadjacent to a_2.

From 5.4 (with a_2-a_1-b_2) it follows that v is strongly antiadjacent to a_1, contrary to the hypothesis that A is dominating. Thus all the hypotheses of 16.1 hold, and the result follows. This proves 16.2. ■

16.3 *Let G be a claw-free trigraph, such that every 5- and 6-hole in G is dominating, and no 6-hole in G has a hub. Let C_0 be a 6-hole in G, with a star-triangle. If some vertex of $V(G) \setminus V(C_0)$ is a hat or a clone with respect to C_0, then G is decomposable.*

Proof. Let C_0 have vertices c_1-$\cdots$-c_6-c_1, and let $A = \{a_1, a_3, a_5\}$ be a star-triangle, where a_1, a_3, a_5 are in positions $1\frac{1}{2}, 3\frac{1}{2}, 5\frac{1}{2}$ respectively.

(1) *There is no hat in position $1\frac{1}{2}, 3\frac{1}{2}$, or $5\frac{1}{2}$ relative to c_1-$\cdots$-c_6-c_1.*

For suppose that h is a hat in position $1\frac{1}{2}$ say. Then h is strongly antiadjacent to a_1, by 9.2; h is strongly antiadjacent to c_3, since $\{c_3, h, a_1, c_4\}$ is not a claw; and similarly h is strongly antiadjacent to c_6. Consequently the 5-hole a_1-c_3-c_4-c_5-c_6-a_1 is not dominating, a contradiction. This proves (1).

(2) *A is dominating.*

For suppose that $v \in V(G) \setminus A$, with no neighbour in A. Then $v \notin V(C_0)$, and so, since there is no hub for C_0, it follows that v is a hat, clone or star relative to C_0. By (1) and 9.2, v is not a hat; and by 9.2 it is not a clone, and not a star in position $1\frac{1}{2}, 3\frac{1}{2}$ or $5\frac{1}{2}$. Thus we may assume v is a star in position $2\frac{1}{2}$ say; but then v-c_3-a_2-c_5-c_6-c_1-v is a 6-hole, and a_1 is a hub for it, a contradiction. This proves (2).

By hypothesis, some vertex $v \in V(G) \setminus V(C_0)$ is either a hat or a clone with respect to C_0, say either a hat in position $\frac{1}{2}$ or a clone in position 1 without loss of generality. By 9.2, v is adjacent to a_1 and antiadjacent to a_3. Since $\{a_1, v, a_5, c_3\}$ is not a claw, v is adjacent to a_5. But then c_1, c_3, c_5, v each have at least two neighbours and at least one antineighbour in A, and only one pair of them is strongly adjacent (namely vc_1) and so the result follows from (1) and 16.2. This proves 16.3. ■

16.4 *Let G be a claw-free trigraph, such that every 5-hole in G is dominating, and there is no 6-hole with a hub or with a star-diagonal. Suppose that some 6-hole has a crown. Then G is decomposable.*

Proof. Let C be a 6-hole with vertices c_1-$\cdots$-c_6-c_1 in order, and let s_1, s_2 be antiadjacent stars in positions $2\frac{1}{2}, 3\frac{1}{2}$ respectively. By 9.2, s_1 is strongly adjacent to c_2, c_3, and s_2 is strongly adjacent to c_3, c_4; and by four applications of 5.3, c_i is strongly adjacent to c_{i+1} for $i = 1, 2, 3, 4$. Also, by 5.4 (with s_2-c_4-c_5), s_2 is strongly adjacent to c_5, and similarly s_1 is strongly adjacent to c_1. Since $\{c_2, c_6, s_1, s_2\}$ is not a claw, c_2 is strongly antiadjacent to c_6, and similarly c_4, c_6 are strongly antiadjacent. Thus the strip $(\{s_1, c_2\}, \emptyset, \{s_2, c_4\})$ is step-connected and parallel to the strip $(\{c_1\}, \{c_6\}, \{c_5\})$. Choose a step-connected strip $(A, \emptyset, B)$ with $s_1, c_2 \in A$ and $s_2, c_4 \in B$, with $A \cup B$ maximal such that c_3 is strongly $A \cup B$-complete and the strips $(A, \emptyset, B)$, $(\{c_1\}, \{c_6\}, \{c_5\})$ are parallel. Suppose that $v \in V(G) \setminus (A \cup B)$, and v has both a neighbour and an antineighbour in A. Then $v \notin \{c_1, c_3, c_5, c_6\}$. Let $N = N_G(v), N^* = N^*_G(v)$. Choose a step a_1-a_2-b_2-b_1-a_1 in the strip $(A, \emptyset, B)$ such that $a_1 \in N$

and $a_2 \notin N^*$. By 5.4, $b_1 \in N^*$. Suppose that $b_2 \in N$. Then 5.4 implies that $c_5 \in N^*$; 5.3 implies that $c_6 \notin N$; 5.4 implies that $c_1 \notin N$; 5.4 implies that $B \subseteq N^*$ and $c_3 \in N^*$; and then v can be added to B, contrary to the maximality of $A \cup B$. Thus $b_2 \notin N$, and so 5.4 implies that $a_1 \in N^*$, and from the symmetry it follows that $a_2 \notin N$. Since c_1-c_6-c_5-b_2-a_2-c_1 is dominating, we may assume from the symmetry that $c_1, c_6 \in N$. If $c_5 \notin N$, then v-c_6-c_5-b_2-a_2-a_1-v is a 6-hole, and b_1 is a hub for it, a contradiction. Thus $c_5 \in N$; but then 5.3 implies that $c_3 \notin N$, and so c_1-c_6-c_5-b_1-c_3-a_2-c_1 is a 6-hole, with a star-diagonal $\{a_1, v\}$, again a contradiction. So there is no such vertex v. We deduce from the symmetry that (A, B) is a homogeneous pair, nondominating because of c_6, and so by 4.3, G is decomposable. This proves 16.4. ■

17 6-holes in non-antiprismatic trigraphs

The next lemma, a consequence of 10.4, is complementary to the last few results.

17.1 *Let G be a claw-free trigraph, containing no hole of length > 6 or long prism, and such that every hole of length 5 or 6 is dominating. Suppose that G contains a 6-hole, but there is no 6-hole in G with a hub, a star-diagonal, or a star-triangle. Then either $G \in \mathcal{S}_3$, or G is decomposable.*

Proof. Since every 5-hole is dominating, no 6-hole has a coronet; by hypothesis, no 6-hole has a hub, star-diagonal or star-triangle; by 16.4, we may assume that none has a crown; and none has a hat-diagonal since G contains no long prism. By 10.4, this proves 17.1. ■

We recall that G is *antiprismatic* if for every $X \subseteq V(G)$ with $|X| = 4$, X is not a claw and there are at least two pairs of members of X that are strongly adjacent. We combine 17.1 with the previous results, to prove the next theorem, which has been the goal of the last several sections.

17.2 *Let G be a claw-free trigraph with a hole of length ≥ 6. Then either $G \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_7$, or G is decomposable.*

Proof. By 8.7, 10.1, 10.3 and 14.3, we may assume that G has no hole of length > 6 or long prism, and every hole of length 5 or 6 is dominating.

(1) *We may assume that there is a 6-hole C in G such that no vertex of G is a hat or clone relative to C, and every two consecutive vertices of C are strongly adjacent.*

For by hypothesis there is a hole of length ≥ 6, and therefore of length 6. If there is no 6-hole in G with a hub, a star-diagonal, or a star-triangle, then either $G \in \mathcal{S}_3$, or G is decomposable, by 17.1. Thus we may assume that there is a 6-hole C with either a hub, a star-diagonal, or a star-triangle, choosing C with a hub if possible. Suppose first that C has a hub. By 13.6 we may assume that no vertex is a clone relative to C, and no two consecutive vertices of C are semiadjacent; and by 15.1 we may assume that no vertex is a hat with respect to C, as claimed. Thus we may assume that C has no hub, and therefore no 6-hole has a hub. Next suppose that C has a star-diagonal. By 13.7, we may assume that no vertex is a clone, and no two consecutive vertices of C are semiadjacent; and by 15.2, no vertex is a hat, as claimed. Finally, suppose that C has a star-triangle. By 16.3, again we may assume that no vertex is a hat or clone with respect to C. It remains to show that no two consecutive vertices of C are semiadjacent. We have shown that every vertex not in $V(C)$ is a strong star relative to C. Let C have vertices c_1-$\cdots$-c_6-c_1 in order, and let s_1, s_3, s_5 be pairwise adjacent stars in positions $1\frac{1}{2}, 3\frac{1}{2}, 5\frac{1}{2}$ respectively. Since $\{s_1, c_6, c_2, c_3\}$ is not a claw, c_2 is strongly adjacent to c_3, and similarly the pairs c_4c_5 and c_6c_1 are strongly adjacent. We assume therefore that c_1, c_2 are semiadjacent. It follows that there are no stars in positions $\frac{1}{2}, 2\frac{1}{2}$. We claim that the triangle $\{s_1, s_3, s_5\}$ is dominating. For certainly it dominates all vertices in C, and all stars in positions $1\frac{1}{2}, 3\frac{1}{2}, 5\frac{1}{2}$, by 9.2, so suppose that there is a star s_4 in position $4\frac{1}{2}$ that is strongly antiadjacent to all of s_1, s_3, s_5. But then c_1-c_2-c_3-s_4-c_5-s_5-c_1 is a 6-hole, and s_3 is a hub relative to it, a contradiction. This proves that $\{s_1, s_3, s_5\}$ is dominating. But c_1, c_2, c_4, c_6 each have at least two neighbours and at least one antineighbour in this triangle, and only one pair of c_1, c_2, c_4, c_6 are strongly adjacent, and the result holds by 16.2. We may therefore assume that no two consecutive vertices of C are semiadjacent. This proves (1).

(2) *There do not exist four pairwise antiadjacent vertices in G.*

For suppose that $a_1, \ldots, a_4$ are pairwise antiadjacent. Not all of $a_1, \ldots, a_4$ belong to C; and each a_i that does not belong to C has exactly four strong neighbours in C, since C is dominating and no vertex is a clone or hat relative to C. We may assume that $a_1 \notin V(C)$. Since it has four strong neighbours in C and is antiadjacent to a_2, a_3, a_4, at most two of a_2, a_3, a_4 belong to C, and we may assume that $a_2 \notin V(C)$. By 5.5, a_1, a_2 do not have exactly the same four neighbours in C, and so at most one vertex of C is antiadjacent to both a_1, a_2; and so not both $a_3, a_4 \in V(C)$, and we may assume that $a_3 \notin V(C)$. Then a_1, a_2, a_3 each have four strong neighbours in C. But they have no common neighbour, and therefore every vertex of C is strongly adjacent to exactly two of them. Consequently $a_4 \notin V(C)$, and therefore a_4 also has four strong neighbours in C; and so some three of $a_1, \ldots, a_4$ have a common neighbour in $V(C)$, a contradiction. This proves (2).

Let C have vertices c_1-$\cdots$-c_6-c_1 in order.

(3) *If there exist stars s_1, s_2, s_3, each in position $1\frac{1}{2}$ or $2\frac{1}{2}$, such that s_3 is antiadjacent to both s_1, s_2, then G is decomposable.*

For suppose that such s_1, s_2, s_3 exist. s_1, s_3 are in different positions, by 9.2, and so are s_2, s_3, and therefore s_1, s_2 are in the same positions. Choose A, B with $A \cup B$ maximal such that:

- A is a set of stars in position $1\frac{1}{2}$
- B is a set of stars in position $2\frac{1}{2}$
- $s_1, s_2, s_3 \in A \cup B$
- let H be the graph with $V(H) = A \cup B$, in which x, y are adjacent if and only if x, y are antiadjacent in G and exactly one of x, y belongs to A; then H is connected.

Suppose that some $v \notin A \cup B$ has a neighbour and an antineighbour in A say. Since H is connected, we may choose $a_1, a_2 \in A$ and $b \in B$ such that v is adjacent to a_1 and antiadjacent to a_2, and b is antiadjacent in G to both a_1, a_2. (Note that a_1, a_2 may be equal.) Since v has a neighbour and an antineighbour in A, it follows that $v \notin V(C)$, and therefore v has exactly four strong neighbours in

C. Since v has an antineighbour in A, it is not a star in position $1\frac{1}{2}$ or a hub in hub-position 2; and from the maximality of $A \cup B$, it is not a star in position $2\frac{1}{2}$. Consequently v is adjacent to c_5. Since $\{v, a_1, b, c_5\}$ is not a claw, v is antiadjacent to b. But v is adjacent to one of c_1, c_2, c_3, say c_i, and then $\{c_i, a_2, b, v\}$ is a claw, a contradiction. Thus there is no such vertex v; and similarly every vertex not in $A \cup B$ is either strongly complete or strongly anticomplete to B. This proves that (A, B) is a homogeneous pair, nondominating because of c_5, and so G is decomposable, by 4.3. This proves (3).

(4) *If there exist a hub t in hub-position 1, and stars s_2, s_3, s_4, each in positions $2\frac{1}{2}$ or $5\frac{1}{2}$, such that s_4 is antiadjacent to s_2, s_3, then G is decomposable.*

For choose A, B with $A \cup B$ maximal such that:

- A is a set of stars in position $2\frac{1}{2}$
- B is a set of stars in position $5\frac{1}{2}$
- $s_2, s_3, s_4 \in A \cup B$
- let H be the graph with $V(H) = A \cup B$, in which x, y are adjacent if and only if x, y are antiadjacent in G and exactly one of x, y belongs to A; then H is connected.

We claim that (A, B) is a homogeneous pair. For let $v \in V(G) \setminus A \cup B$, and suppose it has a neighbour and an antineighbour in A say. Thus $v \notin V(C)$. Since H is connected, we may choose $a_1, a_2 \in A$ (not necessarily distinct) and $b \in B$ such that v is adjacent to a_1 and antiadjacent to a_2, and b is antiadjacent to both a_1, a_2. By 13.1, v is not a hub, and by 9.2 v is not a star in position $2\frac{1}{2}$; and by the maximality of $A \cup B$, v is not a star in position $5\frac{1}{2}$. Hence v is a star in some other position. Consequently v is adjacent to t by 13.1, and v is adjacent to one of c_1, c_4, say c_1. By 13.1, t is antiadjacent to all of a_1, a_2, b. If v is antiadjacent to b, then $\{c_1, v, a_2, b\}$ is a claw, while if v is adjacent to b, then $\{v, a_1, b, t\}$ is a claw, in either case a contradiction. Thus (A, B) is a homogeneous pair. By 13.1, t has no neighbours in $A \cup B$, and so (A, B) is nondominating. By 4.3, G is decomposable. This proves (4).

(5) *If there exist stars $s_1, \dots, s_4$, each in position $1\frac{1}{2}$, $3\frac{1}{2}$ or $5\frac{1}{2}$, and all pairwise antiadjacent except for $s_3 s_4$, then G is decomposable.*

For let B_1, B_2, B_3 be the set of all stars in positions $1\frac{1}{2}$, $3\frac{1}{2}$ and $5\frac{1}{2}$ respectively. By 5.5, B_1, B_2, B_3 are all strong cliques. Let $B = B_1 \cup B_2 \cup B_3$. Because of $s_1, \dots, s_4$, there are two triads in B with two vertices in common. Suppose that T is a triad with $|T \cap B| = 2$; say $T = \{v, b_1, b_2\}$, where $v \notin B$ and $b_1 \in B_1, b_2 \in B_2$. Since every vertex of C is adjacent to one of b_1, b_2 it follows that $v \notin V(C)$, and therefore v has four strong neighbours in C. Since $\{c_2, v, b_1, b_2\}$ is not a claw, v is antiadjacent to c_2 and similarly antiadjacent to c_3; and so it is a star in position $5\frac{1}{2}$, contradicting that $v \notin B$. Thus there is no such triad. By 16.1, it follows that G is decomposable. This proves (5).

