Graph Minors. XXI. Graphs with unique linkages

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Abstract

A *linkage* L in a graph G is a subgraph each component of which is a path, and it is *vital* if $V(L) = V(G)$ and there is no other linkage in G joining the same pairs of vertices. We show that, if G has a vital linkage with p components, then G has tree-width bounded above by a function of p. This is the major step in the proof of the unproved lemma from Graph Minors XIII, and it has a number of other applications, including a constructive proof of the intertwining conjecture.

1 Introduction

A *linkage* in a graph G is a subgraph every component of which is a path. (All graphs in this paper are finite and undirected, and may have loops or parallel edges. *Paths* have at least one vertex, and have no "repeated" vertices or edges.) A vertex of G is a *terminal* of a linkage L in G if $v \in V(L)$ and v has degree ≤ 1 in L. The *pattern* of a linkage L is the partition of its terminals in which two terminals are the same block if and only if they belong to the same component of L. A linkage is a *p-linkage* if it has $\leq p$ terminals, where $p \geq 0$ is an integer. A linkage L in G is *vital* if $V(L) = V(G)$, and no linkage in G different from L has the same pattern as L.

A *tree-decomposition* of a graph G is a pair (T, W), where T is a tree and $W = (W_t : t \in V(T))$ is a family of subgraphs of G, satisfying

1. $\bigcup(W_t : T \in V(T)) = G$, and
2. if $t, t', t'' \in V(T)$ and t' lies on the path of T between t and t'', then $W_t \cap W_{t''} \subseteq W_{t'}$.

It has *width* $\leq w$ if $|V(W_t)| \leq w + 1$ for every $t \in V(T)$, and G has *tree-width* $\leq w$ if some tree-decomposition has width $\leq w$.

The main objective of this paper is to prove the following.

1.1 *For every integer $p \geq 0$ there exists $w \geq 0$ such that every graph with a vital p-linkage has tree-width $\leq w$.*

This has a large number of applications. For instance, in section 11 we use it to obtain a constructive proof of the intertwining conjecture, proved non-constructively in [11]. In the next paper of this series we show that it implies theorem (10.2) of [8], which was left unproved in that paper, and which is needed to justify the main algorithm for the p disjoint paths problem described in that paper; and that it also implies, for example, the main theorem of [5].

Despite all these applications, it seems unlikely that such an apparently innocuous statement should need the elaborate proof that we give in this paper; and perhaps (1.1) has an easy proof that we have missed. If $p \leq 5$ (1.1) is indeed easy, and in fact for $p \leq 5$, every graph with a vital p-linkage has path-width $\leq p$. (*Path-width* is defined in the same way as tree-width except that the tree T is required to be a path.) We shall not need this result, and so we omit its proof, but it follows easily by induction from the following.

1.2 *If L is a vital 5-linkage in a simple graph G, then either some two terminals in different components of L are adjacent in G, or some terminal has the same degree in G and in L, or G is null.*

This raises the question of whether (1.1) is true in general with tree-width replaced by path-width; and indeed it is, as we shall show in section 12.

Our proof of (1.1) is as follows. From a theorem of [8] it follows immediately that every graph with a vital p-linkage has no K_n minor, where $n \geq \frac{5}{2}p + 1$. Consequently we can apply the results of [9, 10] concerning the structure of graphs excluding a fixed minor. They "almost" tell us that G has bounded genus. The remainder of the proof falls into two main parts; first we prove it when G really does have bounded genus, and then we fix the gaps implied by "almost".

2 Some basic lemmas

In this section we establish some lemmas about vital linkages that we shall need repeatedly. We use $\setminus$ to denote the result of deletion; thus, $G \setminus X$ is the graph obtained from G by deleting X.

2.1 *If L is a vital p-linkage in G, and $X \subseteq V(G)$, then $L \setminus X$ is a vital $(p + 2|X|)$-linkage in $G \setminus X$.*

The proof is clear.

A *separation* of G is a pair (A, B) of subgraphs of G with union G and with $E(A \cap B) = \emptyset$; its *order* is $|V(A \cap B)|$.

2.2 *If L is a linkage in G with set of terminals X, and (A, B) is a separation of G, then $L \cap B$ is a linkage in B with set of terminals a subset of $(X \setminus V(A)) \cup V(A \cap B)$.*

Again, the proof is clear, as is the proof of the next lemma.

2.3 *If L is a linkage in G, and (A, B) is a separation of G, and L' is a linkage in B with the same pattern as $L \cap B$, then $(L \cap A) \cup L'$ is a linkage in G with the same pattern as L. In particular, if L is a vital linkage in G then $L \cap B$ is a vital linkage in B.*

We need the operation of "splitting" a vertex of a graph. (For the following to make sense, we regard a graph as a triple consisting of a set of vertices, a set of edges, and an appropriate incidence relation between them.) Let v be a vertex of a graph G, and let $\delta_1, \delta_2 \subseteq E(G)$ with $\delta_1 \cap \delta_2 = \emptyset$, so that $\delta_1 \cup \delta_2$ is the set of all edges of G incident with v. Take two new elements v_1, v_2, and let G' be the graph with vertex set $(V(G) \setminus \{v\}) \cup \{v_1, v_2\}$ and edge set $E(G)$, in which an edge e is incident with a vertex $u \in V(G')$ if either $u \neq v_1, v_2$ and e is incident with u in G, or $u = v_i$ and $e \in \delta_i$ for $i = 1$ or 2. We say that G' is obtained from G by *splitting* v (*according to* δ_1, δ_2).

2.4 *If G' is obtained from G by splitting a vertex v, and G has a vital p-linkage, then G' has a vital $(p+2)$-linkage.*

Proof. Let L be a vital p-linkage in G. Let L' be the subgraph of G' with $V(L') = V(G')$ and $E(L') = E(L)$. Then L' is a vital $(p+2)$-linkage as is easily seen. ■

If A, B are graphs, we write $A \subseteq B$ to denote that A is a subgraph of B.

2.5 *Let $p, k \geq 0$ and let $n = (2p+1)(p+k+1)^{2(p+k)} + 1$. Let L be a vital p-linkage in a graph G, and for $1 \leq i \leq n$ let (A_i, B_i) be a separation of G of order k, such that*

1. *$A_i \subseteq A_j$ and $B_j \subseteq B_i$ for $1 \leq i < j \leq n$, and*
2. *for $1 \leq i < n$, there is a linkage M_i in $B_i \cap A_{i+1}$ with k components, each with one end in $V(A_i \cap B_i)$ and the other in $V(A_{i+1} \cap B_{i+1})$.*

Then there exists i with $1 \leq i < n$ such that $L \cap B_i \cap A_{i+1} = M_i$.

Proof. Let Z be the set of terminals of L.

(1) *At most $2p+1$ of the pairs $(V(A_i) \cap Z, V(B_i) \cap Z)$ $(1 \leq i \leq n)$ are distinct.*

Subproof. Let $c_i = |V(A_i) \cap Z| - |V(B_i) \cap Z|$ $(1 \leq i \leq n)$. Then $-p \leq c_i \leq p$, and so at most $2p+1$ of the integers c_i $(1 \leq i \leq n)$ are mutually distinct. But if $i < j$, then $V(A_i) \cap Z \subseteq V(A_j) \cap Z$ and $V(B_i) \cap Z \supseteq V(B_j) \cap Z$, and so if $c_i = c_j$ then

$$(V(A_i) \cap Z, V(B_i) \cap Z) = (V(A_j) \cap Z, V(B_j) \cap Z)$$

as required. This proves (1).

For $1 \leq i \leq n$, let $V(A_i \cap B_i) = \{v_i^1, \ldots, v_i^k\}$, numbered so that for $1 \leq i < n$, the pattern of M_i is $\{\{v_i^1, v_{i+1}^1\}, \ldots, \{v_i^k, v_{i+1}^k\}\}$. Let $Z_i = V(A_i \cap B_i) \cup (Z \cap V(B_i))$, and let $L_i = L \cap B_i$. By (2.2),

Z_i contains every terminal of L_i. Let π_i be the pattern of L_i, and let $\phi_i : Z_i \to Z_1$ be defined by $\phi_i(v_i^t) = v_1^t$ $(1 \le t \le k)$, and $\phi_i(z) = z$ for all $z \in Z_i \setminus V(A_i \cap B_i)$. Now ϕ_i is an injection, and maps π_i to a partition of a subset of Z_1 in which each block has cardinality 1 or 2. Since $|Z_1| \le p + k$, there are at most $(p+k+1)^{2(p+k)}$ such partitions. Since $n > (2p+1)(p+k+1)^{2(p+k)}$, it follows that there exist distinct i, j with $1 \le i, j \le n$, such that $V(A_i) \cap Z = V(A_j) \cap Z, V(B_i) \cap Z = V(B_j) \cap Z$, and $\phi_i(\pi_i) = \phi_j(\pi_j)$.

(2) *For $1 \le t \le k$, the degree of v_i^t in L_i equals the degree of v_j^t in L_j; and if one of v_i^t, v_j^t is in Z then $v_i^t = v_j^t$.*

Subproof. For the first claim, we observe that v_i^t has degree 0, 1 or 2 in L_i, depending where $\{v_i^t\}$ is a block of π_i, a proper subset of a block of π_i, or not a subset of any block of π_i respectively. Since $\phi_i(v_i^t) = \phi_j(v_j^t)$ and $\phi_i(\pi_i) = \phi_j(\pi_j)$, the first claim follows. For the second, suppose that $v_i^t \in Z$. Since $V(A_i) \cap Z = V(A_j) \cap Z$ it follows that $v_i^t \in V(A_j)$ and similarly $v_i^t \in V(B_j)$. Thus $v_i^t = v_j^{t'}$ for some t'. Now $M_1 \cup \ldots \cup M_{n-1}$ is a linkage with k components, and one of them meets $V(A_i \cap B_i)$ in $\{v_i^t\}$ and meets $V(A_j \cap B_j)$ only in $\{v_j^t\}$. Since $v_i^t = v_j^{t'} \in V(A_j \cap B_j)$ it follows that $t' = t$. This proves (2).

We may assume that $i < j$. Let $M = M_i \cup M_{i+1} \cup \ldots \cup M_{j-1}$, and let the components of M be $P^1, \ldots, P^k$, where P^t has ends v_i^t and v_j^t $(1 \le t \le k)$. Let

$$T = \{t : 1 \le t \le k, \text{ and either } v_i^t = v_j^t \text{ or } v_i^t \text{ has degree 1 in } L_i\}.$$

Let M^* be the subgraph of G formed by the vertices in $V(A_i \cap B_i)$, the vertices in $V(A_j \cap B_j)$, and all the paths P_t $(t \in T)$.

(3) *$M^* \cup L_j$ is a linkage in B_i.*

Subproof. It is clearly a forest, and so it suffices to show that it has maximum degree at most 2. Let $v \in V(M^* \cup L_j)$. Since M^* and L_j both have maximum degree ≤ 2, we may assume that $v \in V(M^* \cap L_j)$, and v has degree ≥ 1 in both M^* and L_j. Hence $v \in V(A_j \cap B_j)$, and so v has degree 1 in M^* and 1 in L_j, from the definition of M^*. This proves (3).

(4) *If P is a component of L_i with ends a and b, then there is a component of $M^* \cup L_j$ with*

ends a and b.