We may assume that G is not antiprismatic. Therefore there are four vertices $a_1, \dots, a_4$, pairwise antiadjacent except possibly for $a_3 a_4$. By (2), a_3, a_4 are strongly adjacent. Suppose first that $a_1, a_2 \in V(C)$. Then, since no two consecutive vertices of C are semiadjacent, at least one of a_3, a_4 is not in $V(C)$, say a_3; and therefore a_3 is strongly adjacent to every vertex of C except a_1, a_2. Since a_1, a_2 are antiadjacent, it follows that a_3 is a hub, and so we may assume that $a_1 = c_1, a_2 = c_4$. Then every other vertex of C is strongly adjacent to one of a_1, a_2, and so $a_4 \notin V(C)$; and therefore a_4 is also a hub, in the same hub-position as c_3. Then G is decomposable, by 13.2.

We may therefore assume that not both $a_1, a_2 \in V(C)$, say $a_1 \notin V(C)$. Consequently a_1 has four strong neighbours in $V(C)$. Assume that $a_2, a_3 \in V(C)$. Then since a_1, a_2, a_3 are pairwise antiadjacent, it follows that a_1 is a hub, and we may assume that $a_2 = c_1, a_3 = c_4$. Since a_4 is adjacent to a_3 and $a_4 \notin V(C)$, it follows that a_4 is a star in position $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, or $5\frac{1}{2}$, or a hub in hub-position 2 or 3. Since a_4 is antiadjacent to $a_2 = c_1$, we may assume from the symmetry that a_4 is a star in position $2\frac{1}{2}$; but then it is strongly adjacent to a_1 by 13.1, a contradiction.

This proves that not both $a_2, a_3 \in V(C)$. Assume that $a_2 \in V(C)$, say $a_2 = c_1$. Then $a_3 \notin V(C)$, and similarly $a_4 \notin V(C)$. Each of a_1, a_3, a_4 is strongly adjacent to four of $c_2, \ldots, c_6$, and is therefore either a star in position $3\frac{1}{2}$ or $4\frac{1}{2}$, or a hub in hub-position 1. If any of them is a hub in hub-position 1, then it is adjacent to both the others by 13.1, a contradiction; and so all three are stars. But then the result follows by (3). So we may assume that $a_2 \notin V(C)$.

Since a_1, a_2 do not have exactly the same neighbours in C by 5.5, it follows that at least one of $a_3, a_4 \notin V(C)$, say a_3. Hence a_1, a_2, a_3 each has four strong neighbours in $V(C)$, and yet they have no common neighbour. Consequently each vertex of C is strongly adjacent to exactly two of a_1, a_2, a_3, and therefore $a_4 \notin V(C)$. Thus a_4 also has exactly four strong neighbours in C, and no vertex is adjacent to all of a_1, a_2, a_4, and therefore a_3, a_4 have the same neighbours in C. By 13.2 we may assume that a_3, a_4 are not hubs, and so we may assume that they are both stars in position $2\frac{1}{2}$ say. Hence a_1, a_2 are both strongly adjacent to both c_5, c_6, and each of c_1, c_2, c_3, c_4 is adjacent to exactly one of a_1, a_2. Thus either one of a_1, a_2 is a star in position $5\frac{1}{2}$ and the other is a hub in hub-position 1, or one of a_1, a_2 is a star in position $4\frac{1}{2}$ and the other is a star in position $\frac{1}{2}$. In the first case the result follows from (4), and in the second case from (5). This proves 17.2. ■

18 Stable sets of size 4

For a trigraph G, we recall that $\alpha(G)$ is the maximum cardinality of stable sets in G. In this section we finish the case that $\alpha(G) \geq 4$. We have already (in 17.2) handled such graphs that have a hole of length at least 6, so it suffices to prove the following.

18.1 *Let G be a claw-free trigraph, such that G has no hole of length > 5, every 5-hole in G is dominating, $\alpha(G) \geq 4$, and G is not decomposable. Then G is either a line trigraph or a long circular interval trigraph.*

The proof of 18.1 falls into several parts, as follows. Let G satisfy the hypotheses of 18.1. We shall prove the following.

- (In 18.7) If some 5-hole has a coronet, then G is a line trigraph.
- (In 18.8) If G contains a $(1,1,1)$-prism, then G is a line trigraph.
- (In 18.9) If G has a 5-hole, but no 5-hole has a coronet, and G contains no $(1,1,1)$-prism, then G is a long circular interval trigraph.
- (In 18.10) If G has a 4-hole but no 5-hole, then G is a line trigraph.

- (In 18.11) It is impossible that G has no holes at all.

We begin with a few lemmas.

18.2 *Let B be a clique in a claw-free trigraph G, and let $a_1, a_2 \in V(G) \setminus B$ be antiadjacent. If a_1, a_2 are not strongly B-complete and not strongly B-anticomplete, then there is a path of length 3 between a_1, a_2 with interior in B.*

Proof. For $i = 1, 2$, let N_i, N_i^* be the set of neighbours and strong neighbours of a_i in B. By hypothesis, $N_i \neq \emptyset$ and $N_i^* \neq B$. Suppose that $N_1 \subseteq N_2^*$. Since $N_1 \neq \emptyset$, there exists $x \in N_1$; and since $N_2^* \neq B$, there exists $y \in B \setminus N_2^*$. But then $\{x, y, a_1, a_2\}$ is a claw, a contradiction. Thus $N_1 \not\subseteq N_2^*$, and similarly $N_2 \not\subseteq N_1^*$. Choose $n_1 \in N_1 \setminus N_2^*$, and $n_2 \in N_2 \setminus N_1^*$. Then a_1-n_1-n_2-a_2 is a path. This proves 18.2. ■

18.3 *Let G be a claw-free trigraph, with no hole of length > 5, not decomposable, and such that every 5-hole is dominating. Let the paths a_1-b_1, a_2-b_2 and a_3-c_3-b_3 form a prism in G, where $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are strong triangles. Then there is a 5-hole in G with a strong centre, and every neighbour of c_3 that is antiadjacent to all of a_1, b_1, a_2, b_2 is strongly adjacent to both of a_3, b_3.*

Proof. Choose a step-connected strip $(A, \emptyset, B)$ with $a_1, a_2 \in A$ and $b_1, b_2 \in B$, parallel to the strip $(\{a_3\}, \{c_3\}, \{b_3\})$, and maximal with this property. Since c_3 is strongly anticomplete to $A \cup B$ and G is not decomposable, 4.3 implies that (A, B) is not a homogeneous pair. Thus we may assume that there exists $v \in V(G) \setminus (A \cup B)$ with a neighbour and an antineighbour in A. Then $v \notin \{a_3, b_3, c_3\}$. Choose a step a_1'-a_2'-b_2'-b_1'-a_1' such that v is adjacent to a_1' and antiadjacent to a_2'. By 5.4, v is strongly adjacent to b_1'. If v is adjacent to b_2', then by 5.4 v is strongly adjacent to b_3; by 5.3 v is strongly antiadjacent to c_3; and by 5.4 v is strongly antiadjacent to a_3. But then v can be added to B, contrary to the maximality of $A \cup B$. Thus v is strongly antiadjacent to b_2'. From the symmetry between a_1' and b_1' it follows that v is strongly adjacent to a_1' and strongly antiadjacent to a_2'. Since the 5-hole a_3-c_3-b_3-b_2'-a_2'-a_3 is dominating, v has a neighbour in the path a_3-c_3-b_3, and therefore is adjacent to at least two consecutive vertices of this path. In particular, v is strongly adjacent to c_3. Since v-c_3-b_3-b_2'-a_2'-a_1'-v is not a 6-hole, v is strongly adjacent to b_3 and similarly to a_3. Hence v is a strong centre for the 5-hole a_3-c_3-b_3-b_1'-a_1'-a_3. Now suppose that d is a neighbour of c_3, antiadjacent to a_1, b_1, a_2, b_2. Hence d has an antineighbour in A. If d also has a neighbour in A, then by exchanging v, d we deduce that d is strongly adjacent to both a_3, b_3 as required. Thus we may assume that d has no neighbour in A, and similarly none in B. From the symmetry, we may assume that d is strongly adjacent to a_3. By 5.4 (with d-a_3-a_2'), v is adjacent to d; and by 5.3 (with $\{d, a_1', b_3\}$) it follows that d is strongly adjacent to b_3 as required. This proves 18.3. ■

18.4 *Let G be a claw-free trigraph, with no hole of length > 5, and such that every 5-hole is dominating. Let C be a 4-hole. If there exist adjacent vertices of $G \setminus V(C)$, both with no neighbour in $V(C)$, then G is decomposable.*

Proof. Let C have vertices c_1-$\cdots$-c_4-c_1 in order. Let $Z \subseteq V(G) \setminus V(C)$ be maximal such that Z is connected and no vertex in Z has a neighbour in $V(C)$, with $|Z| > 1$. Let Y be the set of vertices of $V(G) \setminus Z$ with a neighbour in Z. Then from the maximality of Z, every vertex of Y has a neighbour in $V(C)$; and since G is claw-free, it follows that every vertex in Y is a strong hat relative to C. Let $Y = Y_1 \cup \cdots \cup Y_4$, where for $i = 1, \ldots, 4$, Y_i is the set of vertices in Y that are adjacent to c_i, c_{i+1} (reading subscripts modulo 4).

(1) *$Y_1, \ldots, Y_4$ are strong cliques; and for $1 \le i \le 4$, Y_i is strongly complete to Y_{i+1}.*

The first assertion follows from 5.5. For the second, suppose that $y_1 \in Y_1$ and $y_2 \in Y_2$ say are antiadjacent, and let P be a path between y_1, y_2 with interior in Z. Then y_1-c_1-c_4-c_3-y_2-P-y_1 is a hole of length ≥ 6, a contradiction. This proves (1).

(2) *We may assume that if $y, y' \in Y$ are antiadjacent then every vertex in Z is strongly adjacent to both y, y'.*

For let $y \in Y_1, y' \in Y_3$ say (without loss of generality, by (1)). Let P be a path between y, y' with interior in Z. Since the hole y-c_2-c_3-y'-P-y has length ≤ 5, it follows that P has length 2, and the hole has length 5. Let z be the middle vertex of P. Since every 5-hole is dominating, every vertex in $Z \setminus \{z\}$ has a neighbour in P, and therefore is adjacent to z and to at least one of y, y'. By 18.3, applied to the prism formed by the three paths c_1-c_2, c_4-c_3 and y-z-y', it follows that every member of Z is strongly adjacent to both y, y'. This proves (2).

(3) *For $1 \le i < j \le 4$, if $y_i \in Y_i$ and $y_j \in Y_j$ then y_i, y_j have the same neighbours in Z, and no vertex in Z is semiadjacent to one of y_i, y_j.*

For if y_i, y_j are antiadjacent this follows from (2). If they are strongly adjacent, suppose that $z \in Z$ is adjacent to y_i and antiadjacent to y_j, and choose $c \in V(C)$ adjacent to y_i and antiadjacent to y_j; then $\{y_i, z, y_j, c\}$ is a claw, a contradiction. This proves (3).

If Y is a strong clique then Y is an internal clique cutset and the theorem holds. Thus by (1), we may assume that Y_1 is not strongly complete to Y_3 (and therefore Y_1, Y_3 are nonempty). By (2) and (3) it follows that Y is complete to Z, and therefore Z is a strong clique by 5.5; but then all members of Z are twins. This proves 18.4. ■

18.5 *Let G be a claw-free trigraph, let C be a dominating 5-hole in G, and let $X \subseteq V(G)$ be stable with $|X| = 4$. Then there is a 5-numbering c_1-$\cdots$-c_5-c_1 of C such that either*

- *there are three strong hats in X, in positions $1\frac{1}{2}, 2\frac{1}{2}$ and $3\frac{1}{2}$, or*
- *X consists of two strong hats in positions $1\frac{1}{2}$ and $2\frac{1}{2}$ and two clones in positions $4, 5$, or*
- *c_4, c_5 are semiadjacent, and X consists of c_4, c_5 and two strong hats in positions $1\frac{1}{2}$ and $2\frac{1}{2}$, or*
- *X consists of three strong hats in positions $1\frac{1}{2}, 2\frac{1}{2}$ and $4\frac{1}{2}$ and a strong star in position $4\frac{1}{2}$.*

Proof. Let C have vertices c_1-$\cdots$-c_5-c_1 and let $X = \{v_1, \ldots, v_4\}$. Each member of $X \setminus V(C)$ has at least two strong neighbours in $V(C)$, consecutive in C, since C is dominating; and on the other hand, every vertex of C is adjacent to at most two members of X, since G is claw-free. For

$1 \leq i \leq 5$, not all of $c_i, c_{i+1}, c_{i+2} \in X$, since c_{i+1} is strongly adjacent to at least one of c_i, c_{i+2}; and hence $|X \cap V(C)| \leq 3$. We may therefore assume that $v_1 \notin V(C)$.

Suppose that $v_2, v_3, v_4 \in V(C)$. Now v_1 has two strong neighbours in C, consecutive in C, say c_1, c_2; and so $X \cap V(C) = \{c_3, c_4, c_5\}$, which is impossible as we already saw. Thus we may assume that $v_1, v_2 \notin V(C)$.

Suppose that $v_3, v_4 \in V(C)$. Since v_1, v_2 both have at least two strong neighbours in C, consecutive in C, and since v_1, v_2 are not hats in the same position by 5.5, it follows that v_3, v_4 are consecutive in C and therefore semiadjacent; say $v_3 = c_4, v_4 = c_5$. Hence v_1, v_2 are strongly antiadjacent to c_4, c_5 (since $F(G)$ is a matching); and so by 9.2 it follows that the third outcome of the theorem holds. Thus we may assume that $v_1, v_2, v_3 \notin V(C)$.

Suppose that $v_4 \in V(C)$, say $v_4 = c_5$. Then each of c_1, c_4 is adjacent to at most one of v_1, v_2, v_3, and each of c_2, c_3 is adjacent to at most two of v_1, v_2, v_3. On the other hand, v_1, v_2, v_3 each have at least two strong neighbours in C. Hence equality holds, and therefore v_1, v_2, v_3 are strong hats in positions $1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}$, as required.

We may therefore assume that $v_4 \notin V(C)$. Now $c_1, \dots, c_5$ are each adjacent to at most two of members of X, and every member of X is strongly adjacent to at least two of $c_1, \dots, c_5$. Consequently at least two members of X are strong hats, say v_1, v_2. Suppose that no two members of X are strong hats in consecutive positions. Then we may assume that v_1, v_2 are in positions $1\frac{1}{2}, 3\frac{1}{2}$, and v_3, v_4 are not strong hats; and from counting the edges between $V(C)$ and X, it follows that v_3, v_4 are clones, in positions $1, 4$. But since they are antiadjacent to v_1, v_2, this contradicts 9.2. Thus at least two members of X are strong hats in consecutive positions, and so we may assume that v_1, v_2 are strong hats in positions $1\frac{1}{2}, 2\frac{1}{2}$ respectively. If v_3, v_4 are not strong hats, then they are clones in positions $4, 5$ and the theorem holds. Thus we may assume that v_3 is a strong hat. If it is in position $3\frac{1}{2}$ or $\frac{1}{2}$ then the theorem holds, so we may assume it is in position $4\frac{1}{2}$. If v_4 is a strong hat, then it is in position $3\frac{1}{2}$ or $\frac{1}{2}$ and the theorem holds; and by 9.2 is it not a clone. So we may assume it is a strong star, and hence in position $4\frac{1}{2}$; but then the theorem holds. This proves 18.5. ■

18.6 *Let G be a claw-free trigraph, such that G has no hole of length > 5, every 5-hole in G is dominating, and $\alpha(G) \geq 4$. Then no 5-hole in G has a centre; and G does not contain a $(2,1,1)$-prism.*

Proof. For suppose first that c_1-$\cdots$-c_5-c_1 is a 5-hole C, with a centre z. Since $\alpha(G) \geq 4$, we may assume by 18.5 that there are antiadjacent hats h_1, h_2 in positions $1\frac{1}{2}, 2\frac{1}{2}$ say. Since $\{z, h_1, c_3, c_5\}$ is not a claw, z is antiadjacent to h_1, and similarly it is antiadjacent to h_2. But then $\{c_2, z, h_1, h_2\}$ is a claw, a contradiction. This proves that no 5-hole has a centre. The second assertion of the theorem follows from 18.3. This proves 18.6. ■

The following completes the first step of the proof of 18.1.