Subproof. Since a is a terminal of L_i, it follows that $a \in V(A_i \cap B_i) \cup (Z \setminus V(A_i))$, by (2.2). Since $Z \cap V(A_i) = Z \cap V(A_j)$ it follows that if $a \notin V(A_i \cap B_i)$ then $a \in Z \setminus V(A_j)$. Similarly, either $b \in V(A_i \cap B_i)$ or $b \in Z \setminus V(A_j)$. If $a \in Z \setminus V(A_j)$ let $a' = a$, and if $a \in V(A_i \cap B_i), a = v_i^t$ say, let $a' = v_j^t$. Thus $\phi_j(a') = \phi_i(a)$. Define b' similarly. Now since $\phi_j(\pi_j) = \phi_i(\pi_i)$ and ϕ_i, ϕ_j are injections, and $\{a, b\}$ is a block of π_i, it follows that $\{a', b'\}$ is a block of π_j, that is, there is a component P' of L_j with ends a', b'. There are five cases:

Case 1: $a, b \in Z \setminus V(A_j)$.

Then $a = a'$ and $b = b'$. Now no internal vertex of P' has degree ≥ 1 in M^*, and $a, b \notin V(M^*)$, and so P' is a component of $M^* \cup L_j$ satisfying (4).

Case 2: $V(A_i \cap B_i)$ contains exactly one of a, b, say a.

Then $a \neq b$, and so $a' \neq b' = b$. Let $a = v_i^t$ say; then $a' = v_j^t$, and v_j^t has degree 1 in L_j since it is an end of P' and $E(P') \neq \emptyset$. Consequently $t \in T$, and so $P_t \cup P'$ is a component of $M^* \cup L_j$ with ends a and b, as required.

Case 3: $a \neq b$, and $a, b \in V(A_i \cap B_i)$.

Then again $a' \neq b'$. Let $a = v_i^s, b = v_i^t$; then $a' = v_j^s, b' = v_j^t$. Since a' and b' both have degree 1 in L_j it follows that $s, t \in T$, and so $P_s \cup P_t \cup P'$ is a component of $M^* \cup L_j$ with ends a and b.

Case 4: $a = b \in V(A_i \cap B_i)$ and $a = a'$.

Let $a = b = v_i^t$. Then $v_i^t = a = a' = v_j^t$, and so a has degree 0 in M^*. Moreover, since v_i^t has degree 0 in L_i (because $|V(P)| = 1$) it follows from (2) that v_j^t has degree 0 in L_j. Hence $a = v_j^t$ has degree 0 in $M^* \cup L_j$, and so P is a component of $M^* \cup L_j$ with ends a and b.

Case 5: $a = b \in V(A_i \cap B_i)$ and $a \neq a'$.

Let $a = b = v_i^t$; then $a' = v_j^t \neq v_i^t$. Since v_i^t has degree 0 in L_i, it follows that $t \notin T$, and so a has degree 0 in M^*. Since $a \notin V(L_j)$ we deduce that a has degree 0 in $M^* \cup L_j$, and so again P is a component of $M^* \cup L_j$ with ends a and b.

In each case we have found a component of $M^* \cup L_j$ with ends a and b. This proves (4).

(5) $L \subseteq (L \cap A_i) \cup M^* \cup L_j$.

Subproof. (4) implies that there is a linkage in $M^* \cup L_j$ with the same pattern as L_i. From (2.3), it follows that there is a linkage in $(L \cap A_i) \cup M^* \cup L_j$ with the same pattern as L. Since L is vital, we deduce that $L \subseteq (L \cap A_i) \cup M^* \cup L_j$. This proves (5).

(6) $T = \{1, \ldots, k\}$.

Subproof. Suppose that $1 \leq t \leq k$ and $t \notin T$. Then $v_i^t \neq v_j^t$, and v_i^t has degree 0 or 2 in L_i. Suppose first that v_i^t has degree 0 in L_i. Then v_j^t has degree 0 in L_j by (2), it has degree 0 in M^* since $t \notin T$, and is not a vertex of $L \cap A_i$. Consequently, v_j^t has degree 0 in $(L \cap A_i) \cup M^* \cup L_j$. Since $V(L) \setminus V(G)$, it follows from (5) that v_j^t has degree 0 in L, and so $v_j^t \in Z$, contrary to (2) since $v_i^t \neq v_j^t$. It follows that v_i^t does not have degree 0 in L_i. Now v_i^t has degree ≤ 1 in M^* and is not a vertex of L_j, since $v_i^t \neq v_j^t$. Consequently, v_i^t has degree ≤ 1 in $M^* \cup L_j$. But from (5), $L_i \subseteq M^* \cup L_j$, and hence v_i^t has degree ≤ 1 in L_i, and hence $t \in T$, a contradiction. This proves (6).

From (6) it follows that $M^* = M$.

(7) *M is a subgraph of L.*

Subproof. Let $1 \leq t \leq k$; we claim that P^t is a subgraph of L. If $v_i^t = v_j^t$ this is clear, and so we may assume that $v_i^t \neq v_j^t$. By (2), $v_i^t, v_j^t \notin Z$. Moreover, since $V(A_i) \cap Z = V(A_j) \cap Z$ and the internal vertices of P^t belong to $V(A_j) \setminus V(A_i)$, it follows that no vertex of P_t is a terminal of L, and so they all have degree 2 in L. From (6), v_j^t has degree 1 in L_j, but it has degree 2 in L, and so L contains the edge of P^t incident with v_j^t. From (5) it follows that L contains both edges of P^t incident with any internal vertex of P^t. Consequently $E(P^t) \subseteq E(L)$. This proves (7).

From (5) and (7), $L \cap B_j \cap A_i \subseteq M^* = M \subseteq L \cap B_j \cap A_i$. Consequently, $L \cap B_j \cap A_i = M$, and so $L \cap B_{i+1} \cap A_i = M_i$, as required. ■

Here is a slight strengthening of (2.5).

2.6 *For all integers $p, k \geq 0$ there exists $n \geq 0$ with the following property. Let L be a vital p-linkage in a graph G, and for $1 \leq i \leq n$ let (A_i, B_i) be a separation of G of order $\leq k$, such that*

1. *$A_i \subseteq A_j$ and $B_j \subseteq B_i$, for $1 \le i < j \le n$*

2. *if $1 \le i < i' \le n$, and $|V(A_i \cap B_i)| = |V(A_{i'} \cap B_{i'})| = k'$ say, and $|V(A_j \cap B_j)| > k'$ for all j with $i < j < i'$, then there is a linkage $M_{ii'}$ in $B_i \cap A_{i'}$ with k' components, each with one end in $V(A_i \cap B_i)$ and the other in $V(A_{i'} \cap B_{i'})$.*

Then there exist i, i' as in (ii) such that $L \cap B_i \cap A_{i'} = M_{ii'}$.

Proof. For $0 \le k' \le k$ let $n(k') = (2p+1)(p+k'+1)^{2(p+k')} + 1$, and for $0 \le k' \le k$ let

$$m(k') = n(0)n(1)n(2)\dots n(k').$$

Let $n = m(k)$; we shall show it satisfies the theorem. For let G, L and (A_i, B_i) $(1 \le i \le n)$ and the $M_{ii'}$ be as in the theorem. Since there are at least $m(k)$ values of i with $1 \le i \le n$ such that $|V(A_i \cap B_i)| \le k$, there exists $k' \le k$ minimum such that

$$|\{i : 1 \le i \le n, |V(A_i \cap B_i)| \le k'\}| \ge m(k').$$

If $k' = 0$, then from (2.5) (with k replaced by 0) applied to the sequence of all (A_i, B_i) of order 0, we find the desired $M_{ii'}$, as required. We assume then that $k' > 0$. From the minimality of k', it follows that

$$|\{i : 1 \le i \le n, |V(A_i \cap B_i)| \le k' - 1\}| \le m(k' - 1) - 1,$$

and so

$$|\{i : 1 \le i \le n, |V(A_i \cap B_i)| \le k'\}| \ge n(k')(|\{i : 1 \le i \le n, |V(A_i \cap B_i)| \le k' - 1\}| + 1).$$

By examining the intervals between consecutive members of the second set, it follows that there exist i_1, i_2 with $1 \le i_1 < i_2 \le n$, such that

$$|\{i : i_1 \le i \le i_2, |V(A_i \cap B_i)| = k'\}| \ge n(k')$$
$$\{i : i_1 \le i \le i_2, |V(A_i \cap B_i)| < k'\} = \emptyset.$$

Then the result follows, from (2.5) applied to the sequence (A_i, B_i) $(i_1 \le i \le i_2, |V(A_i \cap B_i)| = k')$. ■

Similarly, we have

2.7 *For all integers $p, k \geq 0$ there exists $n \geq 0$ with the following property. Let L, L' be vital p-linkages in a graph G, and for $1 \leq i \leq n$ let (A_i, B_i) be as in (2.6), satisfying (2.6)(i) and (ii). Then there exist i, i' as in (ii) such that $L \cap B_i \cap A_{i'} = L' \cap B_i \cap A_{i'} = M_{ii'}$.*

Proof. First we prove an analogous version of (2.5) for two linkages L, L' instead of one. We let Z be the terminals of either L or L', and then follow the proof of (2.5), taking

$$n = (4p + 1)(2p + k + 1)^{2(2p+k)} + 1.$$

Statement (1) in the proof of (2.5) holds with $2p + 1$ replaced by $4p + 1$, since $|Z| \leq 2p$. We find distinct i, j with $1 \leq i, j \leq n$ such that $V(A_i) \cap Z = V(A_j) \cap Z, V(B_i) \cap Z = V(B_j) \cap Z, \phi_i(\pi_i) = \phi_j(\pi_j)$, and $\phi_i(\pi'_i) = \phi_j(\pi'_j)$, where π'_i is the pattern of $L' \cap B_i$. Then the proof of (2.5) yields that $L \cap B_i \cap A_{i+1} = M_i$ and $L' \cap B_i \cap A_{i+1} = M_i$, as required.

Now we use this modified version of (2.5) to prove (2.7), by modifying the proof of (2.6) in the obvious way. ■

3 Tangles

A *tangle of order* $\theta \geq 1$ in a graph G is a set $\mathcal{T}$ of separations of G, all of order $< \theta$, such that

(i) one of $(A, B), (B, A)$ belongs to $\mathcal{T}$, for every separation (A, B) of G of order $< \theta$

(ii) if $(A_i, B_i) \in \mathcal{T} (1 \leq i \leq 3)$ then $A_1 \cup A_2 \cup A_3 \neq G$

(iii) if $(A, B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

We write $ord(\mathcal{T}) = \theta$. Tangles were introduced in [6]. We shall need several lemmas about tangles, which we establish in this section. First, the following was shown in theorem (5.2) of [6].

3.1 *If a graph G has tree-width w, then G has a tangle of order $\geq \frac{2}{3}(w + 1)$, and has no tangle of order $> w + 1$.*

If $\mathcal{T}$ is a tangle of order θ in a graph G, and $W \subseteq V(G)$ with $|W| < \theta$, we denote

$$\{(A \setminus W, B \setminus W) : (A, B) \in \mathcal{T}, W \subseteq V(A \cap B)\}$$

by $\mathcal{T} \setminus W$. The following is theorem (8.5) of [6].

3.2 *With $\mathcal{T}, \theta, G, W$ as above, $\mathcal{T} \setminus W$ is a tangle of order $\theta - |W|$ in $G \setminus W$.*

[13, theorem (2.9)] asserts

3.3 *Let $\mathcal{T}$ be a tangle in a graph G, and let $(C, D) \in \mathcal{T}$ of order* 0*. Let*

$$\mathcal{T}' = \{(A \cap D, B \cap D) : (A, B) \in \mathcal{T}\}.$$

Then $\mathcal{T}'$ is a tangle in B of the same order as $\mathcal{T}$.

From [13, theorem (2.3)] or theorem (6.1) of [6] we have

3.4 *Let G' be a subgraph of G, and let $\mathcal{T}'$ be a tangle in G'. Let $\mathcal{T}$ be the set of all separations (A, B) of G of order $< ord(\mathcal{T}')$ such that $(A \cap G', B \cap G') \in \mathcal{T}'$. Then $\mathcal{T}$ is a tangle in G of order $ord(\mathcal{T}')$.*

We also need the following.

3.5 *Let $\mathcal{T}$ be a tangle in G and let $(A_0, B_0) \in \mathcal{T}$. Then there is a tangle $\mathcal{T}_0$ in B_0 of order $ord(\mathcal{T}) - |V(A_0 \cap B_0)|$, such that $(A \cap B_0, B \cap B_0) \in \mathcal{T}_0$ for every $(A, B) \in \mathcal{T}$ of order $< ord(\mathcal{T}) - |V(A_0 \cap B_0)|$.*

Proof. Let $ord(\mathcal{T}) = \theta$, and let $V(A_0 \cap B_0) = W$, where $|W| < \theta$. Then $\mathcal{T} \setminus W$ is a tangle in $G \setminus W$ of order $\theta - |W|$, by (3.2), and $(A_0 \setminus W, B_0 \setminus W) \in \mathcal{T} \setminus W$, and has order 0. Let

$$\mathcal{T}_1 = \{(A \cap (B_0 \setminus W), B \cap (B_0 \setminus W) : (A, B) \in \mathcal{T} \setminus W\}.$$

By [13, theorem (2.9)], $\mathcal{T}_1$ is a tangle in $B_0 \setminus W$ of order $\theta - |W|$. Let $\mathcal{T}_0$ be the tangle of order $\theta - |W|$ in B_0 induced by $\mathcal{T}_1$; this exists, by (3.4). We claim that $\mathcal{T}_0$ satisfies the theorem.

For let $(A, B) \in \mathcal{T}$ with order $< \theta - |W|$. Let A' be the subgraph of G with $E(A') = E(A)$ and $V(A') = V(A) \cup W$, and define B' similarly. Then (A', B') has order $< \theta$, and so $(A', B') \in \mathcal{T}$, by theorem (2.9) of [6]. Consequently, $(A' \setminus W, B' \setminus W) \in \mathcal{T} \setminus W$. From the definition of $\mathcal{T}_1$,

$$((A' \setminus W) \cap (B_0 \setminus W), (B' \setminus W) \cap (B_0 \setminus W)) \in \mathcal{T}_1.$$

But $(A' \setminus W) \cap (B_0 \setminus W) = (A \cap B_0) \setminus W$, and $(B' \setminus W) \cap (B_0 \setminus W) = (B \cap B_0) \setminus W$, and so $((A \cap B_0) \setminus W, (B \cap B_0) \setminus W) \in \mathcal{T}_1$, that is,

$$((A \cap B_0) \cap (B_0 \setminus W), (B \cap B_0) \cap (B_0 \setminus W)) \in \mathcal{T}_1.$$

Since $(A \cap B_0, B \cap B_0)$ is a separation of B_0 of order $< \theta - |W|$, we deduce from the definition of $\mathcal{T}_0$ that $(A \cap B_0, B \cap B_0) \in \mathcal{T}_0$, as required. ■

3.6 *If G' can be obtained from G by splitting a vertex, and $\mathcal{T}$ is a tangle in G of order ≥ 2, there is a tangle in G' of order $ord(\mathcal{T}) - 1$.*

Proof. Let G' be obtained by splitting $v \in V(G)$. By (3.2), $G \setminus v$ has a tangle of order $ord(\mathcal{T}) - 1$, and hence by (3.4), so does G', since $G \setminus v$ is a subgraph of G'. ■

Let us mention also the obvious

3.7 *If $\mathcal{T}$ is a tangle in G, and θ is an integer with $1 \leq \theta \leq ord(\mathcal{T})$, then the set of all members of $\mathcal{T}$ of order $< \theta$ is a tangle in G of order θ.*

We call this tangle the *θ-truncation* of $\mathcal{T}$.

4 Surfaces

A *surface* is a connected compact 2-manifold, possibly with boundary. The boundary of a surface Σ is denoted by $bd(\Sigma)$. The components of $bd(\Sigma)$ are called the *cuffs* of Σ; each cuff is homeomorphic to a circle. An 0-*arc* in Σ is a subset homeomorphic to a circle, and a *line* is a subset homeomorphic to the closed interval [0, 1]. The *ends* of a line are defined in the natural way. If $X \subseteq \Sigma$, its topological closure is denoted by $\overline{X}$. The surface obtained from Σ by pasting a closed disc onto every cuff is denoted by $\hat{\Sigma}$.

If Σ_1 and Σ_2 are surfaces with null boundary, we say that Σ_1 is *simpler* than Σ_2 if Σ_2 can be obtained from Σ_1 by adding handles or crosscaps (at least one). For a general surface Σ we denote the number of cuffs of Σ by $c(\Sigma)$. In this paper we shall prove several different statements about surfaces Σ by a double induction; we assume that the statement is true for all surfaces Σ' with $\hat{\Sigma}'$ simpler than $\hat{\Sigma}$, and we assume it is true for all Σ' with $\hat{\Sigma}'$ homeomorphic to $\hat{\Sigma}$ and with $c(\Sigma') < c(\Sigma)$.

To accomplish this, we shall need to consider cutting surfaces along certain lines and 0-arcs. A line in Σ is *proper* if its ends are in $bd(\Sigma)$ and it has no internal point in $bd(\Sigma)$. The operations we need are: cutting along a proper line, and cutting a surface with null boundary along an O-arc. In both cases we shall only use the operation when it results in another (connected) surface. What we mean by these operations is clear, but the notation is a little tricky. To simplify matters as far as possible, we postulate that cutting along a line or O-arc F in Σ as described above results in another surface Σ' with $\Sigma \cap \Sigma' = \Sigma \setminus F$, such that for every point of F there correspond two points of $\Sigma' \setminus \Sigma$, both in $bd(\Sigma')$, in the natural way. We observe:

4.1 *Let Σ be a surface, and let Σ' be obtained by cutting along $F \subseteq \Sigma$.*

1. *If F is a proper line with ends in different cuffs then $\hat{\Sigma}'$ is homeomorphic to $\hat{\Sigma}$ and $c(\hat{\Sigma}') = c(\hat{\Sigma}) - 1$.*
2. *If F is a proper line with ends in the same cuff and Σ' is connected then $\hat{\Sigma}'$ is simpler than $\hat{\Sigma}$.*
3. *If $bd(\Sigma) = \emptyset$ and F is an O-arc and Σ' is connected then $\hat{\Sigma}'$ is simpler than Σ.*

The proof is straightforward, and we omit it. We shall also have to deal with line and O-arcs F which *separate* Σ (that is, such that $\Sigma \setminus F$ is disconnected), but in these cases there are surfaces $\Sigma_1, \Sigma_2 \subseteq \Sigma$ with $\Sigma_1 \cup \Sigma_2 = \Sigma$ and $\Sigma_1 \cap \Sigma_2 = F$, and we can get away with using these subsurfaces. In these cases, therefore, no cutting is needed, which is convenient for purposes of notation. The corresponding results are:

4.2 *Let Σ be a surface, and let $\Sigma_1, \Sigma_2 \subseteq \Sigma$ be surfaces with $\Sigma_1 \cup \Sigma_2 = \Sigma$ and $\Sigma_1 \cap \Sigma_2 = F$.*

1. *If F is a proper line in Σ with both ends in the same cuff, then either $\hat{\Sigma}_2$ is simpler than $\hat{\Sigma}$ or $\hat{\Sigma}_1$ is a sphere.*
2. *If F is a proper line in Σ with both ends in the same cuff, and $\hat{\Sigma}_1$ is a sphere, then $\hat{\Sigma}_2$ is homeomorphic to $\hat{\Sigma}$; and either $c(\Sigma_2) < c(\Sigma)$, or $c(\Sigma_2) = c(\Sigma)$ and there is a closed disc $\Delta \subseteq \Sigma$ with $F \subseteq bd(\Delta) \subseteq F \cup bd(\Sigma)$.*
3. *If $bd(\Sigma) = \emptyset$ and F is an O-arc in Σ, then either $\hat{\Sigma}_2$ is simpler than $\hat{\Sigma}$ or Σ_1 is a closed disc.*

A *drawing* in a surface Σ is a pair (U, V) where $U \subseteq \Sigma$ is closed, $V \subseteq U$ is finite, $U \cap bd(\Sigma) \subseteq V, U \setminus V$ has only finitely many connected components, called *edges*, and for each edge e, either

1. its closure $\overline{e}$ is an O-arc with $|\overline{e} \cap V| = 1$, or
2. its closure is a line meeting V in precisely its ends.

If $\Gamma = (U, V)$ is a drawing we write $U(\Gamma) = U$ and $V(\Gamma) = V$. A drawing Γ is therefore a graph with vertex set $V(\Gamma)$, and we use graph-theoretic terminology for drawings without further explanation.

If Γ is a drawing in Σ, and $F \subseteq \Sigma$, we say that F is Γ-*normal* if $F \cap U(\Gamma) \subseteq V(\Gamma)$. If F is a Γ-normal O-arc or line which does not separate Σ, and Σ' is obtained by cutting along Σ, then by splitting the vertices of Γ which lie in F in the natural way we obtain a new drawing Γ' in Σ'. Provided that Γ is loopless, this definition agrees with the definition of splitting a vertex discussed for general graphs in section 2.

If $bd(\Sigma) = \emptyset$, a drawing Γ in Σ is 2-*cell* if every region of Γ in Σ is homeomorphic to an open disc. We need the following two well-known lemmas.

4.3 *If $bd(\Sigma) = \emptyset$, a drawing Γ in Σ is 2-cell if and only if $V(\Gamma) \neq \emptyset, \Gamma$ is connected, and for every O-arc $F \subseteq \Sigma$ with $F \cap U(\Gamma) = \emptyset$, there is a closed disc $\Delta \subseteq \Sigma$ with $bd(\Delta) = F$.*

Proof. The "only if" direction is obvious. For "if", let r be a region of Γ in Σ. Since $V(\Gamma) \neq \emptyset$ it follows that r is not a sphere. If every O-arc $F \subseteq r$ bounds a closed disc in r, then r is an open disc by [5, theorem (4.2)]. Suppose that $F \subseteq r$ is an O-arc which bounds no closed disc in r. From the hypothesis, F bounds a closed disc $\Delta \subseteq \Sigma$, but $\Delta \not\subseteq r$, and so $\Delta \cap V(\Gamma) \neq \emptyset$. Since $F \cap U(\Gamma) = \emptyset$ and Γ is connected it follows that $U(\Gamma) \subseteq \Delta$. If Σ is not a sphere, there is a non-null-homotopic O-arc $F' \subseteq \Sigma$, and it can be chosen with $F' \cap \Delta = \emptyset$ since Δ is a disc; but then $F' \subseteq r$ contrary to the hypothesis. If Σ is a sphere, let Δ' be the disc different from Δ bounded by F; then $\Delta' \subseteq r$, contrary to our assumption. In either case we have a contradiction, and so there is no such F, as required. ■

We remind the reader of the following basic fact [1, theorem (1.7)].