18.7 *Let G be a claw-free trigraph, such that G has no hole of length > 5, every 5-hole in G is dominating, $\alpha(G) \geq 4$, and G is not decomposable. If some 5-hole has a coronet then G is a line trigraph.*

Proof. Let c_1-$\cdots$-c_5-c_1 be a 5-numbering of a 5-hole C, such that there is a hat h and a star s both in position $1\frac{1}{2}$. By 9.2, h and s are strongly antiadjacent, and h is a strong hat and s is a strong star. Let $\mathcal{C}$ be the proximity component of order 5 containing C.

(1) *For every a_1-$\cdots$-a_5-a_1 in $\mathcal{C}$, h is a strong hat and s is a strong star, both in position $1\frac{1}{2}$.*

For it suffices to show that if two 5-numberings are proximate, and the claim is true for one of them, then it is true for the other. Thus, suppose that a_1-$\cdots$-a_5-a_1 is a 5-numbering and h is a strong hat and s is a strong star, both in position $1\frac{1}{2}$, relative to a_1-$\cdots$-a_5-a_1. Let $1 \leq i \leq 5$, and let a'_i be a clone in position i relative to a_1-$\cdots$-a_5-a_1. We must show that a_i and a'_i have the same neighbours in $\{h, s\}$. If $i = 1$, then a'_1 is strongly adjacent to s, h by 9.1. If $i = 4$, then a'_4 is strongly antiadjacent to h by 9.1, and strongly antiadjacent to s by 18.6, since otherwise s would be a centre for a_1-a_2-a_3-a'_4-a_5-a_1. Thus from the symmetry we may assume that $i = 5$. Since $\{a'_5, h, s, a_4\}$ is not a claw, it follows that a'_5 is strongly antiadjacent to at least one of h, s. Since $\{a_1, a'_5, h, s\}$ is not a claw, a'_5 is strongly adjacent to at least one of h, s. If a'_5 is adjacent to h and not to s, then the 5-hole h-a_2-s-a_5-a'_5-h has a centre a_1, contrary to 18.6. Thus a'_5 is strongly adjacent to s and strongly antiadjacent to h. This proves (1).

For $1 \leq i \leq 5$, let $A_i = A_i(\mathcal{C})$. From (1), $A_1 \cup A_2$ is strongly complete to both h, s; $A_3 \cup A_5$ is strongly complete to s and strongly anticomplete to h; and A_4 is strongly anticomplete to both h, s. Let $W = A_1 \cup \cdots \cup A_5$. For each $v \in V(G) \setminus \{h, s\}$, let $P(v)$ be the set of all k such that v is in position k relative to some member of $\mathcal{C}$. (Note that since every 5-hole is dominating, and none has a centre, it follows that v has a position relative to each member of $\mathcal{C}$.) If two 5-numberings are proximate, then the positions of v relative to them differ by at most $\frac{1}{2}$, and it follows that $P(v)$ is a set of consecutive $\frac{1}{2}$-integers modulo 5, that is, $P(v)$ is an "interval".

(2) *The sets $A_1, \ldots, A_5$ are pairwise disjoint; and every vertex in $V(G) \setminus W$ is either strongly complete to four of $A_1, \ldots, A_5$ and strongly anticomplete to the fifth, or strongly complete to two consecutive of $A_1, \ldots, A_5$ and strongly anticomplete to the other three.*

For certainly the sets $A_1 \cup A_2$, $A_3 \cup A_5$ and A_4 are pairwise disjoint. Suppose that there exists $v \in A_1 \cap A_2$. Then $1, 2 \in P(v)$, and v is strongly adjacent to h, s. Hence $3, 4, 5 \notin P(v)$, by (1), and since $P(v)$ is an interval, it follows that $1\frac{1}{2} \in P(v)$. So relative to some member of $\mathcal{C}$, v is a hat or star in position $1\frac{1}{2}$. But by 9.2, a hat in position $1\frac{1}{2}$ is antiadjacent to s, and a star in position $1\frac{1}{2}$ is antiadjacent to h, in either case a contradiction. This proves that $A_1 \cap A_2 = \emptyset$. Now assume that there exists $v \in A_3 \cap A_5$. Thus $3, 5 \in P(v)$, and by (1) v is strongly adjacent to s and strongly antiadjacent to h. By (1) $1, 2, 4 \notin P(v)$, contradicting that $P(v)$ is an interval. This proves that $A_1, \ldots, A_5$ are pairwise disjoint. Now if $v \in V(G) \setminus W$, it follows that $P(v)$ contains no integer, and so $P(v)$ has only one member, since it is an interval; and the final assertion of (2) follows. This proves (2).

For $1 \leq i \leq 5$, let H_i be the set of all vertices in $V(G) \setminus W$ that are strongly complete to $A_{i+2} \cup A_{i+3}$ and strongly anticomplete to A_{i-1}, A_i, A_{i+1}, and let S_i be the set of all vertices in $V(G) \setminus W$ that are strongly complete to $W \setminus A_i$ and strongly anticomplete to A_i. By (2), $V(G)$ is the union of $W, H_1, \ldots, H_5$ and $S_1, \ldots, S_5$. Moreover, $h \in H_4$ and $s \in S_4$. From 5.5, each H_i and each S_i is a strong clique.

(3) $A_1 \cup A_2$ *is strongly anticomplete to* A_4*;* A_1 *is strongly anticomplete to* A_3*, and* A_2 *to* A_5*;* A_5 *is strongly complete to* A_1*, and* A_1 *to* A_2*, and* A_2 *to* A_3*; and* $A_i = \{c_i\}$ *for* $i = 1, 2$.

For if $a_1 \in A_1$ and $a_4 \in A_4$, then since $\{a_1, a_4, h, s\}$ is not a claw it follows that a_1, a_4 are strongly antiadjacent. Thus $A_1 \cup A_2$ is strongly anticomplete to A_4. Let a_1-$\cdots$-a_5-a_1 be in $\mathcal{C}$, and suppose that some $v \in A_1$ is adjacent to a_3. Since v is strongly anticomplete to A_4 as we saw, it follows that v is strongly adjacent to a_2; by 9.2 v is strongly antiadjacent to c_5, since it is adjacent to h, and so v is strongly adjacent to a_1, since otherwise v-a_3-a_4-a_5-a_1-h-v would be a 6-hole. Hence v is in position 2 relative to a_1-$\cdots$-a_5-a_1, and so $v \in A_1 \cap A_2$, contrary to (2). This proves that A_1 is strongly anticomplete to A_3, and similarly A_2 is strongly anticomplete to A_5. Now let a_1-$\cdots$-a_5-a_1 be in $\mathcal{C}$, and suppose that some $a_1' \in A_1$ is antiadjacent to a_5. Then $\{s, a_1', a_5, a_3\}$ is a claw, a contradiction. Consequently A_1 is strongly complete to A_5, and similarly A_2 to A_3. Moreover, if $a_1, a_1' \in A_1$ are antiadjacent then $\{s, a_1, a_1', c_3\}$ is a claw, a contradiction, and so A_1 is a strong clique, and similarly so is A_2. Since every vertex in $V(G) \setminus W$ is either complete or anticomplete to A_i for $i = 1, 2$, it follows that (A_1, A_2) is a homogeneous pair, nondominating since $A_4 \neq \emptyset$; and so by 4.3, A_1, A_2 both have cardinality 1, since G is not decomposable. Thus $A_i = \{c_i\}$ for $i = 1, 2$. If c_1, c_2 are antiadjacent then h-c_2-c_3-c_4-c_5-c_1-h is a 6-hole, a contradiction. Thus c_1, c_2 are strongly adjacent. This proves (3).

(4) A_3, A_4, A_5 *are strong cliques.*

For if $a_3, a_3' \in A_3$ then they are strongly adjacent since $\{s, a_3, a_3', c_1\}$ is not a claw, and so A_3 is a strong clique, and similarly so is A_5. Now let a_1-$\cdots$-a_5-a_1 be in $\mathcal{C}$, and let $a_4' \in A_4$ be different from a_4. Since A_4 is disjoint from A_3, A_5, it follows that $3, 5 \notin P(a_4')$; and since $4 \in P(a_4')$ and $P(a_4')$ is an interval, it follows that $P(a_4') \subseteq \{3\frac{1}{2}, 4, 4\frac{1}{2}\}$. In particular, relative to a_1-$\cdots$-a_5-a_1, a_4' has position one of $3\frac{1}{2}, 4, 4\frac{1}{2}$, and therefore is strongly adjacent to a_4. This proves that A_4 is a strong clique, and therefore proves (4).

For $i = 3, 5$, let A_i' be the set of members of A_i with an antineighbour in A_4.

(5) A_3' *is strongly complete to* A_5'*;* A_3' *is strongly anticomplete to* $A_5 \setminus A_5'$*; and* $A_3 \setminus A_3'$ *is strongly anticomplete to* A_5'.

For suppose that $a_3 \in A_3'$ and $a_5 \in A_5'$ are antiadjacent. Each of them is not strongly A_4-complete and not strongly A_4-anticomplete, and therefore by 18.2, there is a path between them of length 3 with interior in A_4. But also a_5-c_1-c_2-a_3 is a path, and the union of these two paths is a 6-hole, contrary to hypothesis. This proves the first assertion of (5). Now suppose that $a_3 \in A_3'$ and $a_5 \in A_5 \setminus A_5'$ are adjacent. Choose $a_4 \in A_4$ antiadjacent to a_3. Since $a_5 \notin A_5'$, it follows that a_4, a_5 are adjacent; but then $\{a_5, a_3, a_4, c_1\}$ is a claw, a contradiction. Thus A_3' is strongly anticomplete to $A_5 \setminus A_5'$, and the third assertion of (5) follows by symmetry. This proves (5).

(6) *One of* A_3', A_5' *is empty.*

For suppose they are both nonempty. Choose $a_3' \in A_3'$ and $a_5' \in A_5'$. Choose $a_4, a_4' \in A_4$ (possibly equal) with a_4 adjacent to a_3' and a_4' antiadjacent to a_3'. Since $\{a_5', a_3', a_4', c_1\}$ is not a claw, a_4' is strongly antiadjacent to a_5', and since $\{a_3', a_5', a_4, c_2\}$ is not a claw, a_4 is strongly adjacent to a_5'.

Thus $a_4 \neq a'_4$. Let $\overline{G}$ be the complement of G. Since $\mathcal{C}$ is connected by proximity, it follows that $\overline{G}|(A_3 \cup A_5)$ is connected, and so $A'_3 \cup (A_5 \setminus A'_5)$ is not strongly complete to $A'_5 \cup (A_3 \setminus A'_3)$. Hence by (4) and (5), there exist $a_3 \in A_3 \setminus A'_3$ and $a_5 \in A_5 \setminus A'_5$, antiadjacent. But then a_3-a'_4-a_5-a'_5-a'_3-a_3 is a 5-hole with a centre a_4, contrary to 18.6. This proves (6).

(7) *$A_i = \{c_i\}$ for $1 \leq i \leq 5$.*

For from (6) we may assume that $A'_5 = \emptyset$. Then (A'_3, A_4) and $(A_3 \setminus A'_3, A_5)$ are both homogeneous pairs, by (3) and (5), and they are both nondominating because of h, and so by 4.3, $A'_3, A_4, A_3 \setminus A'_3, A_5$ all have cardinality at most 1. In particular $A_4 = \{c_4\}$ and $A_5 = \{c_5\}$. Thus we may assume that $|A_3| > 1$, and so $|A'_3| = |A_3 \setminus A'_3| = 1$. Let $A'_3 = \{a'_3\}$ and $A_3 \setminus A'_3 = \{a''_3\}$. Since a'_3 has a neighbour in A_4, it follows that a'_3, c_4 are semiadjacent. If a''_3 is strongly antiadjacent to c_5 then (A_3, A_4) is a nondominating homogeneous pair, a contradiction; so a''_3 is semiadjacent to c_5. If there exists $h_1 \in H_1$, then h_1-c_4-c_5-c_1-c_2-a'_3-h_1 is a 6-hole, a contradiction; so $H_1 = \emptyset$. If there exists $h_2 \in H_2$, then $\{c_5, h_2, a''_3, c_1\}$ is a claw, a contradiction; so $H_2 = \emptyset$. If there exists $s' \in S_2 \cup S_5$, then $\{s', c_4, a'_3, c_1\}$ is a claw, a contradiction; so $S_2 = S_5 = \emptyset$. Since $\alpha(G) \geq 4$, and every stable set contains at most two neighbours of c_2 (since G is claw-free), there are two antiadjacent vertices that are both strongly antiadjacent to c_2; and they therefore both belong to $H_3 \cup \{c_4, c_5\}$. Hence there exists $h_3 \in H_3$. If there exists $s_3 \in S_3$, then $\{c_5, h_3, s_3, a''_3\}$ is a claw, a contradiction; so $S_3 = \emptyset$. If there exist $s_1 \in S_1$ and $s_4 \in S_4$ that are antiadjacent then $\{c_2, s_1, s_4, h\}$ is a claw, a contradiction; so S_1 is strongly complete to S_4. Hence $(H_3 \cup \{c_1\}, H_4, H_5 \cup \{c_2\})$ is a breaker, and 4.4 implies that G is decomposable, a contradiction. This proves (7).

(8) *The following hold:*

- *For $1 \leq i, j \leq 5$, H_i is strongly complete to S_j if $j = i + 1$ or $j = i - 1$, and otherwise H_i is strongly anticomplete to S_j*
- *For $1 \leq i < j \leq 5$, H_i is strongly anticomplete to H_j*
- *For $1 \leq i \leq 5$, if $H_i \neq \emptyset$ then S_i is strongly anticomplete to S_{i-1}, S_{i+1}*
- *For $1 \leq i \leq 5$, if $H_i \neq \emptyset$ then S_i is strongly complete to S_{i-2}, S_{i+2}*
- *For $1 \leq i \leq 5$, if $H_i, S_i \neq \emptyset$ then S_{i-1} is strongly complete to S_{i+1}.*

For the first claim follows from 9.2. No two hats in consecutive positions are adjacent, since otherwise G would contain a 6-hole, and no two hats in distinct nonconsecutive positions are adjacent, by 18.6, since the union of two such adjacent hats with C would be a $(2, 1, 1)$-prism. Hence the second claim holds. The other three claims are trivial if $S_i = \emptyset$, so we may assume that S_i, H_i are both nonempty; and therefore, since S_4, H_4 are nonempty by hypothesis, we may assume that $i = 4$. Since $S_3 \cup S_4 \cup \{h, c_4\}$ includes no claw, S_3 is strongly anticomplete to S_4, and similarly S_4 to S_5, and so the third claim holds. Since $\{c_1, h\} \cup S_2 \cup S_4$ includes no claw, S_2 is strongly complete to S_4 and similarly S_1 is strongly complete to S_4, and therefore the fourth holds. Finally, the fifth holds since $\{c_1, s\} \cup S_3 \cup S_5$ includes no claw. This proves (8).

(9) *If S_2 is strongly complete to S_5 and c_4, c_5 are semiadjacent then G is a line trigraph.*

For if there exists $h_2 \in H_2$, then h_2-c_5-c_1-c_2-c_3-c_4-h_2 is a 6-hole, a contradiction; so $H_2 = \emptyset$. If there exists $s' \in S_1 \cup S_3$ then $\{s', c_4, c_5, c_2\}$ is a claw, a contradiction; so $S_1 = S_3 = \emptyset$. But then G is a line trigraph, by (8). This proves (9)

(10) *If*

- S_i *is strongly anticomplete to* S_{i+1} *for all* $i \in \{1, 2, 5\}$*, and*
- S_i *is strongly complete to* S_{i+2} *for all* $i \in \{1, 5\}$*, and*
- c_3, c_5 *are strongly antiadjacent,*

then G *is a line trigraph.*

For suppose these conditions hold. By (9) we may assume that c_4 is strongly adjacent to c_5 and similarly to c_3. But then G is a line trigraph, by (3) and (8). This proves (10).

(11) *If one of* H_1, H_2 *is nonempty then* G *is a line trigraph.*

For suppose that there exists $h_1 \in H_1$ say. Since $S_2 \cup S_3 \cup \{s, h_1\}$ includes no claw, S_2 is strongly anticomplete to S_3. By (8), S_1 is strongly anticomplete to S_5, S_2 and strongly complete to S_3. Since $\{c_3, c_5, h_1, c_2\}$ is not a claw, c_3 is strongly antiadjacent to c_5. By (10), we may assume that there exist $s_2 \in S_2$ and $s_5 \in S_5$, antiadjacent. Then s-c_2-s_5-c_4-c_5-s is a 5-hole; and relative to this 5-numbering, c_3, h are a star and a hat both in position $2\frac{1}{2}$, and s_2 is a clone in position 5, contrary to (7) applied to this 5-hole. This proves (11).

(12) *If* H_3, H_5 *are both nonempty then* G *is a line trigraph.*

For then (8) implies that S_3 is strongly complete to S_1 and strongly anticomplete to S_2; and S_5 is strongly complete to S_2 and strongly anticomplete to S_1. By (10), we may assume that either S_1 is not strongly anticomplete to S_2, or c_3 is semiadjacent to c_5. In the first case, when S_1 is not strongly anticomplete to S_2, it follows that $S_3 = \emptyset$ since $S_1 \cup S_2 \cup S_3 \cup H_5$ includes no claw, and similarly $S_5 = \emptyset$. In the second case, when c_3 is semiadjacent to c_5, it follows that $S_3 = \emptyset$ since $\{c_5, c_3\} \cup S_3 \cup H_3$ includes no claw, and similarly $S_5 = \emptyset$. Thus in both cases $S_3 = S_5 = \emptyset$. By (11) we may assume that H_1, H_2 are empty. But then $(S_1 \cup \{c_3\}, S_2 \cup \{c_5\})$ is a homogeneous pair, nondominating because of h, and so 4.3 implies that $S_1 = S_2 = \emptyset$. But then G is a line trigraph. This proves (12).

By (7), there are no clones relative to c_1-$\cdots$-c_5-c_1, and so by 18.5 and (11), (12), it follows that the third case of 18.5 holds, and therefore we may assume that $H_5 \neq \emptyset$ and c_4, c_5 are semiadjacent. But then (8) implies that S_2 is strongly complete to S_5, and so G is a line trigraph by (9). This proves 18.7. ■

Let the paths a_i-b_i $(i = 1, 2, 3)$ form a $(1, 1, 1)$-prism. For $1 \leq i \leq 3$, a *hat on* a_i-b_i means a vertex strongly adjacent to a_i, b_i and strongly antiadjacent to the other four vertices in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. The following completes the second step of the proof of 18.1.

18.8 *Let G be a claw-free trigraph, such that G has no hole of length > 5, every 5-hole in G is dominating, $\alpha(G) \geq 4$, and G is not decomposable. If G contains a $(1,1,1)$-prism then G is a line trigraph.*

Proof. Since G is not decomposable and $\alpha(G) \geq 4$, 4.3 implies that G does not admit a coherent W-join. By 18.7, we may assume that no 5-hole has a coronet.

(1) *G contains a $(1,1,1)$-prism with a hat.*

For let the paths a_i-b_i $(i = 1,2,3)$ form a $(1,1,1)$-prism, where $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ are triangles. Suppose first that $A \cup B$ is dominating. By hypothesis, $\alpha(G) \geq 4$, and so there exist pairwise antiadjacent vertices $v_1, \ldots, v_4$. For $1 \leq i \leq 4$, let N_i be the set of neighbours of v_i in $A \cup B$, together with v_i itself if $v_i \in A \cup B$. Thus each $|N_i| \geq 2$ by 5.4, and if $|N_i| = 2$ then v_i is a hat, so we may assume that $|N_i| \geq 3$ for each i. If $|N_i| = 3$, then $v_i \notin A \cup B$ and $N_i = A$ or B; and so by 5.5, $|N_i| = 3$ for at most two values of i. Consequently $|N_1| + |N_2| + |N_3| + |N_4| \geq 14$, and therefore we may assume that a_1 belongs to N_i for at least three values of i. Hence a_1 is strongly adjacent to at least one of $v_1, \ldots, v_4$, and so $a_1 \notin \{v_1, \ldots, v_4\}$; but then G contains a claw, a contradiction. So if $A \cup B$ is dominating then (1) holds.

Now assume that $A \cup B$ is not dominating. Let $z \in V(G)$ have no neighbours in $A \cup B$, and let Y be the set of neighbours of z. For $y \in Y$, let $N(y)$ be the set of neighbours of y in $A \cup B$. By 18.4, $N(y)$ is nonempty; and since G is claw-free, $N(y)$ is a strong clique. We claim we may assume that either $N(y) = A$ or $N(y) = B$. For we may assume that $a_1 \in N(y)$. If $b_1 \in N(y)$ then since $N(y)$ is a strong clique, it follows that y is a hat as required. We assume then that $b_1 \notin N(y)$. By 5.4, $a_2, a_3 \in N(y)$, and since $N(y)$ is a strong clique, we deduce that $N(y) = A$. Thus for every $y \in Y$, $N(y) = A$ or $N(y) = B$. Suppose there exist $y_1, y_2 \in Y$ with $N(y_1) = A$ and $N(y_2) = B$. If y_1, y_2 are antiadjacent, then the paths y_1-z-y_2, a_1-b_1 and a_2-b_2 form a $(2,1,1)$-prism, contrary to 18.6. If y_1, y_2 are adjacent, then the paths y_1-y_2, a_1-b_1, a_2-b_2 form a $(1,1,1)$-prism with a hat z on y_1-y_2, as required. Thus we may assume that $N(y) = A$ for all $y \in Y$. By 5.5, Y is a strong clique.

Let X be the set of all vertices in $V(G) \setminus (Y \cup \{z\})$ with a neighbour in Y. We claim that X is a strong clique. For suppose that $x_1, x_2 \in X$ are antiadjacent. For $i = 1,2$, choose $y_i \in Y$ adjacent to x_i. Since A is a strong clique, not both $x_1, x_2 \in A$, say $x_1 \notin A$. Since y_1 is adjacent to x_1 and to z, 5.4 implies that x_1 is strongly complete to A, and therefore $x_2 \notin A$. If y_1 is adjacent to x_2 then $\{y_1, z, x_1, x_2\}$ is a claw, a contradiction. Thus x_2 is strongly antiadjacent to y_1, and similarly x_1 is strongly antiadjacent to x_2, and in particular $y_1 \neq y_2$. Since $\{a_i, y_2, x_1, b_i\}$ is not a claw, it follows that x_1 is adjacent to b_i for $1 \leq i \leq 3$ and similarly x_2 is complete to B. Hence b_1-x_1-y_1-y_2-x_2-b_1 is a 5-hole with a centre a_1, contrary to 18.6. Thus X is a strong clique, and therefore X is an internal clique cutset (unless $Y = \emptyset$, when G is expressible as a 0-join). Hence G is decomposable, a contradiction. This proves (1).

(2) *G contains a $(1,1,1)$-prism with hats on two different paths.*

For by (1) we may choose paths a_i-b_i $(i = 1,2,3)$ forming a $(1,1,1)$-prism, where $A = \{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are triangles, such that there is a hat h on a_3-b_3. Choose a step-connected strip $(A, \emptyset, B)$ with $a_1, a_2 \in A$ and $b_1, b_2 \in B$, parallel to $(\{a_3\}, \{h\}, \{b_3\})$, and with $A \cup B$ maximal with this property. Since (A, B) is not a nondominating homogeneous pair, by 4.3, we may assume there is a vertex $v \notin A \cup B$ with a neighbour and an antineighbour in A. Let N, N^* be the set of neighbours and strong neighbours of v, and let a_1'-a_2'-b_2'-b_1'-a_1' be a step with $a_1' \in N$ and $a_2' \notin N^*$. By 5.4, $b_1' \in N^*$. If $b_2' \in N$, then by 5.4, $b_3 \in N^*$; by 5.3, $h \notin N$; by 5.4, $B \subseteq N^*$; by 5.4, $a_3 \notin N$; and then v can be added to B, contrary to the maximality of $A \cup B$. Thus $b_2' \notin N$. From the symmetry it follows that $a_1' \in N^*$ and $a_2' \notin N$. Suppose that $h \in N$. Since v-h-a_3-a_2'-b_2'-b_1'-v is not a 6-hole, it follows that $a_3 \in N^*$, and similarly $b_3 \in N^*$. But then v-a_3-a_2'-b_2'-b_1'-v is a 5-hole, and $\{a_1', h\}$ is a coronet for it, a contradiction. Thus $h \notin N$. From 5.4, $a_3, b_3 \notin N$; and so h, v are both hats for the prism formed by a_1'-b_1', a_2'-b_2' and a_3-b_3, on different paths. This proves (2).

From (2), we may choose $k \geq 3$, and disjoint strong cliques $A_1, \ldots, A_k, B_1, \ldots, B_k$ and $C_1, \ldots, C_k$ with the following properties (let $A = A_1 \cup \cdots \cup A_k$, $B = B_1 \cup \cdots \cup B_k$ and $C = C_1 \cup \cdots \cup C_k$):

- $A_1, \ldots, A_{k-1}, B_1, \ldots, B_{k-1}$ and $C_1, \ldots, C_{k-1}$ are all nonempty; and if $k = 3$ then A_3, B_3 are both nonempty
- A and B are strong cliques
- for $1 \leq i, j \leq k$ with $i \neq j$, A_i is strongly anticomplete to B_j
- for $1 \leq i \leq k-1$, A_i is strongly complete to B_i
- every vertex in A_k has a neighbour in B_k, and every vertex in B_k has a neighbour in A_k; and if C_k is nonempty then A_k, B_k are both nonempty and are strongly complete to each other
- for $1 \leq i \leq k$, C_i is strongly complete to $A_i \cup B_i$, and strongly anticomplete to $A \cup B \setminus (A_i \cup B_i)$
- $A \cup B \cup C$ is maximal with these properties.

Note that if C_k is nonempty then there is symmetry between C_k and $C_1, \ldots, C_{k-1}$ (this will be used in the case analysis below).

(3) *$C_1, \ldots, C_k$ are pairwise strongly anticomplete.*

For suppose not; then from the symmetry we may assume that $c_1 \in C_1$ is adjacent to $c_2 \in C_2$. Choose $a_i \in A_i$ and $b_i \in B_i$ for $i = 1, 2, 3$, such that a_3, b_3 are adjacent (this is possible even if $k = 3$). Then c_1-c_2-b_2-b_3-a_3-a_1-c_1 is a 6-hole, a contradiction. This proves (3).

(4) *For every $v \in V(G) \setminus (A \cup B \cup C)$, let N, N^* be the set of neighbours and strong neighbours of v in $A \cup B \cup C$; then $N = N^* = \emptyset, A, B$ or $A \cup B$.*

For suppose first that $N \cap C \neq \emptyset$; there exists $c_1 \in N \cap C_1$, say. Suppose that N meets both $A \setminus A_1$ and $B \setminus B_1$. By 5.3, $N \cap (A \setminus A_1)$ is strongly complete to $N \cap (B \setminus B_1)$, and so there exists i with $2 \leq i \leq k$ such that $N \cap A \subseteq A_1 \cup A_i$ and $N \cap B \subseteq B_1 \cup B_i$. Choose $a_i \in N \cap A_i$ and $b_i \in N \cap B_i$, necessarily adjacent. Choose $j \neq i$ with $2 \leq j \leq k$, and choose $a_j \in A_j$ and $b_j \in B_j$, adjacent. For $a_1 \in A_1$, v-c_1-a_1-a_j-b_j-b_i-v is not a 6-hole, and so $a_1 \in N$. But then v-a_1-a_j-b_j-b_i-v is a 5-hole, and $\{a_i, c_1\}$ is a coronet for it, a contradiction. Hence N does not have nonempty intersection with both $A \setminus A_1$ and $B \setminus B_1$. Suppose next that N meets $A \setminus A_1$ (and therefore does not meet $B \setminus B_1$). If $A \setminus A_1 \not\subseteq N^*$, we may choose distinct i, j with $2 \leq i, j \leq k$, such that $a_i \in N$ and $a_j \notin N^*$; but then

$\{a_i, a_j, v\} \cup B_i$ includes a claw, a contradiction. Thus $A \setminus A_1 \subseteq N^*$. 5.4 (with A_1-A_2-B_2) implies that $A_1 \subseteq N^*$. 5.3 (with C_1, C_2, A_3) implies that $N \cap C_2 = \emptyset$, and similarly $N \cap C \subseteq C_1$. If $b_1 \in B_1$ is antiadjacent to v, then v-c_1-b_1-b_3-a_3-v is a 5-hole (where $a_3 \in A_3$ and $b_3 \in B_3$ are adjacent), and it does not dominate the vertices in C_2, a contradiction. Thus $B_1 \subseteq N^*$. By 5.4 (with C_1-B_1-B_2), $C_1 \subseteq N^*$; but then v can be added to A_1, a contradiction. Finally, if N meets neither of $A \setminus A_1$ and $B \setminus B_1$, then $A_2 \cup A_3 \cup B_2 \cup B_3$ includes a 4-hole that does not dominate either of v, c_1, contrary to 18.4. This proves that $N \cap C = \emptyset$.

Next assume that $N \cap A_1 \neq \emptyset$. 5.4 (with C_1-A_1-A_i) implies that $A \setminus A_1 \subseteq N^*$. In particular, $N \cap A_2 \neq \emptyset$, and so 5.4 (with C_2-A_2-A_1) implies that $A \subseteq N^*$. If N intersects $B \setminus B_k$, then the same argument implies that $B \subseteq N^*$ and the claim holds. We assume then that $N \cap B \subseteq B_k$. If $N \cap B_k = \emptyset$ then again the theorem holds; and otherwise v can be added to A_k, a contradiction.

Thus we may assume that $N \cap A \subseteq A_k$ and $N \cap B \subseteq B_k$; and since we may assume that $N \neq \emptyset$, it follows that $C_k = \emptyset$. By 5.4 (with A_1-$(N \cap A_k)$-$B_k \setminus N$), it follows that $N \cap A_k$ is strongly anticomplete to $B_k \setminus N$, and similarly $N \cap B_k$ is strongly anticomplete to $A_k \setminus N$. Also, $N \cap A_k$ is strongly complete to $N \cap B_k$, for otherwise G contains a $(2,1,1)$-prism, contrary to 18.6. Let $C'_k = \{v\}$, $A'_k = A_k \cap N$, $B'_k = B_k \cap N$, $A'_{k+1} = A_k \setminus N$, and $B'_{k+1} = B_k \setminus N$ (and set $A'_i = A_i$ and so on, for $1 \leq i < k$); then this contradicts the maximality of $A \cup B \cup C$. This proves (4).