4.4 *If $bd(\Sigma) = \emptyset$, an O-arc $F \subseteq \Sigma$ bounds a closed disc in Σ if and only if F is null-homotopic in Σ.*

Let Γ be a drawing in Σ, and let $\mathcal{T}$ be a tangle in Γ. We say that $\mathcal{T}$ is *respectful* if Γ is connected and every Γ-normal O-arc $F \subseteq \hat{\Sigma}$ with $|F \cap V(\Gamma)| < ord(\mathcal{T})$ bounds a disc $\Delta \subseteq \hat{\Sigma}$ such that

$$(\Gamma \cap \Delta, \Gamma \cap \overline{\hat{\Sigma} \setminus \Delta}) \in \mathcal{T}.$$

(If $\Sigma' \subseteq \Sigma$ is a surface, and $bd(\Sigma')$ is Γ-normal, we denote the drawing $(U(\Gamma) \cap \Sigma', V(\Gamma) \cap \Sigma')$ in Σ' by $\Gamma \cap \Sigma'$.) If Δ is related to F as above we write $\Delta = ins(F)$. If there is a respectful tangle in a drawing Γ in Σ, then Γ is automatically 2-cell in $\hat{\Sigma}$, by (4.3).

The main result of this section is the following.

4.5 *For every surface Σ with $bd(\Sigma) = \emptyset$ and every integer $\theta' \geq 1$ there is an integer $\theta \geq 1$ such that, if Γ is a drawing in Σ with a tangle of order $\geq \theta$, and Γ' is obtained from Γ by deleting some vertices and edges, all incident with one region of Γ in Σ, then Γ' has a tangle of order $\geq \theta'$.*

Proof. We proceed by induction on Σ, and assume inductively that the result holds for every surface with null boundary simpler than Σ. Consequently,

(1) *For every $\phi \geq 1$ there exists $f(\phi) \geq \phi$ such that for every surface Σ' with $bd(\Sigma') = \emptyset$ simpler than Σ, if Γ is a drawing in Σ' with a tangle of order $\geq f(\phi)$, and Γ' is obtained from Γ by deleting some vertices and edges, all incident with one region, then Γ' has a tangle of order $\geq \phi$.*

Given $\theta' \geq 1$, let $\theta = \theta' + 6 + f(f(\theta'))$; we shall show that θ satisfies the theorem. Let Γ be a drawing in Σ, let $\mathcal{T}$ be a tangle in Γ of order $\geq \theta$, let r be a region of Γ in Σ, and let Γ' be a drawing obtained from Γ by deleting some vertices and edges of Γ all incident with r.

There exists $(\Gamma_1, \Gamma_2) \in \mathcal{T}$ of order 0 such that Γ_2 is connected, by theorem (2.8) of [6] applied to the 1-truncation of $\mathcal{T}$. By (3.3) there is a tangle in Γ_2 of order $\geq \theta$, and $\Gamma' \cap \Gamma_2$ is obtained from Γ_2 by deleting some vertices and edges all on one region of Γ_2 in Σ. If $\Gamma' \cap \Gamma_2$ has a tangle of order $\geq \theta'$, then so does Γ' by (3.4). Consequently it suffices to prove the theorem for Γ_2; in other words, we may assume that Γ is connected.

Let $\mathcal{T}_1$ be the $(\theta' + 6)$-truncation of $\mathcal{T}$. If $\mathcal{T}_1$ is respectful, then by theorem (7.8) of [7] (with $k = 3$), there is a tangle in Γ' of order θ', since $\theta' + 6 \geq 9$, as required. We may assume therefore that $\mathcal{T}_1$ is not respectful. Since Γ is connected, we deduce

(2) *There is a Γ-normal O-arc $F \subseteq \Sigma$ with $|F \cap V(\Gamma)| < \theta' + 6$, such that there is no closed disc $\Delta \subseteq \Sigma$ bounded by F with $(\Gamma \cap \Delta, \Gamma \cap \overline{\Sigma \setminus \Delta}) \in \mathcal{T}$.*

It follows that Σ is not a sphere. There are two cases, depending on whether F separates Σ or not. We assume first that it does not. Let Σ' be obtained from Σ by cutting along F. Then $\hat{\Sigma}'$ is simpler than Σ by (4.1)(iii). Let Γ'' be obtained from Γ by deleting all vertices in $F \cap V(\Gamma)$. By (3.2), Γ'' has a tangle of order

$$ord(\mathcal{T}) - |F \cap V(\Gamma)| \geq \theta - (\theta' + 6) = f(f(\theta')).$$

Now Γ'' is a drawing in $\hat{\Sigma}'$ and there are one or two regions of Γ'' in $\hat{\Sigma}'$, say r_1 and r_2 where possibly $r_1 = r_2$, such that $\Gamma' \cap \Gamma''$ is obtained from Γ'' by deleting some vertices and edges incident with either r_1 or r_2. By two applications of (1), we deduce that $\Gamma' \cap \Gamma''$ has a tangle of order $\geq \theta'$, and hence so does Γ' by (3.4), as required.

In the second case, we assume that F separates Σ. Let $\Sigma_1, \Sigma_2 \subseteq \Sigma$ be surfaces with $\Sigma_1 \cup \Sigma_2 = \Sigma$ and $\Sigma_1 \cap \Sigma_2 = F$. Let $\Gamma_i = \Gamma \cap \Sigma_i$ $(i = 1, 2)$. Since (Γ_1, Γ_2) is a separation of Γ of order $< \theta' + 6 \leq ord(\mathcal{T})$ (by (2)) we may assume from the symmetry that $(\Gamma_1, \Gamma_2) \in \mathcal{T}$. From (2), Σ_1 is not a disc, and so from (4.2)(iii), $\hat{\Sigma}_2$ is simpler that Σ. From (3.5) there is a tangle in Γ_2 of order $\geq ord(\mathcal{T}) - |F \cap V(\Gamma)| \geq f(f(\theta')) \geq f(\theta')$.

Now $\Gamma_2 \cap \Gamma'$ is obtained from Γ_2 by deleting some vertices and edges all incident with one region of Γ_2 in $\hat{\Sigma}_2$, and so from (1), $\Gamma_2 \cap \Gamma'$ has a tangle of order $\geq \theta'$. By (3.4), so does Γ', as required. ■

By ρ repeated applications of (4.5), we deduce

4.6 *For any surface Σ with $bd(\Sigma) = \emptyset$, and all integers $\theta' \geq 1$ and $\rho \geq 0$, there exists $\theta \geq 1$ such that, if Γ is a drawing in Σ with a tangle of order $\geq \theta$, and $r_1, \ldots, r_\rho$ are regions of Γ in Σ, and Γ' is obtained from Γ by deleting some vertices and edges each incident with one of $r_1, \ldots, r_\rho$, then Γ' has a tangle of order $\geq \theta'$.*

5 Linkages on surfaces

In this section we prove (1.1) for graphs which can be drawn on a fixed surface. We need some further definitions.

If C is a cuff of Σ and Γ is a drawing in Σ, there is a unique region of Γ in $\hat{\Sigma}$ which includes $C \setminus V(\Gamma)$, and we call it the *cuff region of* Γ *in* $\hat{\Sigma}$ *corresponding to* C. An *atom* of a drawing Γ in a surface Σ is either a region of Γ in $\hat{\Sigma}$, or an edge of Γ, or a set $\{v\}$ where v is a vertex. The set of atoms is denoted by $A(\Gamma)$, or $A_\Sigma(\Gamma)$ in cases of ambiguity. If Γ is a drawing in Σ and $\mathcal{T}$ is a respectful tangle in Γ, then $\mathcal{T}$ defines a metric on $A(\Gamma)$ discussed in [7], defined as follows. Let K be a drawing in Γ such that $V(\Gamma) \subseteq V(K)$, every region of Γ includes a unique vertex of K, and $U(\Gamma) \cap U(K) = V(\Gamma) \cap V(K)$; and so that for every region r of Γ, the vertex of K it contains is adjacent in K to every vertex v of Γ incident with r, by multiple edges if r is incident with v more than once, in the natural sense. For a closed walk W of K of length $< 2\, ord(\mathcal{T})$, let $ins(W)$ be the union of the atoms of K in W together with all sets $ins(U(C))$ where C is a circuit of K whose edges all occur in W. For atoms a, b of Γ, let a', b' be the corresponding atoms of K in the natural sense; we say $d(a, b) = 0$ if $a = b$, $d(a, b) = ord(\mathcal{T})$ if $a \neq b$ and there is no closed walk W of length $< 2\, ord(\mathcal{T})$ with $a', b' \in ins(W)$, and otherwise $d(a, b)$ is half the minimum length of such a walk. (See [7] for further discussion.) If $\{v\}$ is an atom of Γ we often write $d(v, a)$ for $d(\{v\}, a)$. We call d the *metric* of $\mathcal{T}$.

We need the following lemma.

5.1 *For every surface Σ and integer $p \geq 0$, there exists $\theta > p$ with the following property. Let Γ be a drawing in Σ with $|V(\Gamma) \cap bd(\Sigma)| = p$, and let $\mathcal{T}$ be a respectful tangle in Γ of order $\geq \theta$. Suppose that*

- *(i) there is no Γ-normal O-arc $F \subseteq \Sigma$ such that $|F \cap V(\Gamma)| < |C \cap V(\Gamma)|$ and $C \subseteq ins(F)$ for some cuff C of Σ, and*
- *(ii) for every two cuffs C_1, C_2, the corresponding cuff regions r_1, r_2 satisfy $d(r_1, r_2) \geq \theta$ where d is the metric of $\mathcal{T}$.*

Then there is no vital linkage in Γ with set of terminals $V(\Gamma) \cap bd(\Sigma)$.

Proof. Choose θ' so that theorem (3.2) of [7] holds, with Σ, t, z, and θ replaced by $\hat{\Sigma}, c(\Sigma), p$ and θ' respectively. We may assume that $\theta' \geq p + 9$, by increasing θ' if necessary. Let $\theta = 2\theta' + 1$. We claim that θ satisfies (5.1). For let $\Gamma, \mathcal{T}$ be as in (5.1), satisfying (i) and (ii). Let d be the metric of $\mathcal{T}$. Now Γ is 2-cell in $\hat{\Sigma}$ since $\mathcal{T}$ is respectful.

(1) *There is a vertex v of Γ such that $d(v, r) \geq \theta'$ for every cuff region r.*

Subproof. We may assume that there is at least one cuff region r_1 say. By [4, theorem (8.9)], there is an edge e of Γ so that $d(e, r_1) = ord(\mathcal{T})$. Let $v_1, \ldots, v_n$ be a sequence of vertices of Γ such that v_1 is incident with r_1, v_n is incident with e, and for $1 \leq i < n$ some region of Γ in $\hat{\Sigma}$ is incident with v_i and $v_i + 1$. Then $d(r_1, v_1) \leq 1, d(e, v_n) \leq 2$, and $d(v_i, v_i + 1) \leq 2$ for $1 \leq i < n$. Since $d(r_1, e) = ord(\mathcal{T})$, it follows (since d is a metric) that $d(r_1, v_n) \geq ord(\mathcal{T}) - 2 \geq \theta'$. Consequently we may choose i with $1 \leq i \leq n$ minimum such that $d(r_1, v_i) \geq \theta'$. Since $d(r_1, v_1) \leq 1$ and $\theta' \geq 2$ it follows that $i \geq 2$. From the minimality of $i, d(r_1, v_{i-1}) < \theta'$. Since $d(v_{i-1}, v_i) \leq 2$ it follows that $d(r_1, v_i) \leq \theta' + 1$. For any cuff region $r_2 \neq r_1$,

$$\theta \leq d(r_1, r_2) \leq d(r_1, v_i) + d(r_2, v_i) \leq \theta' + 1 + d(r_2, v_i)$$

by (ii), and so $d(r_2, v_i) \geq \theta - \theta' - 1 = \theta'$. Thus setting $v = v_i$ satisfies (1). This proves (1).