Let A_0, B_0, M, Z be the sets of vertices $v \in V(G) \setminus (A \cup B \cup C)$ whose set of neighbours in $A \cup B \cup C$ is $A, B, A \cup B$ and $\emptyset$ respectively. By 5.5, A_0, B_0, M are strong cliques. Suppose that there exist adjacent $a \in A_0$ and $b \in B_0$. If $C_k = \emptyset$, we can add a to A_k and b to B_k, and if $C_k \neq \emptyset$, we can define $A_{k+1} = \{a\}$ and $B_{k+1} = \{b\}$, in either case contradicting the maximality of $A \cup B \cup C$. Thus A_0 is strongly anticomplete to B_0. Since $A_1 \cup C_1 \cup A_0 \cup M$ includes no claw, M is strongly complete to A_0 and similarly to B_0. Suppose that there exists $z \in Z$, and let N be the set of neighbours of z. Then by 18.4, $N \subseteq A_0 \cup B_0 \cup M$, and $N \cap M = \emptyset$ since $M \cap A_1 \cup B_2 \cup \{z\}$ includes no claw. If N meets both A_0 and B_0, then G contains a $(2,1,1)$-prism, contrary to 18.6, so we may assume that $N \subseteq A_0$. Since G is claw-free and Z is stable by 18.4, no other member of Z has a neighbour in N. Hence every vertex in $V(G) \setminus (N \cup \{z\})$ is strongly $\{z\}$-anticomplete, and either strongly complete or strongly anticomplete to N. By 4.2, applied to $N, \{z\}$, it follows that G is decomposable, a contradiction. This proves that $Z = \emptyset$. Moreover, (A_k, B_k) is a homogeneous pair, nondominating since $C_1 \neq \emptyset$, and so A_k, B_k both have cardinality ≤ 1. Also each of the sets A_i, B_i, C_i $(1 \leq i \leq k-1)$ is a homogeneous set, and so they all have cardinality 1; and also the sets A_0, B_0, M are homogeneous sets and therefore have cardinality ≤ 1. But then G is a line trigraph. This proves 18.8. ■

The following completes the third step of the proof of 18.1.

18.9 *Let G be a claw-free trigraph, such that G has a 5-hole, G has no hole of length > 5, every 5-hole in G is dominating, $\alpha(G) \geq 4$, and G is not decomposable. If no 5-hole has a coronet, and G contains no $(1,1,1)$-prism, then G is a long circular interval trigraph.*

Proof. By 10.4 it suffices to show that no 5-hole has a coronet, crown, hat-diagonal, star-diagonal or centre. Let C be a 5-hole. By hypothesis, C has no coronet. Also, if $\{s_1, s_2\}$ is a crown for C, then $G|(V(C) \cup \{s_1, s_2\})$ contains a $(1,1,1)$-prism (delete the middle of the three common neighbours of s_1, s_2 in C), a contradiction. C has no hat-diagonal since by 18.6, G contains no $(2,1,1)$-prism. By 18.6, C has no centre; so it remains to prove that C has no star-diagonal.

Suppose that it does; let C have vertices c_1-$\cdots$-c_5-c_1 in order, and let s_1, s_2 be adjacent stars, adjacent respectively to $c_1, \ldots, c_4$ and to c_3, c_4, c_5, c_1. Since $\{s_1, c_1, c_3, c_4\}$ is not a claw, c_3 is strongly adjacent to c_4; since C has no coronet, there are no hats in positions $2\frac{1}{2}, 4\frac{1}{2}$; and there is not both a hat and a star in position $3\frac{1}{2}$. Consequently, the first, third and fourth outcomes of 18.5 are impossible, and so 18.5 implies that there is a stable set X with $|X| = 4$, consisting of two hats x_1, x_2 in positions $\frac{1}{2}$ and $1\frac{1}{2}$ respectively, and two clones x_3, x_4 in positions $3, 4$ respectively. By 9.2, s_1 is adjacent to x_2, x_3 and antiadjacent to x_1, and s_2 is adjacent to x_1, x_4 and antiadjacent to x_2. If x_3 is adjacent to s_2 then $\{s_2, x_1, x_3, x_4\}$ is a claw, while if x_3 is antiadjacent to s_2 then $\{s_1, s_2, x_2, x_3\}$ is a claw, in either case a contradiction. Hence C has no star-diagonal, and 10.4 implies that G is a long circular interval trigraph. This proves 18.9. ■

For the fourth step of the proof of 18.1, we use the following.

18.10 *Let G be a claw-free trigraph, such that G has a hole of length 4, G has no hole of length > 4, $\alpha(G) \geq 4$, and G is not decomposable. Then G is a line trigraph.*

Proof. By 18.8, we may assume that G contains no $(1,1,1)$-prism. Let c_1-$\cdots$-c_4-c_1 be a 4-hole. It is dominating, by 10.3, since G contains no $(1,1,1)$-prism. By hypothesis, there is a stable set X with $|X| = 4$. Thus each member of X either belongs to $\{c_1, \ldots, c_4\}$ or has at least two strong neighbours in this set, by 10.2. If $c_1, c_2 \in X$, and so c_1, c_2 are semiadjacent, then the other two members of X are not in $V(C)$, and are both adjacent to c_3, c_4 and antiadjacent to c_1, c_2, and therefore are strongly adjacent to each other by 5.5, a contradiction. Thus $|X \cap V(C)| \leq 1$. If $c_1 \in X$, then $c_2, c_4 \notin X$, and each is adjacent to at most one member of $X \setminus \{c_1\}$, which is impossible. Thus $c_1, \ldots, c_4 \notin X$. Also, $c_1, \ldots, c_4$ each are adjacent to at most two members of X, and so equality holds, and therefore each member of X is a strong hat relative to c_1-$\cdots$-c_4-c_1, all in different positions. Let $X = \{x_1, \ldots, x_4\}$, where x_i is a strong hat adjacent to c_i, c_{i+1}.

Consequently there are four nonempty strong cliques $A_1, \ldots, A_4$, pairwise disjoint, such that:

- A_i is strongly complete to A_{i+1} and strongly anticomplete to A_{i+2} for $1 \leq i \leq 4$ (reading subscripts modulo 4)
- x_i is strongly complete to A_i, A_{i+1} and strongly anticomplete to A_{i+2}, A_{i+3}, for $1 \leq i \leq 4$.

Choose $A_1, \ldots, A_4$ with maximal union W. Let B be the set of all vertices $v \in V(G) \setminus W$ that are strongly W-complete. For $i = 1, 2, 3, 4$, let H_i be the set of all $v \in V(G) \setminus W$ such that v is strongly complete to $A_i \cup A_{i+1}$ and strongly anticomplete to $A_{i+2} \cup A_{i+3}$. Thus $x_i \in H_i$ $(1 \leq i \leq 4)$.

(1) $V(G) = W \cup B \cup H_1 \cup H_2 \cup H_3 \cup H_4$.

For suppose that $v \in V(G) \setminus W$. We claim that $v \in B \cup H_1 \cup H_2 \cup H_3 \cup H_4$. For let N, N^* be the sets of neighbours and strong neighbours of v respectively. Since every 4-hole is dominating, we may assume that $A_1 \subseteq N^*$. 5.4 (with A_4-A_1-A_2) implies that N^* includes one of A_4, A_2, and from the symmetry we may assume that $A_2 \subseteq N^*$. Suppose that $N \cap A_3 \neq \emptyset$ and $A_3 \not\subseteq N^*$. Choose $a_3, a'_3 \in A_3$ (possibly equal) such that $a_3 \in N$ and $a'_3 \notin N^*$. Then 5.4 (with x_1-A_2-a'_3) implies that $x_1 \in N^*$; 5.3 implies that $x_4 \notin N$; 5.4 (with a'_3-A_4-x_4) implies that $N \cap A_4 = \emptyset$; 5.4 (with x_2-a_3-A_4) implies that $x_2 \in N^*$; and then v-x_2-a'_3-a_4-a_1-v is a 5-hole (where $a_1 \in A_1$ and $a_4 \in A_4$), a contradiction. Thus either $A_3 \subseteq N^*$ or $A_3 \cap N = \emptyset$, and the same holds for A_4. If N is disjoint from both A_3, A_4 then $v \in H_1$ as claimed, and if N^* includes both A_3, A_4 then $v \in B$ as claimed. We assume therefore that N^* includes just one of them, say A_3, and N is disjoint from A_4. By 5.4, $x_1, x_2 \in N^*$, and by 5.3, $x_3, x_4 \notin N$, and so v can be added to A_2, contrary to the maximality of W. This proves (1).

It follows from (1) that for $1 \leq i \leq 4$, all members of A_i are twins, and therefore $|A_i| = 1$, and so $A_i = \{c_i\}$. For $1 \leq i \leq 4$, H_i is strongly anticomplete to H_{i+1}, since G has no 5-hole, and H_i is strongly anticomplete to H_{i+2} since G contains no $(1,1,1)$-prism. Thus $H_1, \ldots, H_4$ are pairwise strongly anticomplete. By 5.5, each H_i is a strong clique. Let B_1 be the set of all $v \in B$ that are strongly complete to $H_1 \cup H_3$ and strongly anticomplete to $H_2 \cup H_4$, and let B_2 be those that are strongly complete to $H_2 \cup H_4$ and strongly anticomplete to $H_1 \cup H_3$. We claim that $B = B_1 \cup B_2$. For let $b \in B$, and let N, N^* be the sets of its neighbours and strong neighbours. 5.4 (with H_1-c_2-H_2) implies that N^* includes one of H_1, H_2, say H_1. By 5.3, N is disjoint from at least two of H_2, H_3, H_4. By 5.4 (with H_2-c_3-H_3 and H_3-c_3-H_4), $H_3 \subseteq N^*$, and so $N \cap (H_2 \cup H_4) = \emptyset$. Thus $v \in B_1$. This proves that $B = B_1 \cup B_2$. Consequently all members of H_i are twins, and so $H_i = \{x_i\}$ for $1 \leq i \leq 4$. Now if $b_1 \in B_1$ and $b_2 \in B_2$ then $\{b_1, b_2, x_1, x_3\}$ is not a claw, and so b_1, b_2 are strongly antiadjacent. Thus B_1 is strongly anticomplete to B_2. By 5.5, B_1, B_2 are strong cliques, and so for $i = 1, 2$, all members of B_i are twins. Hence $|B_1|, |B_2| \leq 1$. But then G is a line trigraph. This proves 18.10. ■

Finally, we handle graphs without any holes at all, in the following.

18.11 *Let G be a claw-free trigraph, such that G has no holes and $\alpha(G) \geq 4$. Then G is decomposable.*

Proof. For a contradiction, suppose that G is not decomposable.

(1) *There do not exist distinct $x_1, \ldots, x_4 \in V(G)$ such that x_1 is adjacent to x_2, and x_3 is adjacent to x_4, and $\{x_1, x_2\}$ is strongly anticomplete to $\{x_3, x_4\}$.*

For suppose that such $x_1, \ldots, x_4$ exist. Choose connected sets A_1, A_2 with $A_1 \cup A_2$ maximal such that $x_1, x_2 \in A_1$, $x_3, x_4 \in A_2$, $A_1 \cap A_2 = \emptyset$, and A_1 is strongly anticomplete to A_2. Let X be the set of vertices in $V(G) \setminus (A_1 \cup A_2)$ with a neighbour in $A_1 \cup A_2$. We claim that X is a strong clique; for let $u, v \in X$. By the maximality of $A_1 \cup A_2$, both u, v have neighbours in both A_1 and A_2; and so for $i = 1, 2$, there is a path P_i between u, v with interior in A_i. If u, v are antiadjacent, $P_1 \cup P_2$ is a hole, a contradiction. This proves that X is a strong clique, and therefore it is an internal clique cutset, since $|A_1|, |A_2| > 1$, a contradiction. This proves (1).

Say a subset $Y \subseteq V(G)$ is *split* if $|Y| \geq 4$ and every connected subset $C \subseteq Y$ satisfies $|C| \leq |Y| - 2$. Since $\alpha(G) \geq 4$, there is a split subset $Y \subseteq V(G)$. Choose Y maximal, and let the components of $G|Y$ be $C_1, \ldots, C_k$. Let $V(G) \setminus Y = X$. For each $x \in X$, we observe that x has neighbours in at most two of $C_1, \ldots, C_k$, since G is claw-free; and if it has neighbours in at most one of $C_1, \ldots, C_k$, then $Y \cup \{x\}$ is split, a contradiction. Thus each $x \in X$ has neighbours in exactly two of $C_1, \ldots, C_k$. By (1) we may assume that $|C_i| = 1$ for $1 \leq i \leq k - 1$.

(2) *$k = 3$, and $|C_k| > 1$, and every $x \in X$ has a neighbour in C_k.*

For since Y is split and $|C_i| = 1$ for $1 \leq i < k$, it follows that $k \geq 3$. Since G is not decomposable, it does not admit a 0-join, and so $X \neq \emptyset$. Choose $x_0 \in X$, with neighbours in C_i, C_j say. Since $Y \cup \{x_0\}$ is not split, it follows that $|Y \setminus (C_i \cup C_j)| \leq 1$, and so $k = 3$. Since $|C_i| = 1$ for $1 \leq i < k$, and $|Y| \geq 4$, it follows that $|C_k| \geq 2$. Every $x \in X$ therefore has a neighbour in C_k, since $Y \cup \{x\}$ is not split. This proves (2).

For $i = 1, 2$ let X_i be the set of vertices in X with a neighbour in C_i. Thus $X = X_1 \cup X_2$. If $x \in X_1 \cap X_2$ then since x has a neighbour in C_3 it follows that G contains a claw, a contradiction. Thus $X_1 \cap X_2 = \emptyset$. Let $x_i \in X_i$ $(i = 1, 2)$. Since x_i, c_i are adjacent for $i = 1, 2$, it follows from (1) that x_1, x_2 are adjacent. Moreover, if $c \in C_3$ is adjacent to x_1, then since $\{x_1, c, c_1, x_2\}$ is not a claw, it follows that c is strongly adjacent to x_2; and so every vertex in C_3 is either strongly adjacent to both x_1, x_2 or strongly antiadjacent to both. Since $X_1, X_2 \neq \emptyset$ (because G does not admit a 0-join) and the same holds for all choices of x_1, x_2, we deduce that $C_3 = M \cup N$, where N, M are the sets of vertices in C_3 that are strongly complete and strongly anticomplete to X respectively. If $n_1, n_2 \in N$ are antiadjacent then $\{x_1, n_1, n_2, c_1\}$ is a claw, where $x_1 \in X_1$; so N is a strong clique. By 4.2 it follows that G is decomposable. This proves 18.11, and therefore completes the proof of 18.1. ■

19 Non-antiprismatic trigraphs

In view of 18.1 and 17.2, to complete the proof of 3.1 it remains to study non-antiprismatic trigraphs G with $\alpha(G) \leq 3$ and with no hole of length > 5, and that is the topic of this section. We need a number of lemmas before the main theorem.

19.1 *Let G be a claw-free trigraph with $\alpha(G) \leq 3$, and let $x, y \in V(G)$ be semiadjacent, such that no vertex is strongly adjacent to both x, y. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$ or G is decomposable.*

Proof. Let C be the set of vertices of G that are antiadjacent to both x, y. Then C is a strong clique since $\alpha(G) \leq 3$, and the result follows from 11.1. ■

19.2 *Let G be a claw-free trigraph, such that there is no hole in G of length > 5, every hole of length 5 is dominating, and $\alpha(G) \leq 3$. Let C be a 5-hole in G with vertices c_1-$\cdots$-c_5-c_1, and let there be hats in positions $1\frac{1}{2}, 2\frac{1}{2}$ respectively. Then G is decomposable.*

Proof. For $i = 1, \ldots, 5$, let C_i be the set of all clones in position i, and let $H_{i+\frac{1}{2}}, S_{i+\frac{1}{2}}$ be the set of all hats and stars in position $i + \frac{1}{2}$ respectively. (These sets are not necessarily disjoint.) Since G has no 6-hole, $H_{i-\frac{1}{2}}$ is strongly anticomplete to $H_{i+\frac{1}{2}}$ for $i = 1, \ldots, 5$. By hypothesis, we may choose $h_1 \in H_{1\frac{1}{2}}$ and $h_2 \in H_{2\frac{1}{2}}$.

(1) *There is no centre for C.*

For suppose that z is a centre for C. Since $\{z, h_1, c_3, c_5\}$ is not a claw, z is antiadjacent to h_1, and similarly z is antiadjacent to h_2. But then $\{c_2, h_1, h_2, z\}$ is a claw, a contradiction. This proves (1).