Let v be as in (1).

(2) *If* $\Gamma \setminus v$ *is not* 2*-cell in* $\hat{\Sigma}$ *then there is no vital linkage in* Γ *with set of terminals* $V(\Gamma) \cap bd(\Sigma)$*.*

Subproof. Suppose that L is a vital linkage in Γ with set of terminals $V(\Gamma) \cap bd(\Sigma)$, and that r is a region of $\Gamma \setminus v$ in $\hat{\Sigma}$ which is not homeomorphic to an open disc. Consequently $v \in r$. Since r is not homeomorphic to an open disc there is by [5, theorem (4.2)] an O-arc $F \subseteq r$ which bounds no disc in r; and it can be chosen so that $F \cap U(\Gamma) = \{v\}$. Since $|F \cap V(\Gamma)| \leq 1$ it follows that $ins(F)$ exists, and $ins(F) \not\subseteq r$. Consequently, $ins(F) \cap U(\Gamma \setminus v) \neq \emptyset$. Since L is vital there exists a component of P of L with $V(P) \cap ins(F) \not\subseteq F$, and it follows that one end s of P is in $ins(F) \setminus F$, since $V(P) \cap F \subseteq \{v\}$. Let r_1 be the cuff region with $s \in \overline{r}_1$. Then $\overline{r}_1 \cap (ins(F) \setminus F) \neq \emptyset$, and so $d(r_1, v) \leq 1$, contrary to (1). This proves (2).

Let $\Gamma' = \Gamma \setminus v$. By (2), we may assume that Γ is 2-cell in $\hat{\Sigma}$. Since $d(\{v\}, e) \leq 2$ for every edge e of Γ not in Γ', and $ord(\mathcal{T}) \geq \theta' \geq 7$, it follows from theorem (7.8) of [7] that

(3) *There is a respectful tangle* $\mathcal{T}'$ *in* Γ' *of order* $ord(\mathcal{T}) - 4$*, such that*

(i) $(A \cap \Gamma', B \cap \Gamma') \in \mathcal{T}'$ *for every* $(A, B) \in \mathcal{T}$ *of order* $< ord(\mathcal{T}) - 4$*, and*

(ii) if $a, b \in A(\Gamma, \hat{\Sigma})$ *and* $a', b' \in A(\Gamma', \hat{\Sigma})$ *satisfy* $a \subseteq a'$ *and* $b \subseteq b'$*, then*

$$d(a, b) \geq d'(a', b') \geq d(a, b) - 8,$$

where d' *is the metric of* $\mathcal{T}'$*.*

Let ins' be the function derived from $\mathcal{T}'$ analogous to ins.

(4) *If* $F \subseteq \Sigma$ *is a* Γ'*-normal* O*-arc with* $|F \cap V(\Gamma')| < |C \cap V(\Gamma')|$ *for some cuff* C*, then* $C \not\subseteq ins'(F)$*.*

Subproof. Let r_1 be the cuff region of Γ in $\hat{\Sigma}$ corresponding to C. Since $d(r_1, v) \geq 2$ it follows that v is not incident with r_1 and so r_1 is also the cuff region of Γ' in $\hat{\Sigma}$ corresponding to C. Let r be the region of Γ' with $v \in r$. If $F \cap r \neq \emptyset$, then

$$d'(r, r_1) \leq |F \cap V(\Gamma')| < |C \cap V(\Gamma')| \leq |bd(\Sigma) \cap V(\Gamma)| = p$$

and so by (3)(ii), $d(v, r_1) \leq p+8$, contrary to (1) since $\theta' \geq p+9$. Consequently $F \cap r = \emptyset$, and so F is Γ-normal, and $|F \cap V(\Gamma)| < |C \cap V(\Gamma)|$. From (i) it follows that $C \not\subseteq ins(F)$. Let $\Delta = ins(F)$. Then

$(\Gamma \cap \Delta, \Gamma \cap \overline{\Sigma \setminus \Delta}) \in \mathcal{T}$, and has order $< k \leq ord(\mathcal{T}) - 4$ and so by (3)(i), $(\Gamma' \cap \Delta, \Gamma' \cap \overline{\Sigma \setminus \Delta}) \in \mathcal{T}'$. Consequently, $\Delta = ins'(F)$. This proves (4).

(5) *For every two distinct cuffs C_1, C_2, the corresponding cuff regions r_1, r_2 of Γ' in $\hat{\Sigma}$ are distinct and satisfy $d'(r_1, r_2) \geq \theta'$.*

Subproof. As we saw in (4), r_1 and r_2 are cuff regions of Γ in $\hat{\Sigma}$. Since $d(r_1, r_2) \geq \theta$ by hypothesis, it follows that $r_1 \neq r_2$. By (3)(ii),

$$d'(r_1, r_2) \geq d(r_1, r_2) - 8 \geq \theta - 8 \geq \theta'.$$

This proves (5).

Suppose that L is a linkage in Γ with set of terminals $V(\Gamma) \cap bd(\Sigma)$. From (4), (5) and theorem (3.2) of [7], there is a linkage in $\Gamma \setminus v$ with the same pattern as L, from the choice of θ'. But then L is not vital. The result follows. ■

The main result of this section is the following.

5.2 *For every surface Σ with $bd(\Sigma) = \emptyset$ and every integer $p \geq 0$ there exists $\theta \geq 1$ such that every drawing in Σ with a tangle of order $\geq \theta$ has no vital p-linkage.*

Proof. Let Σ_0 be a surface with $bd(\Sigma_0) = \emptyset$, and assume that the result holds for all pairs Σ', p' where Σ' is simpler than Σ_0. We shall prove that it holds for Σ_0 and all p. By cutting at most p small holes in Σ_0, one at each terminal, we deduce that it suffices to prove the following.

($*$) *For every surface Σ with $\hat{\Sigma}$ homeomorphic to Σ_0 and every integer $p \geq 0$, there exists $\theta \geq 1$ such that every drawing Γ in Σ with $|V(\Gamma) \cap bd(\Sigma)| \leq p$ and with a tangle of order $\geq \theta$ has no vital linkage with all its terminals in $bd(\Sigma)$.*

We shall prove ($*$) for all p by induction on $c(\Sigma)$, and then, with $c(\Sigma)$ fixed, by induction on p. Our three inductive hypothesis may be summarized as follows.

(1) *For all $p' \geq 0$ there exists $\theta_1(p') \geq 1$ such that for every surface Σ' with $bd(\Sigma') = \emptyset$ which is simpler than Σ_0, every drawing in Σ' with a tangle of order $\geq \theta_1(p')$ has no vital p'-linkage.*

(2) *For all* $p' \geq 0$ *there exists* $\theta_2(p') \geq 1$ *such that for every surface* Σ' *with* $\hat{\Sigma}'$ *homeomorphic to* Σ_0 *and* $c(\Sigma') < c(\Sigma)$, *every drawing* Γ' *in* Σ' *with* $|V(\Gamma') \cap bd(\Sigma')| \leq p'$ *with a tangle of order* $\geq \theta_2(p')$ *has no vital linkage with all its terminals in* $bd(\Sigma')$.

(3) *There exists* $\theta_3 \geq 1$ *such that for every surface* Σ' *homeomorphic to* Σ, *every drawing* Γ' *in* Σ' *with* $|V(\Gamma') \cap bd(\Sigma')| < p$ *with a tangle of order* $\geq \theta_3$ *has no vital linkage with all its terminals in* $bd(\Sigma')$.

Choose $\theta_4 > p$ such that (5.1) holds (with θ replaced by θ_4). Let

$$\theta = 2\theta_4 + \max(\theta_1(p + 3\theta_4), \theta_2(p + 3\theta_4), \theta_3).$$

We claim that θ satisfies $(*)$. For suppose not, and let Γ be a drawing in Σ with $|V(\Gamma) \cap bd(\Sigma)| \leq p$, let $\mathcal{T}$ be a tangle in Γ of order $\geq \theta$, and let L be a vital linkage in Γ with all its terminals in $bd(\Sigma)$. Choose $\Gamma, \mathcal{T}, L$ so that Γ is minimal.

(4) Γ *is connected and loopless.*

Subproof. If Γ is not connected, there exists $(\Gamma_1, \Gamma_2) \in \mathcal{T}$ of order 0 with $\Gamma_2 \neq \Gamma$. By (3.3), Γ_2 has a tangle of order $\geq \theta$, and $|V(\Gamma_2) \cap bd(\Sigma)| \leq p$, and $L \cap \Gamma_2$ is a vital linkage in Γ_2 with all its terminals in $bd(\Sigma)$. This contradicts the minimality of Γ. Thus, Γ is connected, and by theorem (8.4) of [6] and the minimality of Γ it is also loopless. This proves (4).

(5) *Suppose that* $(A, B) \in \mathcal{T}$ *has order* $\leq \theta_4$, *and* Γ' *is a drawing in a surface* Σ', *such that* Γ' *can be obtained from* B *by splitting* $\leq \theta_4$ *vertices of* B. *Then* $\hat{\Sigma}'$ *is not simpler than* Σ_0.

Subproof. From (3.3), B has a tangle of order $\geq \theta - \theta_4$, and so Γ' has a tangle of order $\geq \theta - 2\theta_4$. But from (2.2) and (2.3), B has a vital $(p + \theta_4)$-linkage, and so from (2.4), Γ' has a vital $(p + 3\theta_4)$-linkage. Since $\theta - 2\theta_4 \geq \theta_1(p + 3\theta_4)$, it follows from (1) that $\hat{\Sigma}'$ is not simpler than Σ_0. This proves (5).

(6) *Suppose that* $(A, B) \in \mathcal{T}$ *has order* $\leq \theta_4$, *and* Γ' *is a drawing in a surface* Σ', *such that* Γ' *can be obtained from* B *by splitting* $\leq \theta_4$ *vertices of* B. *Suppose, moreover, that*

(a) $\hat{\Sigma}'$ *is homeomorphic to* Σ_0*, and*

(b) for $v \in V(\Gamma'), v \in bd(\Sigma')$ *if and only if either* $v \in V(A \cap B)$*, or* $v \in V(\Gamma) \cap bd(\Sigma)$*, or* $v \notin V(\Gamma)$ *(that is,* v *is a new vertex produced by splitting).*

Then $c(\Sigma') \geq c(\Sigma)$*, and if equality holds then* $|V(\Gamma') \cap bd(\Sigma')| \geq p$*.*

Subproof. As in (5), Γ' has a tangle of order $\geq \theta - 2\theta_4$, and has a vital linkage with all its terminals in $bd(\Sigma')$ (from (b)). But

$$|V(\Gamma') \cap bd(\Sigma')| \leq |V(A \cap B)| + |V(\Gamma) \cap bd(\Sigma)| + 2\theta_4 \leq p + 3\theta_4$$

and $\theta - 2\theta_4 \geq \theta_2(p + 3\theta_4)$, and so from (2) and (a), $c(\Sigma') \geq c(\Sigma)$. If equality holds then Σ' is homeomorphic to Σ, and since $\theta - 2\theta_4 \geq \theta_3$ it follows that $|V(\Gamma') \cap bd(\Sigma')| \geq p$, from (3). This proves (6).