(2) *The following hold:*

- *$C_1 \cup \{c_1\}$ is strongly antiadjacent to $H_{2\frac{1}{2}}$, and $C_3 \cup \{c_3\}$ is strongly antiadjacent to $H_{1\frac{1}{2}}$, and in particular $C_2 \cap (H_{1\frac{1}{2}} \cup H_{2\frac{1}{2}}) = \emptyset$*
- *$H_{\frac{1}{2}}, H_{3\frac{1}{2}}$ are empty;*
- *at least one of $H_{4\frac{1}{2}}, S_{4\frac{1}{2}}$ is empty; and*
- *$C_4 \cup \{c_4\}$ is strongly complete to $C_5 \cup \{c_5\}$.*

For let $c_1' \in C_1 \cup \{c_1\}$. Then c_1' is adjacent to h_1, c_5, and since $\{c_1', h_1, c_5, h\}$ is not a claw, it follows that c_1', h are strongly antiadjacent for all $h \in H_{2\frac{1}{2}}$. Thus $C_1 \cup \{c_1\}$ is strongly antiadjacent to $H_{2\frac{1}{2}}$, and in particular $C_2 \cap H_{2\frac{1}{2}} = \emptyset$. Similarly $C_3 \cup \{c_3\}$ is strongly antiadjacent to $H_{1\frac{1}{2}}$, and $C_2 \cap H_{1\frac{1}{2}} = \emptyset$. This proves the first assertion. For the second, suppose that there exists $h_3 \in H_{3\frac{1}{2}}$ say. Since $H_{2\frac{1}{2}}$ is strongly anticomplete to $H_{3\frac{1}{2}}$, it follows that h_2, h_3 are strongly antiadjacent. But h_2 is strongly antiadjacent to c_1, as we saw, and similarly to c_4, and so since every 5-hole is dominating, h_1-h_3-c_4-c_5-c_1-h_1 is not a 5-hole (because h_2 has no neighbours in it). Hence h_1, h_3 are antiadjacent. But then $\{h_1, h_2, h_3, c_5\}$ is stable, contradicting that $\alpha(G) \leq 3$. This proves the second assertion. Next, suppose that $h \in H_{4\frac{1}{2}}$ and $s \in S_{4\frac{1}{2}}$. By 9.2, s is strongly antiadjacent to h, h_1, h_2. If h is antiadjacent to both h_1, h_2 then $\{s, h, h_1, h_2\}$ is stable, a contradiction; if h is adjacent to say h_1 and strongly antiadjacent to h_2 then s-c_4-h-h_1-c_1-s is a 5-hole and h_2 has at most one neighbour in it, a contradiction; while if h is adjacent to both h_1, h_2 then $\{h, h_1, h_2, c_4\}$ is a claw, a contradiction. Thus not both $H_{4\frac{1}{2}}, S_{4\frac{1}{2}}$ are nonempty, and this proves the third assertion of (2). For the fourth assertion, suppose that $x \in C_4 \cup \{c_4\}$ and $y \in C_5 \cup \{c_5\}$ are antiadjacent. By 9.2, x is antiadjacent to h_1 and y is antiadjacent to h_2. Since $\{x, y, h_1, h_2\}$ is not stable, we may assume that x is strongly adjacent to h_2, and so $x \neq c_4$; but then x-c_4-y-c_1-c_2-h_2-x is a 6-hole, a contradiction. This proves (2).

Let

$$\begin{aligned}
B_1 &= H_{1\frac{1}{2}} \cup C_1 \cup \{c_1\} \cup S_{\frac{1}{2}} \cup S_{2\frac{1}{2}} \\
B_2 &= H_{2\frac{1}{2}} \cup C_3 \cup \{c_3\} \cup S_{3\frac{1}{2}} \cup S_{1\frac{1}{2}} \\
B_3 &= C_4 \cup C_5 \cup \{c_4, c_5\} \cup S_{4\frac{1}{2}} \cup H_{4\frac{1}{2}} \\
B &= B_1 \cup B_2 \cup B_3.
\end{aligned}$$

(3) *B_1, B_2, B_3 are strong cliques.*

First we show that B_1 is a strong clique. By 9.2, $H_{1\frac{1}{2}} \cup C_1 \cup \{c_1\} \cup S_{\frac{1}{2}}$ is a strong clique, and $S_{2\frac{1}{2}}$ is a strong clique. We must show that every $s \in S_{2\frac{1}{2}}$ is strongly adjacent to every $t \in H_{1\frac{1}{2}} \cup C_1 \cup \{c_1\} \cup S_{\frac{1}{2}}$. But every such t is adjacent to c_2, and antiadjacent to h_2 by (2), and since $\{c_2, h_2, s, t\}$ is not a claw, it follows that s, t are strongly adjacent. This proves that B_1 is a strong clique, and similarly so is B_2. By 5.5, the sets $C_4 \cup \{c_4\}, C_5 \cup \{c_5\}, S_{4\frac{1}{2}}, H_{4\frac{1}{2}}$ are strong cliques; by (2), it follows that $C_4 \cup C_5 \cup \{c_4, c_5\}$ and $S_{4\frac{1}{2}} \cup H_{4\frac{1}{2}}$ are strong cliques; and by 9.2, $C_4 \cup C_5 \cup \{c_4, c_5\}$ is strongly complete to $S_{4\frac{1}{2}} \cup H_{4\frac{1}{2}}$, and therefore B_3 is a strong clique. This proves (3).

(4) *There is no triad T with $|T \cap B| = 2$.*

For suppose that $\{x, y, z\}$ is a triad, where $x, y \in B$ and $z \notin B$. Since C is dominating and has no centre, and $H_{\frac{1}{2}}, H_{3\frac{1}{2}}$ are empty, it follows that $z \in C_2 \cup \{c_2\}$. By 9.2, z is strongly complete to all of $H_{1\frac{1}{2}}$, $H_{2\frac{1}{2}}$, $S_{1\frac{1}{2}}$, $S_{2\frac{1}{2}}$, and so $x, y \notin H_{1\frac{1}{2}} \cup H_{2\frac{1}{2}} \cup S_{1\frac{1}{2}} \cup S_{2\frac{1}{2}}$. If $x \in C_1 \cup \{c_1\}$, then x is adjacent to h_1 by 9.2, and so x-h_1-z-c_3-c_4-c_5-x is a 6-hole, a contradiction. Thus $x \notin C_1 \cup \{c_1\}$, and similarly $x, y \notin C_1 \cup C_3 \cup \{c_1, c_3\}$.

Since B is the union of the three cliques B_1, B_2, B_3, and there is symmetry between B_1, B_2, we may assume that $x \in B_1$, and therefore $x \in S_{\frac{1}{2}}$. Moreover, $y \in B_2 \cup B_3$, and so

$$y \in C_4 \cup C_5 \cup \{c_4, c_5\} \cup S_{3\frac{1}{2}} \cup S_{4\frac{1}{2}} \cup H_{4\frac{1}{2}}.$$

Since $x \in S_{\frac{1}{2}}$, it follows that $z \neq c_2$, and so $z \in C_2$; and $y \neq c_4, c_5$. By 9.2, $y \notin C_5 \cup H_{4\frac{1}{2}}$. Since x, y, z have no common neighbour (since G is claw-free) it follows that y is strongly antiadjacent to c_1, c_2, and so $y \notin S_{3\frac{1}{2}} \cup S_{4\frac{1}{2}}$. We deduce that $y \in C_4$. By 9.2, x is adjacent to h_1, and y is antiadjacent to h_1; but then x-h_1-z-c_3-y-c_5-x is a 6-hole, a contradiction. This proves (4).

Now $\{h_1, h_2, c_4\}$ and $\{h_1, h_2, c_5\}$ are triads, both contained in B and sharing two vertices. From 16.1, we deduce that G is decomposable. This proves 19.2. ∎

Let G be a trigraph. We say a triple (A_1, A_2, A_3) is a *spread* in G if

- A_1, A_2, A_3 are nonempty strong cliques, pairwise disjoint and pairwise anticomplete
- $|A_1| + |A_2| + |A_3| \geq 4$
- every vertex in $V(G) \setminus (A_1 \cup A_2 \cup A_3)$ is anticomplete to at most one of A_1, A_2, A_3.

If (A_1, A_2, A_3) is a spread, no vertex has neighbours in all three of A_1, A_2, A_3 since G is claw-free. For $1 \leq i, j \leq 3$ with $i \neq j$, let $M_{i,j}$ be the set of all vertices in $V(G) \setminus (A_1 \cup A_2 \cup A_3)$ that are strongly complete to $A_i \cup A_j$, and let $N_{i,j}$ be the set of all vertices in $V(G) \setminus (A_1 \cup A_2 \cup A_3)$ that are strongly complete to A_i and have both a neighbour and an antineighbour in A_j. Thus $M_{i,j} = M_{j,i}$ but $N_{i,j}$ and $N_{j,i}$ are disjoint. If $\{i, j, k\} = \{1, 2, 3\}$, then $M_{i,j}, N_{i,j}$ are both strongly anticomplete to A_k, since no vertex has neighbours in all three of A_1, A_2, A_3.

19.3 *Let G be claw-free, with $\alpha(G) \leq 3$, with no hole of length > 5, and such that every 5-hole in G is dominating; and let (A_1, A_2, A_3) be a spread. Then*

- *the sets A_1, A_2, A_3, $M_{i,j}$ $(1 \leq i < j \leq 3)$ and $N_{i,j}$ $(1 \leq i \neq j \leq 3)$ are pairwise disjoint and have union $V(G)$*
- *if $i, j, k \in \{1, 2, 3\}$ are distinct, then $N_{i,j}$ is strongly anticomplete to $M_{j,k} \cup N_{j,k}$*
- *if $i, j \in \{1, 2, 3\}$ are distinct, then $N_{i,j}$ is a strong clique*
- *if $i, j, k \in \{1, 2, 3\}$ are distinct, and $M_{j,k} \cup N_{j,k} \cup N_{k,j} \neq \emptyset$, then $N_{i,j}$ is strongly complete to $N_{i,k}$*

- *if $i,j,k \in \{1,2,3\}$ are distinct, and $M_{j,k} \cup N_{j,k} \cup N_{k,j} \neq \emptyset$, then either $N_{j,i}$ is strongly complete to $N_{k,i}$ or G is decomposable*
- *if $i,j,k \in \{1,2,3\}$ are distinct, and some $x \in M_{i,j} \cup N_{j,i}$ has an antineighbour $y \in N_{i,k}$, and G is not decomposable, then $N_{k,j} = M_{j,k} = \emptyset$, and x,y are strongly complete to $N_{j,k}$.*

Proof. For the first claim, clearly these sets are pairwise disjoint. Let $v \in V(G) \setminus (A_1 \cup A_2 \cup A_3)$; we must show that v belongs to one of the given sets. Since no vertex has neighbours in all of A_1, A_2, A_3, we may assume that v has no neighbour in A_3. If it has both an antineighbour $a_1 \in A_1$ and an antineighbour $a_2 \in A_2$, then $\{v, a_1, a_2, a_3\}$ is a stable set of size 4 (for any $a_3 \in A_3$), contradicting that $\alpha(G) \leq 3$. Thus we may assume that v is strongly A_1-complete. From the third condition in the definition of a spread, v has a neighbour in A_2. If v is strongly A_2-complete then $v \in M_{1,2}$, and otherwise $v \in N_{1,2}$, and in either case the theorem holds. This proves the first claim of the theorem.

For the second claim, suppose that $x \in N_{i,j}$ is adjacent to $y \in M_{j,k} \cup N_{j,k}$. Choose $a_j \in A_j$ antiadjacent to x, and choose $a_k \in A_k$ adjacent to y. Then $\{y, x, a_j, a_k\}$ is a claw, a contradiction. This proves the second statement.

For the third, let $i,j,k \in \{1,2,3\}$ be distinct, and suppose that $x,y \in N_{i,j}$ are antiadjacent. Let $a_i \in A_i$ and $a_k \in A_k$. By 18.2, there is a path x-p-q-y with $p,q \in A_j$. Then x-p-q-y-a_i-x is a 5-hole, and a_k has no strong neighbour in it, and therefore has no neighbour in it since G is claw-free, a contradiction. This proves the third claim.

For the fourth claim, suppose that $x \in N_{i,j}$ is antiadjacent to $y \in N_{i,k}$, and there exists $z \in M_{j,k} \cup N_{j,k} \cup N_{k,j}$. There is a path between x,z with interior in A_j, and a path between z,y with interior in A_k; let P be the path formed by the union of these two paths. Let $a_i \in A_i$; then P can be completed to a hole C via y-a_i-x. Since G has no hole of length > 5, C has length ≤ 5, and so P has length ≤ 3. Since z belongs to P, we may assume that no vertex of A_j is in P. Let $a_j \in A_j$ be an antineighbour of x. Then a_j has at most one strong neighbour in C, and therefore it has no neighbour in C at all; and since every 5-hole is dominating, it follows that C has length 4. Consequently P is x-z-y. Now z is strongly complete to one of A_j, A_k, say A_j; and so $z \in M_{j,k} \cup N_{j,k}$, and yet $x \in N_{i,j}$ and x,z are adjacent, contrary to the second assertion above. This proves the fourth claim.

For the fifth claim, suppose that $x \in N_{i,k}$ is antiadjacent to some $y \in N_{j,k}$. By hypothesis there exist $a_i \in A_i$ and $a_j \in A_j$, and a vertex $z \in M_{j,k} \cup N_{j,k} \cup N_{k,j}$ adjacent to a_i, a_j. Hence there is a path P between x,y with interior in $\{a_i, a_j, z\}$, using z. Since x,y are both not strongly complete and not strongly anticomplete to A_k, it follows from 18.2 that there is a path Q of length 3 between x,y with interior in A_k. The union of P,Q is a hole, and since G has no hole of length > 5 it follows that P has length 2, and therefore x,y are adjacent to z. Relative to this 5-hole, a_i, a_j are hats in consecutive positions, and therefore G is decomposable by 19.2. This proves the fifth claim.

For the sixth claim, suppose that $x \in M_{i,j} \cup N_{j,i}$ and $y \in N_{i,k}$ are antiadjacent. Choose $a_i \in A_i$ adjacent to x. Choose $z \in M_{j,k} \cup N_{j,k} \cup N_{k,j}$ (if there is no such z then the claim is vacuously true). Suppose first that z is antiadjacent to x. Let $a_j \in A_j$ be adjacent to z, and take a path P between y,z with interior in A_k. Then x-a_i-y-P-z-a_j-x is a hole, and since every hole has length at most five, it follows that P has length 1, and so y,z are adjacent. But in that case the hole has length five, and since every 5-hole is dominating, 10.2 implies that every vertex in A_k has two consecutive strong neighbours in the hole, and in particular, every vertex in A_k is strongly complete to y, contradicting that $y \in N_{i,k}$. This proves that x is strongly adjacent to z. Suppose that y,z are antiadjacent. If z is not strongly complete to A_k, there is a three-edge path between y,z with interior in A_k, by 18.2, and it can be completed via z-x-a_i-y to a 6-hole, a contradiction. Hence z is strongly complete to A_k. Choose $a_k \in A_k$ strongly adjacent to y (this exists since y is anticomplete to only one of A_1, A_2, A_3, namely A_j), and $a'_k \in A_k$ antiadjacent to y. Thus a_k, a'_k are distinct, and x-a_i-y-a_k-z-x is a 5-hole, and a'_k, a_j are hats in consecutive positions, (where $a_j \in A_j$), and the result follows from 19.2. We may therefore assume that y, z are strongly adjacent. By the second claim above, $N_{i,k}$ is strongly anticomplete to $M_{j,k} \cup N_{k,j}$, and so $z \in N_{j,k}$. This proves that $N_{k,j} = M_{j,k} = \emptyset$, and that x, y are strongly complete to $N_{j,k}$, and therefore proves 19.3. ■

With notation as before, a spread (A_1, A_2, A_3) is *poor* if $M_{1,2} = N_{1,2} = N_{2,1} = \emptyset$.