(7) *There is no* Γ*-normal proper line* $F \subseteq \Sigma$ *with ends in different cuffs such that* $|F \cap V(\Gamma)| \leq \theta_4$*.*

Subproof. If there is such an F, let A be null and $B = \Gamma$; let Σ' be obtained from Σ by cutting along F, and let Γ' be obtained from Γ by splitting appropriately the vertices of Γ in F. (Since Γ is loopless we can do so.) By (4.1)(i), $\hat{\Sigma}'$ is homeomorphic to Σ_0 and $c(\Sigma') = c(\Sigma) - 1$, contrary to (6). This proves (7).

Let $\mathcal{T}_1$ be the θ_4-truncation of $\mathcal{T}$.

(8) $\mathcal{T}_1$ *is respectful.*

Subproof. Certainly Γ is connected, by (4). Let $F \subseteq \hat{\Sigma}$ be a Γ-normal O-arc with $|F \cap V(\Gamma)| < \theta_4$. Suppose first that F does not separate $\hat{\Sigma}$, let Σ' be obtained from $\hat{\Sigma}$ by cutting along F, and let Γ' be the drawing in Σ' obtained from Γ by splitting appropriately the vertices of Γ in F. By (4.1)(iii) $\hat{\Sigma}'$ is simpler than Σ_0, contrary to (5). Thus F separates $\hat{\Sigma}$. Let $\Sigma_1, \Sigma_2 \subseteq \hat{\Sigma}$ be surfaces such that $\Sigma_1 \cup \Sigma_2 = \hat{\Sigma}$ and $\Sigma_1 \cap \Sigma_2 = F$. Let $\Gamma_i = \Gamma \cap \Sigma_i$ $(i = 1, 2)$. Since (Γ_1, Γ_2) is a separation of Γ of order $< \theta_4$, we may assume that $(\Gamma_1, \Gamma_2) \in \mathcal{T}_1 \subseteq \mathcal{T}$. By (5), $\hat{\Sigma}_2$ is not simpler than Σ_0, and so by (4.2)(iii), Σ_1 is a disc. This proves (8).

If $F \subseteq \hat{\Sigma}$ is a Γ-normal O-arc with $|F \cap V(\Gamma)| < \theta_4$, we define $ins(F)$ as usual.

(9) *If $F \subseteq \hat{\Sigma}$ is a Γ-normal O-arc with $|F \cap V(\Gamma)| < \theta_4$, then $C \cap ins(F) \neq \emptyset$ for at most one cuff C.*

Subproof. Let $A = \Gamma \cap ins(F)$ and choose $B \subseteq \Gamma$ so that (A, B) is a separation of Γ and $V(A \cap B) = V(\Gamma) \cap F$. If $C \cap ins(F) \neq \emptyset$ for ≥ 2 cuffs C, then by splitting ≤ 2 vertices of B, we can obtain from B a drawing Γ' in a surface Σ' with $\hat{\Sigma}'$ homeomorphic to Σ_0 and with $c(\Sigma') < c(\Sigma)$, satisfying (6)(b), contrary to (6). This proves (9).

Let d_1 be the metric of $\mathcal{T}_1$.

(10) *If r_1, r_2 are the cuff regions corresponding to distinct cuffs C_1, C_2 then $d_1(r_1, r_2) \geq \theta_4$.*

Subproof. Suppose not. From (7), (9) and the definition of the metric, and exchanging C_1 and C_2 if necessary, there is a Γ-normal O-arc $F_1 \subseteq \Sigma \setminus bd(\Sigma)$ with $|F_1 \cap V(\Gamma)| < \theta_4$, and with $r_1 \subseteq ins(F_1) \setminus F_1$, such that $ins(F_1) \cap bd(\Sigma) = C_1$. Moreover, there is also either

- (i) a Γ-normal line $F_0 \subseteq \Sigma$ with one end in F_1, the other end in C_2, and with no internal point in $bd(\Sigma) \cup ins(F_1)$, with $|(F_0 \cup F_1) \cap V(\Gamma)| < \theta_4$, or
- (ii) a Γ-normal O-arc $F_2 \subseteq \Sigma \setminus bd(\Sigma)$ with $|(F_1 \cup F_2) \cap V(\Gamma)| < \theta_4$ and with $r_2 \subseteq ins(F_2) \setminus F_2$, such that $ins(F_2) \cap bd(\Sigma) = C_2$ and $|ins(F_1) \cap ins(F_2)| = 1$, or
- (iii) a Γ-normal O-arc $F_2 \subseteq \Sigma \setminus bd(\Sigma)$ and a Γ-normal line $F_0 \subseteq \Sigma \setminus bd(\Sigma)$ with $|(F_0 \cup F_1 \cup F_2) \cap V(\Gamma)| < \theta_4$, such that $r_2 \subseteq ins(F_2) \setminus F_2, ins(F_2) \cap bd(\Sigma) = C_2, ins(F_1) \cap ins(F_2) = \emptyset$, one end of F_0 is in F_1, the other end is in F_2, and no internal point of F_0 is in $ins(F_1)$ or $ins(F_2)$.

If (i) holds, let $A = \Gamma \cap ins(F_1)$, and let $(A, B) \in \mathcal{T}$ where $V(A \cap B) = F \cap V(\Gamma)$; then (6) is contradicted, by splitting all the vertices of B in F_0. If (ii) or (iii) holds, let

$$A = (\Gamma \cap ins(F_1)) \cup (\Gamma \cap ins(F_2))$$

and let $(A, B) \in \mathcal{T}$ where $V(A \cap B) = (F_1 \cup F_2) \cap V(\Gamma)$; then again (6) is contradicted, in (ii) by splitting the vertex of B in $F_1 \cap F_2$ if there is one, and in (iii) by splitting all the vertices of B in F_0. This proves (10).

(11) *If* $F \subseteq \Sigma$ *is a* Γ*-normal O-arc with* $|F \cap V(\Gamma)| < \theta_4$*, and* $ins(F)$ *includes a unique cuff* C*, then* $|F \cap V(\Gamma)| \geq |C \cap V(\Gamma)|$.

Subproof. By (9), $ins(F) \cap bd(\Sigma) = C$. Let $A = \Gamma \cap ins(F)$, and let $(A, B) \in \mathcal{T}$ where $V(A \cap B) = F \cap V(\Gamma)$. Let Σ' be obtained from Σ by deleting $\Sigma \cap (ins(F) \setminus F)$. Then Σ' is homeomorphic to Σ, and so by (6),

$$|V(B) \cap bd(\Sigma')| \geq p \geq |V(\Gamma) \cap bd(\Sigma)|.$$

Consequently $|F \cap V(\Gamma)| \geq |C \cap V(\Gamma)|$. This proves (11).

(12) *Every vertex in* $V(\Gamma) \cap bd(\Sigma)$ *is a terminal of* L*.*

Subproof. If $v \in bd(\Sigma) \setminus V(\Gamma)$ is not a terminal of L, let Σ' be obtained from Σ by slightly enlarging Σ in the neighbourhood of v. Then $|V(\Gamma) \cap bd(\Sigma')| < p$, contrary to (3). This proves (12).

But (8), (10), (11) and (12) contradict (5.1). Thus our assumption that θ does not satisfy $(*)$ was false, and the proof is complete. ■

6 Presentations

If Γ is a drawing in a surface Σ such that $|V(\Gamma) \cap C| \geq 2$ for every cuff C, we call the connected components of $bd(\Sigma) \setminus V(\Gamma)$ the *spaces* of Γ in Σ. For every space s there are two vertices u, v such that $s \cup \{u, v\}$ is a line; we call u, v the *ends* of s. A *support* of Γ in Σ is either a set $\{v\}$ where $v \in V(\Gamma) \cap bd(\Sigma)$, or a line $F \subseteq bd(\Sigma)$ with both ends in $V(\Gamma)$ (where possibly some internal points of F belong to $V(\Gamma)$). The *ends* of a support F are a and b, where $a = b = v$ if $F = \{v\}$, and a and b are the ends of the line F if F is a line.

Let Γ, as above, be a subgraph of some graph G, which is not required to be a drawing. Let $F_1^*, \ldots, F_r^*$ be mutually disjoint supports of Γ in Σ with $V(\Gamma) \cap bd(\Sigma) \subseteq F_1^* \cup \cdots \cup F_r^*$. For $1 \leq j \leq r$, let $H_j \subseteq G$, so that

(V1) $H_1 \cup \ldots \cup H_r \cup \Gamma = G$*;* $H_1, \ldots, H_r$ *are mutually vertex-disjoint; and for* $1 \leq j \leq r, V(H_j \cap \Gamma) = V(\Gamma) \cap F_j^*$*, and* $E(H_j \cap \Gamma) = \emptyset$*.*

For $1 \leq j \leq r$, let $q_j \geq 1$ be an integer; for each $u \in V(\Gamma) \cap F_j^*$ let $\mu(u)$ be an edge-less subgraph of H_j with q_j vertices and with $V(\mu(u) \cap \Gamma) = \{u\}$; and for each space s with $s \subseteq F_j^*$, let $\mu(s) \subseteq H_j$, satisfying the following. (For a support F of Γ in $\Sigma, \mu(F)$ denotes the union of $\mu(s)$ over all spaces $s \subseteq F$ and $\mu(u)$ over all $u \in F \cap V(\Gamma)$.)

(V2) *For* $1 \leq j \leq r, H_j = \mu(F_j^*)$.

(V3) *For* $1 \leq j \leq r$, *if* $s \subseteq F_j^*$ *is a space with ends* u_1, u_2, *then* $V(\mu(s) \cap \Gamma) = \{u_1, u_2\}$, *and* $\mu(u_1), \mu(u_2) \subseteq \mu(s)$ *and there are* q_j *mutually vertex-disjoint paths of* $\mu(s)$ *between* $V(\mu(u_1))$ *and* $V(\mu(u_2))$.

(V4) *For* $1 \leq j \leq r$, *if* $s_1, s_2 \subseteq F_j^*$ *are spaces, and* $u \in F_j^* \cap V(\Gamma)$ *lies between them in* F_j^*, *then* $\mu(s_1) \cap \mu(s_2) \subseteq \mu(u)$ *(and consequently* $E(\mu(s_1) \cap \mu(s_2)) = \emptyset$*).*

In these circumstances, we call $\Gamma, F_1^*, \ldots, F_r^*, H_1, \ldots, H_r, \mu$ a *presentation* of G in Σ, with *defect* r. Its *depth* is $\max(q_j : 1 \leq j \leq r)$, or 0 if $r = 0$ (which implies $bd(\Sigma) = \emptyset$). Our next goal is to prove a form of (1.1) for graphs with presentations with given defect and depth, in a fixed surface. We use the following lemma.

6.1 *Let* $\Gamma, F_1^*, \ldots, F_r^*, H_1, \ldots, H_r, \mu$ *be a presentation of* G *in* Σ. *Let* $a, b, a', b' \in F_1^* \cap V(\Gamma)$ *be in order and let* $I, I' \subseteq F_1^*$ *be the supports with ends* a, b *and* a', b' *respectively. Let* $|\mu(a)| = q$; *let* $P_1, \ldots, P_q$ *be mutually vertex-disjoint paths of* $\mu(I)$ *between* $V(\mu(a))$ *and* $V(\mu(b))$; *and let* $P'_1, \ldots, P'_q$ *be mutually vertex-disjoint paths of* $\mu(I')$ *between* $V(\mu(a'))$ *and* $V(\mu(b'))$.

(i) For $1 \leq i, i' \leq q$, *if* $v \in V(P_i \cap P'_{i'})$, *then* $v \in V(\mu(b) \cap \mu(a'))$ *and* v *is an end of* P_i *and of* $P'_{i'}$.