19.4 *Let G be claw-free, with $\alpha(G) \leq 3$, with no hole of length > 5 and such that every 5-hole in G is dominating. If G has a poor spread then either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$ or G is decomposable.*

Proof. We assume that G is not decomposable. Choose a poor spread (A_1, A_2, A_3) with $|A_3|$ maximum, and define $M_{i,j}$ etc. as before. If some vertex in A_1 is semiadjacent to some vertex in A_2, then they have no common neighbours, and the result follows from 19.1. Thus we may assume that A_1, A_2 are strongly anticomplete.

(1) *$N_{3,1}, N_{3,2}$ are both empty.*

For suppose that $N_{3,1}$ is nonempty, and choose $x \in N_{3,1}$ with as few strong neighbours in A_1 as possible. Let Y be the set of vertices in A_1 strongly adjacent to x. Let X be the set of all vertices in $N_{3,1}$ that are strongly complete to Y and anticomplete to $A_1 \setminus Y$; thus, $x \in X$. Define $A'_3 = A_3 \cup X$, $A'_1 = A_1 \setminus Y$, and $A'_2 = A_2$. We claim that (A'_1, A'_2, A'_3) is a poor spread. For certainly A'_1, A'_2 are strong cliques, and so is A'_3 from the third statement of 19.3; and since $Y \neq A_1$, it follows that A'_1, A'_2, A'_3 are all nonempty. Moreover, A'_1, A'_2, A'_3 are pairwise anticomplete. Suppose that $v \in V(G) \setminus (A'_1 \cup A'_2 \cup A'_3)$, and is anticomplete to two of A'_1, A'_2, A'_3 (and therefore strongly complete to the third, since $\alpha(G) \leq 3$; let the third be A'_i say). Consequently $v \notin A_1 \setminus Y, A_2, A_3$. Moreover, every vertex in Y is strongly adjacent to x and to each vertex in A'_1, and so $v \notin Y$. Hence $v \notin A_1, A_2, A_3$, and therefore v has strong neighbours in two of A_1, A_2, A_3 since (A_1, A_2, A_3) is a spread. Since this spread is poor, v has a strong neighbour in A_3 and in $A_1 \cup A_2$. Hence v has a strong neighbour in A'_3, and so $i = 3$ and v is strongly complete to A'_3 and anticomplete to A'_1, A'_2. Since $A'_2 = A_2$, v has no strong neighbour in A_2, and therefore it has a strong neighbour in A_1; and since it has none in A'_1, it follows that $v \in N_{3,1}$, and every strong neighbour of v in A_1 belongs to Y. From the choice of x, v is strongly complete to Y, and so $v \in X$, contradicting that $v \notin A'_1$. This proves that (A'_1, A'_2, A'_3) is a spread. Since (A_1, A_2, A_3) is poor, $M_{1,2} = N_{1,2} = N_{2,1} = \emptyset$. Since also A_1, A_2 are stronly anticomplete, it follows that no vertex of G has neighbours in both A'_1, A'_2, and therefore the spread (A'_1, A'_2, A'_3) is poor. But this contradicts the maximality of $|A_3|$. Hence $N_{3,1} = \emptyset$, and similarly $N_{3,2} = \emptyset$. This proves (1).

Choose $a_i \in A_i$ for $i = 1, 2$. For $i = 1, 2$, let P_i be the set of members of $M_{i,3}$ with an antineighbour in $N_{i,3}$, and let Q_i be the set of members of $N_{i,3}$ with an antineighbour in $M_{i,3}$. Note that, by the second assertion of 19.3, $N_{1,3}$ is strongly anticomplete to $M_{2,3}$, and $N_{2,3}$ is strongly anticomplete to $M_{1,3}$.

(2) *P_1 is strongly complete to $M_{2,3}$, and P_2 is strongly complete to $M_{1,3}$. Moreover, Q_1 is strongly*

complete to $N_{2,3}$, and Q_2 is strongly complete to $N_{1,3}$.

For if $p_1 \in P_1$ has an antineighbour $x \in M_{2,3}$, choose $q_1 \in Q_1$ antiadjacent to p_1, and let $a_3 \in A_3$ be adjacent to q_1. Then $\{a_3, p_1, q_1, x\}$ is a claw, a contradiction. This proves the first assertion, and the second follows by symmetry. For the third, suppose that $q_1 \in Q_1$ has an antineighbour $x \in N_{2,3}$; let $p_1 \in P_1$ be antiadjacent to q_1, and let $a_3 \in A_3$ be adjacent to q_1. Then a_1-p_1-a_3-q_1-a_1 is a 4-hole, and since x, a_2 are adjacent and a_2 has no strong neighbour in this 4-hole, it follows that x has two strong neighbours in this 4-hole, by 18.4 and 10.2. But x is antiadjacent to q_1, p_1, a_1, a contradiction. This proves the third claim, and the fourth follows by symmetry. This proves (2).

(3) *Either $M_{1,3}$ is strongly complete to $N_{1,3}$ or $M_{2,3}$ is strongly complete to $N_{2,3}$.*

For suppose not; then P_1, Q_1, P_2, Q_2 are all nonempty. For $i = 1, 2$ choose $p_i \in P_i$ and $q_i \in Q_i$, antiadjacent. By (2), p_1 is adjacent to p_2 and q_1 to q_2. But then a_1-p_1-p_2-a_2-q_2-q_1-a_1 is a 6-hole, a contradiction. This proves (3).

(4) *We may assume that $N_{1,3}, N_{2,3}$ are both nonempty, and $M_{1,3}, M_{2,3}$ are both strong cliques.*

For suppose that, say, $N_{2,3} = \emptyset$. If also $M_{2,3} = \emptyset$, then since G admits no 0-join, it follows that there exist vertices in A_2, A_3 that are semiadjacent; but these two vertices have no common neighbours, and the theorem holds by 19.1. Thus we may assume that there exists $m \in M_{2,3}$. Let S, T be the set of all $v \in M_{1,3} \cup N_{1,3} \cup A_1 \cup A_3$ that are strongly $M_{2,3}$-complete and strongly $M_{2,3}$-anticomplete respectively. Thus $A_3 \subseteq S$ and $A_1 \cup N_{1,3} \subseteq T$. We claim that (S, T) is a homogeneous pair. First let us see that S, T are strong cliques. If $s_1, s_2 \in S$ are antiadjacent, then $\{m, s_1, s_2, a_2\}$ is a claw, a contradiction; so S is a strong clique. If $t_1, t_2 \in T$ are antiadjacent, then since $A_1 \cup N_{1,3}$ is a strong clique, it follows that at least one of $t_1, t_2 \in M_{1,3}$, and therefore t_1, t_2 have a common neighbour in A_3, say a_3; but then $\{a_3, t_1, t_2, m\}$ is a claw, a contradiction. This proves that S, T are both strong cliques. Now suppose that $v \in V(G) \setminus (S \cup T)$. We claim that v is either strongly S-complete or strongly S-anticomplete, and either strongly T-complete or strongly T-anticomplete. Since $v \notin S \cup T$ it follows that $v \notin A_3 \cup A_1 \cup N_{1,3}$, and if $v \in A_2 \cup M_{2,3}$ the claim holds, so we may assume that $v \in M_{1,3}$. Since $v \notin T$, it has a neighbour $x \in M_{2,3}$ say; and since every $s \in S$ is adjacent to x, and $\{x, s, v, a_2\}$ is not a claw, it follows that v is strongly complete to S. Since $v \notin S$, it has an antineighbour $y \in M_{2,3}$. If $t \in T$ is antiadjacent to v, then $t \notin A_1$, and so t has a neighbour $a_3 \in A_3$; then $\{a_3, v, t, y\}$ is a claw, a contradiction. Thus v is strongly T-complete. This proves that (S, T) is a homogeneous pair, nondominating because $A_2 \neq \emptyset$. By 4.3, it follows that $|S|, |T| \leq 1$. Hence $|A_1| = |A_3| = 1$ and $N_{1,3} = \emptyset$. By exchanging A_1, A_2, we deduce that $|A_2| = 1$, contradicting the definition of a spread.

This proves that $N_{1,3}, N_{2,3}$ are both nonempty. If there exist $x, y \in M_{1,3}$, antiadjacent, choose $z \in N_{2,3}$, let $a_3 \in A_3$ be a neighbour of z, and then $\{a_3, x, y, z\}$ is a claw, a contradiction. Thus $M_{1,3}$ is a strong clique, and similarly $M_{2,3}$ is a strong clique. This proves (4).

(5) *$M_{i,3} \subseteq P_i$ for $i = 1, 2$.*

For by (3) and the symmetry, we may assume that $M_{2,3}$ is strongly complete to $N_{2,3}$. Define

$V_1 = (M_{1,3} \setminus P_1) \cup M_{2,3}$, and $V_2 = V(G) \setminus V_1$. If $V_1 = \emptyset$ then the claim holds, so we may assume that $V_1 \neq \emptyset$; and clearly $V_2 \neq \emptyset$. We claim that G is the hex-join of $G|V_1$ and $G|V_2$. For V_1 is the union of the two strong cliques $M_{1,3} \setminus P_1$ and $M_{2,3}$, and V_2 is the union of the three strong cliques $N_{2,3} \cup A_2$, $N_{1,3} \cup A_1$ and $P_1 \cup A_3$. Since $M_{1,3} \setminus P_1$ is strongly anticomplete to $N_{2,3} \cup A_2$ and strongly complete to $N_{1,3} \cup A_1$ and $P_1 \cup A_3$, and $M_{2,3}$ is strongly anticomplete to $N_{1,3} \cup A_1$ and strongly complete to $N_{2,3} \cup A_2$ and $P_1 \cup A_3$, it follows that G is a hex-join and therefore decomposable, a contradiction. This proves (5).

(6) *$M_{1,3} = M_{2,3} = \emptyset$.*

For from (3) we may assume that $P_2 = \emptyset$, and therefore from (5) $M_{2,3} = \emptyset$. Suppose that $M_{1,3} \neq \emptyset$. By (5), $P_1 \neq \emptyset$, and therefore $Q_1 \neq \emptyset$. Choose $p_1 \in P_1$ and $q_1 \in Q_1$, antiadjacent. If $x \in N_{1,3}$ and $y \in N_{2,3}$ are adjacent, and $a_3 \in A_3$, then since $\{x, a_1, a_3, y\}$ and $\{y, a_3, a_2, x\}$ are not claws, it follows that a_3 is either strongly complete or strongly anticomplete to $\{x, y\}$. Consequently x, y have the same neighbours in A_3, for every such adjacent pair x, y. Let Z be the set of neighbours of q_1 in A_3 (so $Z \neq \emptyset$ since $q_1 \in N_{1,3}$). By (2), q_1 is strongly complete to $N_{2,3}$, and therefore every vertex in $N_{2,3}$ is strongly complete to Z and strongly anticomplete to $A_3 \setminus Z$. In particular, every vertex in A_3 is either strongly complete or strongly anticomplete to $N_{2,3}$. We claim that every vertex $x \in V(G) \setminus N_{2,3}$ is either strongly complete or strongly anticomplete to $N_{2,3}$. For suppose not; then $x \in N_{1,3} \setminus Q_1$. Since x has a neighbour in $N_{2,3}$, it follows as before that x is strongly complete to Z and strongly anticomplete to $A_3 \setminus Z$. Let $y \in N_{2,3}$ be antiadjacent to x. Choose $z \in Z$, and $a_3 \in A_3$ antiadjacent to x (such a vertex a_3 exists since $x \in N_{1,3}$.) Thus $a_3 \notin Z$, and so y is antiadjacent to a_3; but then $\{z, a_3, x, y\}$ is a claw, a contradiction. This proves our claim that every vertex in $V(G) \setminus N_{2,3}$ is either strongly complete or strongly anticomplete to $N_{2,3}$. Hence every vertex in $V(G) \setminus (N_{2,3} \cup A_2)$ is either strongly complete or strongly anticomplete to $N_{2,3}$, and anticomplete to A_2. Suppose that there exist $a_2' \in A_2$ and $a_3' \in A_3$ that are semiadjacent. If $a_3' \in Z$, then a_3' is adjacent to q_1, and so $\{a_3', p_1, q_1, a_2'\}$ is a claw, a contradiction. If $a_3' \notin Z$, then a_2', a_3' have no common neighbours and the result follows from 19.1. Thus we may assume that A_2 is strongly anticomplete to A_3. By 4.2 it follows that G is decomposable, a contradiction. Hence $M_{1,3} = \emptyset$. This proves (6).

(7) *If A_1, A_2 are strongly anticomplete to A_3 then the result holds.*

For then A_1, A_2 are both homogeneous sets, and so have cardinality one; and for $i = 1, 2$, the set of neighbours of a_i is $N_{i,3}$, which is a strong clique. Moreover, the set of vertices antiadjacent to both a_1, a_2 is A_3, which is also a strong clique, and the result follows from 11.2.

In view of (7) we may assume that $a_1 \in A_1$ is semiadjacent to some $a_3 \in A_3$. Now if a_3 is adjacent to some $v \in N_{2,3}$ (and hence a_3, v are strongly adjacent since $F(G)$ is a matching) choose $u \in A_3$ antiadjacent to v (this exists, since $v \in N_{2,3}$, and is different from a_3 since v is strongly adjacent to a_3); then $\{a_3, u, v, a_1\}$ is a claw, a contradiction. Hence a_3 is strongly anticomplete to $N_{2,3}$. Let S be the set of neighbours of a_3 in $N_{1,3}$. Since $S \cup N_{2,3} \cup \{a_3, a_1\}$ includes no claw, it follows that S is strongly anticomplete to $N_{2,3}$. Since we may assume that A_3 is not an internal clique cutset, by 4.1, it follows that $S \neq N_{1,3}$. Let $n_1 \in N_{1,3} \setminus S$, and let Z be the set of neighbours of n_1 in A_3. Thus $a_3 \notin Z \neq \emptyset$. For each $z \in Z$, a_1-n_1-z-a_3-a_1 is a 4-hole, and by 18.4 and 10.2, applied to the pair n_2a_2, it follows that every $n_2 \in N_{2,3}$ has two strong neighbours in this 4-hole, and therefore is strongly adjacent to n_1, z. Hence $N_{2,3}$ is strongly complete to $N_{1,3} \setminus S$. If $x \in N_{1,3} \setminus S$ and $y \in N_{2,3}$, and $a_3' \in A_3$, then since $\{x, a_1, a_3', y\}$ and $\{y, a_3', a_2, x\}$ are not claws, it follows that a_3' is either strongly complete or strongly anticomplete to $\{x, y\}$. Consequently x, y have the same neighbours and the same strong neighbours in A_3, for every such pair x, y. Hence $N_{1,3} \setminus S, N_{2,3}$ are both strongly complete to Z and strongly anticomplete to $A_3 \setminus Z$. But then $(N_{1,3} \setminus S) \cup Z$ is an internal clique cutset and the result follows from 4.1. This proves 19.4. ■

Now we can prove the main result of this section.

19.5 *Let G be a claw-free trigraph, with $\alpha(G) \leq 3$, with no hole of length > 5 and such that every 5-hole in G is dominating. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6 \cup \mathcal{S}_7$, or G is decomposable.*

Proof. We assume that G is not decomposable and not antiprismatic. We claim that G contains a spread. For since G is not antiprismatic, and $\alpha(G) \leq 3$, it follows that there are three strong cliques A_1, A_2, A_3, all nonempty and pairwise disjoint and anticomplete, such that $|A_1 \cup A_2 \cup A_3| \geq 4$. Choose three such cliques with $|A_3|$ maximum (thus $|A_3| \geq 2$), and subject to that with $A_1 \cup A_2 \cup A_3$ maximal. Since $\alpha(G) \leq 3$, every vertex $v \notin A_1 \cup A_2 \cup A_3$ is strongly complete to at least one of A_1, A_2, A_3, and therefore from the maximality of $A_1 \cup A_2 \cup A_3$, v has strong neighbours in two of A_1, A_2, A_3. Consequently (A_1, A_2, A_3) is a spread. Define the sets $M_{i,j}, N_{i,j}$ as before. By 19.4, we may assume that the spreads (A_1, A_2, A_3), (A_2, A_3, A_1), (A_3, A_1, A_2) are not poor.