(ii) If $a' = b$, *then* $P_1, \ldots, P_q$ *can be renumbered so that* $P_1 \cup P'_1, \ldots, P_q \cup P'_q$ *are mutually vertex-disjoint paths of* $\mu(I \cup I')$ *between* $V(\mu(a))$ *and* $V(\mu(b'))$.

(iii) If $|\mu(a) \cup \mu(b')| = |\mu(a') \cup \mu(b)| = k$ *say, then* $\{P_1, \ldots, P_q, P'_1, \ldots, P'_q\}$ *has cardinality* k, *and its members are* k *mutually vertex-disjoint paths of* $\mu(I) \cup \mu(I')$ *between* $V(\mu(a) \cup \mu(b'))$ *and* $V(\mu(b) \cup \mu(a'))$.

Proof. First we prove (i). Let $v \in V(P_i \cap P'_{i'})$, where $1 \leq i, i' \leq q$. We claim that $v \in V(\mu(b))$. If $a = b$ this is clear since

$$v \in V(P_i) \subseteq V(\mu(I)) = V(\mu(b))$$

and so we assume that $a \neq b$. Consequently there is a space $s \subseteq I$ with $v \in V(\mu(s))$. Let $J \subseteq F_1^*$ be the support with ends b and b'. Since $I' \subseteq J$ it follows that $v \in V(\mu(J))$. If $b = b'$ then $v \in V(\mu(b))$ by the same argument as above applied to J instead of I, and so we may assume that $b \neq b'$. Hence there is a space $s' \subseteq J$ with $v \in V(\mu(s'))$. Now b lies in F_1^* between s and s', and so $v \in V(\mu(b))$ by (V4). This proves our claim that $v \in V(\mu(b))$.

Now since $|V(\mu(b))| = q$ and each of $P_1, \ldots, P_q$ has an end in $V(\mu(b))$, it follows that each vertex of $\mu(b)$ is an end of one of $P_1, \ldots, P_q$, and in particular, v is an end of one of $P_1, \ldots, P_q$. Since $v \in V(P_i)$ and $P_1, \ldots, P_q$ are mutually vertex-disjoint, it follows that v is an end of P_i. Similarly, $v \in V(\mu(a'))$ and v is an end of $P'_{i'}$. This proves (i).

For (ii), let $P_1, \ldots, P_q$ and $P'_1, \ldots, P'_q$ be numbered so that for $1 \leq i \leq q$, P_i and P'_i have a common end in $V(\mu(b))$. For $1 \leq i, i' \leq q$, if $i \neq i'$ it follows from (i) that P_i is vertex-disjoint from $P'_{i'}$; and so $P_1 \cup P'_1, \ldots, P_q \cup P'_q$ are mutually vertex-disjoint paths satisfying (ii).

For (iii), let $\mu(a) = \{a_1, \ldots, a_q\}, \mu(b) = \{b_1, \ldots, b_q\}, \mu(a') = \{a'_1, \ldots, a'_q\}, \mu(b') = \{b'_1, \ldots, b'_q\}$, numbered so that for $1 \leq i \leq q$, P_i has ends a_i and b_i, and P'_i has ends a'_i and b'_i, and for $1 \leq i, i' \leq q$, if $i \neq i'$ then $b_i \neq a'_{i'}$. Suppose that $1 \leq i, i' \leq q$ and $a_i = b'_{i'}$. By (i), $i = i'$ and $a_i = b_i = a'_i = b'_i$, and so $P_i = P'_{i'}$. In particular, for $1 \leq i, i' \leq q$, if $i \neq i'$ then P_i and $P'_{i'}$ are vertex-disjoint. Thus

$$\begin{aligned}\{i : 1 \leq i \leq q, a_i = b'_{i'}\} &\subseteq \{i : 1 \leq i \leq q, P_i = P'_i\} \\ &\subseteq \{i : 1 \leq i \leq q, V(P_i \cap P'_i) \neq \emptyset\} = \{i : 1 \leq i \leq q, a'_i = b_i\}.\end{aligned}$$

But from the hypothesis of (iii), $|\mu(a) \cup \mu(b')| = |\mu(a') \cup \mu(b)|$, and so

$$|\{i : 1 \leq i \leq q, a_i = b'_i\}| = |\{i : 1 \leq i \leq q, a'_i = b_i\}|.$$

We therefore have equality throughout, and in particular for $1 \leq i \leq q$, if P_i meets P'_i then $P_i = P'_i$. Then (iii) follows. ■

6.2 *Let $\Gamma, F_1^*, \ldots, F_r^*, H_1, \ldots, H_r, \mu$ be a presentation of a graph G in some surface Σ. Let $I \subseteq F_1^*$ be a support with ends a, b. Then there are $|V(\mu(a))|$ mutually vertex-disjoint paths of $\mu(I)$ between $V(\mu(a))$ and $V(\mu(b))$.*

Proof. This follows by induction on the number of spaces included in I, applying (6.1)(ii) if I has an internal point in $V(\Gamma)$, and applying (V3) otherwise. ■

Let $\Gamma, F_1^*, \ldots, F_r^*, H_1, \ldots, H_r, \mu$ be a presentation of G in Σ, and let L be a vital linkage in G. If $\Gamma \subseteq L$ and every terminal of L is in one of $H_1, \ldots, H_r$, we say that L is *exhaustive* (for the presentation). If L is exhaustive then it follows that Γ is a forest and every vertex of Γ not in $bd(\Sigma)$ has degree 2 in Γ. Let us say a Σ-*jump* is a path in Γ with distinct ends both in $bd(\Sigma)$ and with no internal vertex in $bd(\Sigma)$. If L is exhaustive, then every edge of Γ is in a unique Σ-jump, and any two Σ-jumps have no common vertices except possibly one end. The main result of this section is the following.

6.3 *For every surface Σ and all integers $p, r \geq 0$ and $q \geq 1$, there exists $\lambda \geq 0$ such that, if G is a graph with a presentation in Σ of depth $\leq q$ and defect $\leq r$, and there is an exhaustive vital p-linkage in G, then there are at most λ Σ-jumps.*

The most difficult step in the proof of (6.3) is where Σ is a disc and $r = 1$. Then the cases when Σ is a disc and $r = 2, 3$ and at least 4 are successively easier, and finally the case when Σ is not a disc is also quite easy.

6.4 *For all integers $p \geq 0$ and $q \geq 1$ there exists $\lambda \geq 0$ such that, if G has a presentation in a disc Σ of depth $\leq q$ and defect 1, and there is an exhaustive vital p-linkage in G, then there are at most λ Σ-jumps.*

Proof. Choose n so that (2.6) holds, with k replaced by $2q$. Let $\lambda = n^n$. We claim that (6.2) is satisfied. For let Γ, F^*, H, μ be a presentation of G in a disc Σ with depth $\leq q$ and let L be a vital p-linkage which is exhaustive for the presentation.

A line $F \subseteq \Sigma$ is *good* if it is proper, both its ends are in $V(\Gamma)$, and no internal point is in $U(\Gamma)$. If F is a good line, there are two lines $J(F), K(F) \subseteq bd(\Sigma)$ with the same ends as F, where $K(F) \subseteq F^*$. Let $\Delta(F)$ be the closed disc in Σ bounded by $F \cup K(F)$. Define

$$A(F) = (\Gamma \cap \overline{\Sigma \setminus \Delta(F)}) \cup \mu(J(F))$$
$$B(F) = (\Gamma \cap \Delta(F)) \cup \mu(K(F))$$
$$X(F) = V(A(F) \cap B(F)).$$

(1) *$(A(F), B(F))$ is a separation of G, and $X(F) = V(\mu(u_1) \cup \mu(u_2))$ where F has ends u_1, u_2.*

The proof is similar to that of theorem (2.1) of [10] and we omit it.

Let F, F' be good lines with $\Delta(F') \subseteq \Delta(F)$. Let F have ends u_1, u_2, and let F' have ends u'_1, u'_2, so that u_1, u'_1, u'_2, u_2 are in order in F^*.

(2) *$A(F) \subseteq A(F')$ and $B(F') \subseteq B(F)$.*

This is immediate since $\Delta(F') \subseteq \Delta(F)$.

Let $I_1 \subseteq F^*$ be the support with ends u_1, u'_1, and let $I_2 \subseteq F^*$ have ends u'_2, u_2. From (6.2), there are q_1 mutually vertex-disjoint paths of $\mu(I_1)$ between $V(\mu(u_1))$ and $V(\mu(u'_1))$, and similarly for I_2, where the presentation has depth q_1. From (6.1)(iii), we deduce that

(3) *If $|X(F)| = |X(F')|$ there are $|X(F)|$ mutually vertex-disjoint paths of $B(F) \cap A(F')$ between $X(F)$ and $X(F')$, each using no edge of Γ.*

(4) *If $F_1, \ldots, F_n$ is a sequence of good lines such that $\Delta(F_{i+1}) \subseteq \Delta(F_i)$ for $1 \leq i < n$, there exists i with $1 \leq i < n$ such that every edge of $\Gamma \cap \Delta(F_i)$ is an edge of $\Gamma \cap \Delta(F_{i+1})$.*

Subproof. From (1), (2), (3) and (2.6) (with k replaced by $2q$), we deduce that there exists i, i' with $1 \leq i < i' \leq n$, such that $|X_i| = |X_{i'}|$ and $L \cap B_i \cap A_{i'}$ uses no edge of Γ. But every edge of $B_i \cap A_{i'}$ which is in Γ is also in L, since L is exhaustive, and so $E(\Gamma \cap B_i \cap A_{i'}) = \emptyset$. Then every edge of $\Gamma \cap \Delta(F_i)$ is also an edge of $\Gamma \cap \Delta(F_{i'})$ and hence of $\Gamma \cap \Delta(F_{i+1})$. This proves (4).

Let us say that two distinct regions r_1, r_2 of Γ in Σ *touch* if there is an edge e of Γ with $e \subseteq \overline{r}_1 \cap \overline{r}_2$. It follows that if r_1, r_2 touch then $\overline{r}_1 \cap \overline{r}_2 = U(J)$ for some Σ-jump J, and if they do not touch then $|\overline{r}_1 \cap \overline{r}_2| \leq 1$. In particular, the touching relation defines a graph T with vertex set the set of regions of Γ in Σ, and it is a tree. From (4), every vertex of T has degree at most n, and every path of T has at most n vertices. Hence $|E(T)| \leq n^n = \lambda$. But $E(T)$ is in 1-1 correspondence with the Σ-jumps, and the result follows. ■

6.5 *For all integers $p \geq 0$ and $q \geq 1$ there exists $\lambda \geq 0$ such that, if G has a presentation in a disc Σ of depth $\leq q$ and defect 2, and there is an exhaustive vital p-linkage in G, then there are at most λ Σ-jumps.*