(1) *$N_{1,2} \cup N_{2,1} \cup M_{1,2}$ is a strong clique.*

For suppose that there are two antiadjacent vertices in this set, say x, y. Since x, y both have neighbours in A_1, and both have neighbours in A_2, there is a hole C containing x, y with $V(C) \subseteq A_1 \cup A_2 \cup \{x, y\}$. No vertex of A_3 has a strong neighbour in C, and since G has no hole of length > 5 and every 5-hole is dominating, it follows that C has length 4. But this contradicts 18.4 and 10.2 (applied to two vertices in A_3). This proves (1).

(2) *$N_{3,1} = N_{3,2} = \emptyset$; $N_{1,2}$ is strongly complete to $M_{1,3}$; and $N_{2,1}$ is strongly complete to $M_{2,3}$.*

For suppose that there exists $x \in N_{3,1}$. Choose $a_1 \in A_1$ antiadjacent to x. Then the strong cliques $\{a_1\}$, A_2, and $A_3 \cup \{x\}$ are pairwise disjoint, and pairwise anticomplete, contradicting the maximality of $|A_3|$. Hence $N_{3,1} = N_{3,2} = \emptyset$. Now suppose that $x \in N_{1,2}$ has an antineighbour $y \in M_{1,3}$. Let $a_2 \in A_2$ be an antineighbour of x. Then the three strong cliques $\{x\}$, $\{a_2\}$ and $A_3 \cup \{y\}$ again contradict the choice of A_1, A_2, A_3. This proves (2).

(3) *Either $M_{1,3}$ is strongly complete to $N_{1,3}$, or $M_{2,3}$ is strongly complete to $N_{2,3}$.*

For suppose that for $i = 1, 2$ there exist $m_i \in M_{i,3}$ and $n_i \in N_{i,3}$, antiadjacent, and choose $a_i \in A_i$ for $i = 1, 2$. Now n_1, n_2 are adjacent, by the fifth assertion of 19.3. If m_1, m_2 are adjacent, then m_1-a_1-n_1-n_2-a_2-m_2-m_1 is a 6-hole, a contradiction. Thus m_1, m_2 are antiadjacent, and so m_1-a_1-n_1-n_2-a_2-m_2 is a path P of length 5. Choose $a_3 \in A_3$ antiadjacent to n_1. Then since $\{n_2, n_1, a_3, a_2\}$ is not a claw, a_3 is antiadjacent to n_2; and so P can be completed to a 7-hole via m_2-a_3-m_1, a contradiction. This proves (3).

For $i = 1, 2$, let X_i be the set of all vertices in $M_{1,2}$ with an antineighbour in $N_{i,3}$. Let $X_0 = M_{1,2} \setminus (X_1 \cup X_2)$.

(4) *$X_1 \cap X_2 = \emptyset$, and one of $X_1, M_{2,3} = \emptyset$, and one of $X_2, M_{1,3} = \emptyset$.*

For if $x \in X_1$, then by the sixth claim of 19.3, $M_{2,3} = \emptyset$, and x is strongly complete to $N_{2,3}$; and therefore $x \notin X_2$. This proves (4).

(5) *At least one of $M_{1,3}, M_{2,3}$ is nonempty.*

For suppose not. Since $N_{1,3} \cup N_{2,3}$ is a strong clique, by the third and fifth claims of 19.3, and since G is not decomposable, it follows from 4.1 that $N_{1,3} \cup N_{2,3}$ is not an internal clique cutset. Hence some vertex in A_3 is semiadjacent to some vertex in $A_1 \cup A_2$, say $a_3 \in A_3$ is semiadjacent to $a_1 \in A_1$. By 19.1, there exists $n_1 \in N_{1,3}$ adjacent to both a_1, a_3. Choose $n_2 \in N_{2,3}$ (and therefore adjacent to n_1), and choose $a_3' \in A_3$ antiadjacent to n_2. If n_2, a_3 are antiadjacent then $\{n_1, a_1, a_3, n_2\}$ is a claw, and if n_2, a_3 are adjacent then $\{a_3, n_2, a_3', a_1\}$ is a claw, in either case a contradiction. This proves (5).

(6) *It is not the case that $M_{1,3}$ is strongly complete to $N_{1,3}$ and $M_{2,3}$ is strongly complete to $N_{2,3}$.*

For let $B_1 = A_1 \cup N_{1,2} \cup N_{1,3} \cup X_2$, $B_2 = A_2 \cup N_{2,1} \cup N_{2,3} \cup X_1$, and $B_3 = A_3$. Then B_1, B_2, B_3 are disjoint strong cliques, by 19.3 and (4), and their union is not $V(G)$, by (5); and since $\{a_1, a_2, a_3\}$ is a triad for each choice of $a_i \in A_i$ ($i = 1, 2, 3$), and there are at least two such vertices a_3, it follows from 16.1 that there is a triad $\{t_1, t_2, t_3\}$ with $t_1, t_2 \in B_1 \cup B_2 \cup B_3$ and $t_3 \notin B_1 \cup B_2 \cup B_3$ (and therefore $t_3 \in M_{1,3} \cup M_{2,3} \cup X_0$). Not both $t_1, t_2 \in B_3$, so we may assume from the symmetry that $t_1 \in B_1$. Since X_0 is strongly complete to B_1, it follows that $t_3 \notin X_0$, and so t_3 is strongly complete to A_3, and therefore $t_2 \notin A_3$. Hence $t_2 \in B_2$, and $t_3 \in M_{1,3} \cup M_{2,3}$, and from the symmetry we may assume that $t_3 \in M_{1,3}$. By (4), $X_2 = \emptyset$, and since $M_{1,3}$ is strongly complete to $N_{1,3} \cup A_1$, it follows that $t_1 \in N_{1,2}$. Since $t_3 \in M_{1,3}$, we deduce that $N_{1,2}$ is not strongly complete to $M_{1,3}$, contrary to (2). This proves (6).

In view of (3) and (6), we may assume that $M_{1,3}$ is strongly complete to $N_{1,3}$ and $M_{2,3}$ is not strongly complete to $N_{2,3}$. In particular, $M_{2,3} \neq \emptyset$, and so $X_1 = \emptyset$ by (4).

(7) *$|A_1| = 1$ and $N_{1,2} = N_{2,1} = \emptyset$.*

For choose $n_2 \in N_{2,3}$ and $m_2 \in M_{2,3}$, antiadjacent; let $a_3 \in A_3$ be adjacent to n_2, and choose $a_2 \in A_2$. Then a_2-m_2-a_3-n_2-a_2 is a 4-hole; no member of A_1 has a strong neighbour in it, and no member of $N_{1,2}$ has two strong members in it, by the second assertion of 19.3; and so by 18.4 and 10.2 it follows that $|A_1| = 1$ and $N_{1,2} = \emptyset$. Since every member of $N_{2,1}$ has a strong neighbour in A_1 and an antineighbour in A_1, it follows that $N_{2,1} = \emptyset$. This proves (7).

Let $A_1 = \{a_1\}$, and let Y be the set of all vertices in $M_{2,3}$ with an antineighbour in $N_{2,3}$.

(8) *Y is strongly complete to $M_{1,3}$; Y is strongly anticomplete to X_0; and Y is a strong clique.*

For let $y \in Y$, and choose $x \in N_{2,3}$ antiadjacent to y. If y is antiadjacent to some $m \in M_{1,3}$, choose $a_3 \in A_3$ adjacent to x; then $\{a_3, x, y, m\}$ is a claw, a contradiction. Thus Y is strongly complete to $M_{1,3}$. If y is adjacent to some $m \in X_0$, then $\{m, x, y, a_1\}$ is a claw, a contradiction. Now suppose that there exist antiadjacent $y_1, y_2 \in Y$. Since the spread (A_3, A_1, A_2) is not poor, one of $M_{1,3}, N_{1,3}$ is nonempty. If there exists $n \in N_{1,3}$, let $a_3 \in A_3$ be adjacent to n; then $\{a_3, n, y_1, y_2\}$ is a claw, a contradiction. Thus there exists $m \in M_{1,3}$, adjacent to y_1, y_2 since Y is complete to $M_{1,3}$. But then $\{m, a_1, y_1, y_2\}$ is a claw, a contradiction. Thus Y is a strong clique. This proves (8).

Let $B_1 = A_1 \cup N_{1,3} \cup X_2$, $B_2 = A_2 \cup N_{2,3}$ and $B_3 = A_3 \cup Y$. By (1), (2), (4), (8) and 19.3, these three sets are all strong cliques.

(9) *$B_1 \cup B_2 \cup B_3 = V(G)$.*

For suppose not. Since $\{a_1, a_2, a_3\}$ is a triad for all $a_i \in A_i\,(i = 2, 3)$, 16.1 implies that there is a triad $\{t_1, t_2, t_3\}$ with $t_1, t_2 \in B_1 \cup B_2 \cup B_3$ and $t_3 \notin B_1 \cup B_2 \cup B_3$. It follows that $t_3 \in X_0 \cup M_{1,3} \cup (M_{2,3} \setminus Y)$. Now X_0 is strongly complete to $B_1 \cup B_2$, and $M_{1,3}$ is strongly complete to $B_1 \cup B_3$, by (4) and (8); and therefore $t_3 \in M_{2,3} \setminus Y$. Hence t_3 is strongly complete to B_2, and so we may assume that $t_1 \in B_1$ and $t_2 \in B_3$. Since t_3 is strongly complete to A_3, it follows that $t_2 \in Y$. If there exists $n \in N_{1,3}$, let $a_3 \in A_3$ be adjacent to n, and then $\{a_3, n, t_2, t_3\}$ is a claw, a contradiction. Thus $N_{1,3} = \emptyset$. Since (A_3, A_1, A_2) is not poor, there exists $m_1 \in M_{1,3}$. By (4), $X_2 = \emptyset$, and so $X_0 = M_{1,2}$. For $a \in A_2 \cup A_3$, since $\{a, a_1, t_2, t_3\}$ is not a claw, it follows that a_1, a are strongly antiadjacent; and so a_1 is strongly anticomplete to A_2, A_3. Choose $m_2 \in M_{1,2}$. Let $Z = A_2 \cup A_3 \cup M_{2,3} \cup N_{2,3}$; thus, A_1 is strongly anticomplete to Z. Let P be the set of all vertices in Z strongly complete to $M_{1,2}$ and strongly anticomplete to $M_{1,3}$, and let Q be the set of all vertices in Z that are strongly complete to $M_{1,3}$ and strongly anticomplete to $M_{1,2}$. Since m_1, m_2 exist, it follows that $P \cap Q = \emptyset$. Moreover, $A_2 \cup N_{2,3} \subseteq P$, and $A_3 \cup Y \subseteq Q$, by (8). If $p_1, p_2 \in P$ are antiadjacent, then $\{m_2, a_1, p_1, p_2\}$ is a claw, while if $q_1, q_2 \in Q$ are antiadjacent then $\{m_1, a_1, q_1, q_2\}$ is a claw, in either case a contradiction; thus, P, Q are strong cliques. We claim that (P, Q) is a homogeneous pair. For let $v \in V(G) \setminus (P \cup Q)$. We claim that v is either strongly complete or strongly anticomplete to P, and either strongly complete or strongly anticomplete to Q. This is true if $v \notin Z$, so we assume that $v \in Z$, and consequently $v \in Z \setminus (P \cup Q) \subseteq M_{2,3} \setminus Y$. Suppose first that v has an antineighbour $p \in P$. Since v is strongly complete to $A_2 \cup A_3 \cup N_{2,3}$, it follows that $p \in M_{2,3}$. If v has a neighbour $x \in M_{1,2}$, then $\{x, a_1, p, v\}$ is a claw, while if v has an antineighbour $x \in M_{1,3}$ then $\{a_3, x, p, v\}$ is a claw, in either case a contradiction; and otherwise v is strongly complete to $M_{1,3}$ and strongly anticomplete to $M_{1,2}$, and therefore belongs to Q, a contradiction. Thus v is strongly complete to P. Suppose that v has an antineighbour $q \in Q$. Since v is strongly complete to $A_2 \cup A_3 \cup N_{2,3}$, it follows that $q \in M_{2,3}$. If v has a neighbour $x \in M_{1,3}$ then $\{x, a_1, v, q\}$ is a claw, and if v has an antineighbour $x \in M_{1,2}$ then $\{a_2, x, v, q\}$ is a claw, in either case a contradiction; and otherwise v is strongly anticomplete to $M_{1,3}$ and strongly complete to $M_{1,2}$, and therefore belongs to P, a contradiction. This proves that (P, Q) is a homogeneous pair, nondominating since $A_1 \neq \emptyset$. Since $A_3 \subseteq Q$ and $|A_3| \geq 2$, 4.3 implies that G is decomposable, a contradiction. This proves (9).

From (9) it follows that $X_0 = M_{1,3} = \emptyset$ and $Y = M_{2,3}$. Since (A_3, A_1, A_2) is not poor, $N_{1,3}$ is nonempty; and we have already seen that $M_{2,3}$ is not strongly complete to $N_{2,3}$, and therefore both Y and $N_{2,3}$ are nonempty. If $x \in N_{1,3}$ and $y \in N_{2,3}$, then x, y are adjacent by the fifth claim of

19.3; and if $a_3 \in A_3$, then since $\{x, a_1, a_3, y\}$ and $\{y, a_2, a_3, x\}$ are not claws, it follows that a_3 is adjacent to both or neither of x, y. Consequently x, y have the same neighbours in A_3, and they are both strongly adjacent to all their neighbours in A_3. Since this holds for all choices of x, y, and since $N_{1,3}, N_{2,3}$ are both nonempty, it follows that there exists $Z \subseteq A_3$ such that every vertex in $N_{1,3} \cup N_{2,3}$ is strongly complete to Z and strongly anticomplete to $A_3 \setminus Z$. Since every vertex in $N_{1,3}$ has a neighbour and an antineighbour in A_3, it follows that $\emptyset \neq Z \neq A_3$. If $a_3 \in A_3 \setminus Z$, then no vertex is strongly adjacent to both a_1, a_3, and so a_1, a_3 are not semiadjacent by 19.1; and a_3, a_2 are not semiadjacent where $a_2 \in A_2$ since $\{a_2, a_3, x_2, n_2\}$ is not a claw, where $x_2 \in X_2$ and $n_2 \in N_{2,3}$ are antiadjacent. Thus $A_3 \setminus Z$ is strongly anticomplete to $A_1 \cup A_2$, and so all members of $A_3 \setminus Z$ are twins; and therefore $|A_3 \setminus Z| = 1$. Let $A_3 \setminus Z = \{a_3\}$ say. We claim that all neighbours of a_3 are strongly adjacent to a_3 and to each other. For the set of neighbours of a_3 is $Z \cup Y$, and $Z \cup Y \cup \{a_3\}$ is indeed a strong clique. Moreover, all neighbours of a_1 are strongly adjacent to a_1 and to each other; for a_1 is strongly antiadjacent to all $a_2 \in A_2$ (since $\{a_2, a_1, x, y\}$ is not a claw, where $y \in Y$ and $x \in N_{2,3}$ are antiadjacent), and so the set of neighbours of a_1 is $N_{1,3} \cup X_2$, and $N_{1,3} \cup X_2 \cup \{a_1\}$ is indeed a strong clique. But then the hypotheses of 11.2 are satisfied by the pair a_1, a_3, and the result follows from 11.2. This proves 19.5. ■

Finally, let us explicitly prove the main theorem.

Proof of 3.1. If some hole has length ≥ 6, the result follows from 17.2, so we assume that every hole has length at most five, and in particular, G contains no long prism. By 14.3, we may assume that every 5-hole is dominating. If $\alpha(G) \geq 4$, the result follows from 18.1, and otherwise it follows from 19.5. This proves 3.1. ■

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