Proof. Choose $n \geq 0$ as in (2.5) with k replaced by $2q$. Choose λ' so that (6.4) holds, with p, q, λ replaced by $p+2q, q, \lambda'$. Let $\lambda = n(2\lambda'+1)$. We claim that λ satisfies (6.5). For let $\Gamma, F_1^*, F_2^*, H_1, H_2, \mu$ be a presentation of G in the disc Σ, of depth $\leq q$, and let L be an exhaustive vital p-linkage. Let the two spaces not included in $F_1^* \cup F_2^*$ be s_1^* and s_2^*. Let $r_1, \ldots, r_t$ be all the regions of Γ in Σ which are incident both with a vertex in F_1^* and with a vertex in F_2^*, numbered in order, so that for $1 \leq i < t, r_i$ lies between s_1^* and r_{i+1}, in the natural sense. For $1 \leq i \leq t$, choose $a_i \in V(\Gamma) \cap F_1^*$ and $b_i \in V(\Gamma) \cap F_2^*$ so that r_i is incident with a_i and b_i; and let F_i be a good line with ends a_i, b_i and with interior in r_i. Let J_i and K_i be the two lines in $bd(\Sigma)$ with ends a_i and b_i, where $s_1^* \subseteq J_i$ and $s_2^* \subseteq K_i$. Then $J_1 \subseteq J_2 \subseteq \ldots \subseteq J_t$ and $K_t \subseteq K_{t-1} \subseteq \ldots \subseteq K_1$. Let A_i be the union of $\mu(J_i)$ and $\Gamma \cap \Delta$, where $\Delta \subseteq \Sigma$ is the disc bounded by $F_i \cup J_i$; and define B_i similarly using K_i instead of J_i. Then (A_i, B_i) is a separation of G, and $A_i \cap B_i = \mu(a_i) \cup \mu(b_i)$, as is easily seen. Since $\mu(a_i) \cap \mu(b_i) = \emptyset$, it follows that $|V(A_i \cap B_i)| = q_1 + q_2$, where $|\mu(a)| = q_j$ for all $a \in F_j^* \cap V(\Gamma)$ $(j = 1, 2)$. From (6.2), for $1 \leq i < t$, there are q_j mutually vertex-disjoint paths of $H_j \cap B_i \cap A_{i+1}$ between $V(A_i \cap B_i)$ and $V(A_{i+1} \cap B_{i+1})$ for $j = 1$ and 2, and since H_1 and H_2 are disjoint, it follows that there are $q_1 + q_2$ mutually vertex-disjoint paths of $(H_1 \cup H_2) \cap B_i \cap A_{i+1}$ between $V(A_i \cap B_i)$ and $V(A_{i+1} \cap B_{i+1})$. But for $1 \leq i < t, \Gamma \cap B_i \cap A_{i+1}$ has an edge since $r_i \neq r_{i+1}$ and it belongs to L since L is exhaustive, and so $L \cap B_i \cap A_{i+1} \not\subseteq H_1 \cup H_2$. From (2.5), it follows that $t \leq n$.

For $1 \leq i < t$, let P_i be the Σ-jump with $\overline{r}_i \cap \overline{r}_i + 1 = U(P_i)$, and let $\Gamma' = P_1 \cup \cdots \cup P_{t-1}$. Let $R_1, \ldots, R_t$ be the regions of Γ' in Σ, where $r_i \subseteq R_i$ $(1 \leq i \leq t)$. Now let us fix i with $1 \leq i \leq t$. We claim that there are at most $2\lambda'$ Σ-jumps J with $U(J) \subseteq R_i$. For $j = 1, 2$, let $I_j \subseteq F_j^*$ be the support $\overline{r}_i \cap F_j^*$; then $\overline{r}_j$ is a closed disc bounded by $I_1 \cup U(P_{i-1}) \cup I_2 \cup U(P_i)$ (replacing $U(P_{i-1})$ by s_1^* if $i = 1$, and replacing $U(P_i)$ by s_2^* if $i = t$). For $j = 1, 2$, let Γ_j be the drawing in Σ formed by all the vertices in $V(\Gamma) \cap I_j$ and all the Σ-jumps J with $U(J) \subseteq R_i \cup I_1 \cup I_2$ with both ends in I_j. We claim that Γ_j includes at most λ' Σ-jumps. For if $|I_j| = 1$ this is trivial, since Γ_j includes no Σ-jumps. If $|I_j| \neq 1$ then $\Gamma_j, I_j, \mu(I_j)$, and the restriction of μ to I_j, is a presentation of $\Gamma_j \cup \mu(I_j)$ in a disc with depth $\leq q$ and defect 1; and $L \cap (\Gamma_j \cup \mu(I_j))$ is an exhaustive vital $(p + 2q)$-linkage, and so by (6.4) it has at most λ' Σ-jumps. This proves our claim that Γ_1 and Γ_2 both include at most λ' Σ-jumps. Since every Σ-jump J with $U(J) \subseteq R_i \cup I_1 \cup I_2$ either has both ends in I_1 or has both ends in I_2 (because r_i is the only region of Γ included in R_i incident with both F_1^* and F_2^*) it follows that R_i includes at most $2\lambda'$ Σ-jumps. Counting $P_1, \ldots, P_{t-1}$, and using the fact that $t \leq n$, we deduce that there are at most $2\lambda' t + t - 1 \leq \lambda$ Σ-jumps altogether, as required. ■

6.6 *For all integers $p \geq 0$ and $q \geq 1$ there exists $\lambda \geq 0$ such that, if G has a presentation in a disc Σ of depth $\leq q$ and defect 3, and there is an exhaustive vital p-linkage G, then there are at most λ Σ-jumps.*

Proof. Choose $\lambda' \geq 0$ so that (6.5) holds with p, q, λ replaced by $p+2q, q, \lambda'$, and let $\lambda = 3\lambda'$. We claim that λ satisfies (6.6). For let $\Gamma, F_1^*, F_2^*, F_3^*, H_1, H_2, H_2, \mu$ be a presentation of G in the disc Σ, of depth $\leq q$, and let there be an exhaustive vital p-linkage. For $i = 1, 2, 3$, choose $v_i \in V(\Gamma) \cap F_i^*$ such that there is a region r of Γ in Σ incident with v_1, v_2 and v_3. (It is an easy exercise to prove that this choice is possible, since $V(\Gamma) \cap bd(\Sigma) \subseteq F_1^* \cup F_2^* \cup F_3^*$.) Choose $v_0 \in r \setminus bd(\Sigma)$, and for $i = 1, 2, 3$ let $F_i \subseteq r \cup \{v_i\}$ be a line with ends v_0 and v_i, so that F_1, F_2 and F_3 are mutually disjoint except for v_0, and $F_i \cap bd(\Sigma) = \{v_i\}$ $(i = 1, 2, 3)$. For $i = 1, 2, 3$ let $J_i \subseteq bd(\Sigma)$ be the support with ends $\{v_1, v_2, v_3\} \setminus \{v_i\}$ which does not include F_i^*. Let $\Delta_1 \subseteq \Sigma$ be the closed disc bounded by $J_1 \cup F_2 \cup F_3$ and define Δ_2, Δ_3 similarly. Now

$$\Gamma \cap \Delta_1, F_2^* \cap J_1, F_3^* \cap J_1, \mu(F_2^* \cap J_1), \mu(F_3^* \cap J_1)$$

and the restriction of μ to $(F_2^* \cup F_3^*) \cap J_1$, is a presentation of $\Gamma \cap \Delta_1 \cup \mu(F_2^* \cup J_1) \cup \mu(F_3^* \cup J_1)$ of depth $\leq q$ and defect 2, and this graph has an exhaustive vital $(p+2q)$-linkage. Consequently, Δ_1 includes at most λ' Σ-jumps. Similarly so do Δ_2 and Δ_3, and since every Σ-jump belongs to one of these discs, it follows that there are at most $3\lambda' = \lambda$ Σ-jumps altogether, as required. ■

6.7 *For all integers $p, r \geq 0$ and $q \geq 1$ there exists $\lambda \geq 0$ such that, if G has a presentation in a disc Σ of depth $\leq q$ and defect $\leq r$, and there is an exhaustive vital p-linkage in G, then there are at most λ Σ-jumps.*

Proof. We prove the result for all p and q by induction on r. By (6.4), (6.5), (6.6) we may assume that $r \geq 4$, and for all $p' \geq 0$ and $q' \geq 1$ there exists $\lambda(p', q') \geq 0$ such that, if G has a presentation in a disc Σ of depth $\leq q'$ and defect $< r$, and there is an exhaustive vital p'-linkage in G, then there are at most $\lambda(p', q')$ Σ-jumps. Let $\lambda = 2\lambda(p+2q, q)$. We claim that λ satisfies (6.7). For let $\Gamma, F_1^*, \ldots, F_r^*, H_1, \ldots, H_r, \mu$ be a presentation of G in the disc Σ of depth $\leq q$ and let there be an exhaustive vital p-linkage in G. Choose a region r incident with vertices in at least three of $F_1^*, \ldots, F_r^*$, as in the proof of (6.6). Since two of these three are non-consecutive, because $r \geq 4$, it follows that there is a proper line $F \subseteq \Sigma$ with ends $a, b \in V(\Gamma)$, and with $F \cap U(\Gamma) = \{a, b\}$, such that both of the lines $J_1, J_2 \subseteq bd(\Sigma)$ with ends a and b are disjoint from at least one of $F_1^*, \ldots, F_r^*$.

For $i = 1, 2$ let $\Delta_i \subseteq \Sigma$ be the closed disc bounded by $F \cup J_i$. Now $(\Gamma \cap \Delta_j) \cup \mu(J_i)$ has a presentation in Σ with depth $\leq q$ and defect $< r$, and it has an exhaustive vital $(p + 2q)$-linkage, and so Δ_j includes at most $\lambda(p + 2q, q)$ Σ-jumps, from the inductive hypothesis. But every Σ-jump belongs to one of Δ_1, Δ_2, and so there are at most $2\lambda(p + 2q, q) = \lambda$ Σ-jumps altogether, as required. ■

Proof of (6.3). We proceed by induction on Σ and by (6.3), we may assume that Σ is not a disc. We make the inductive hypothesis that

(1) *For all integers $p', r' \geq 0$ and $q' \geq 1$ there exists $\lambda(p, q, r) \geq 0$ such that for every surface Σ', if either $\hat{\Sigma}'$ is simpler then $\hat{\Sigma}$, or $\hat{\Sigma}'$ is homeomorphic to $\hat{\Sigma}$ and $c(\Sigma') < c(\Sigma)$, the following is true. If G' is a graph with a presentation in Σ' of depth $\leq q'$ and defect $\leq r'$, and there is an exhaustive vital p-linkage in G', then there are at most $\lambda(p, q, r)$ Σ'-jumps.*

Let $\lambda = \max(\lambda(p + 4q, q, r + 2), 2\lambda(p + 2q, q, r + 1))$. We claim that λ satisfies the theorem. For let $\Gamma, F_1^*, \ldots, F_r^*, H_1, \ldots, H_r, \mu$ be a presentation of a graph G in Σ, with depth $\leq q$.

(2) *We may assume that every proper line $F \subseteq \Sigma$ with ends in $V(\Gamma)$ and with no internal point in $U(\Gamma)$ separates Σ.*

Subproof. Suppose that $F \subseteq \Sigma$ is a proper line with ends $a, b \in V(\Gamma)$ and with no internal point in $U(\Gamma)$, and F does not separate Σ. Let Σ' be obtained from Σ by cutting along F. By (4.1)(i) and (4.1)(ii), either $\hat{\Sigma}'$ is simpler than $\hat{\Sigma}$, or $\hat{\Sigma}'$ is homeomorphic to $\hat{\Sigma}$ and $c(\Sigma') < c(\Sigma)$. Let Γ' be the drawing in Σ' obtained from Γ by splitting appropriately the vertices of Γ in F. Let $a \in F_1^*, b \in F_2^*$ say. Let F_1^* have ends u_1, u_2, and let $I_i \subseteq F_1^*$ be the support with ends u_1, a. For each $v \in V(\mu(a))$, let $\delta_i(v)$ be the set of edges of $\mu(I_i)$ incident with v $(i = 1, 2)$. By splitting v according to $\delta_1(v), \delta_2(v)$, for each $v \in V(\mu(a))$, and similarly splitting each $v \in V(\mu(b))$, we obtain a graph G' which has a presentation in Σ' (using the drawing Γ') of depth $\leq q$ and defect $\leq r + 2$. But by (2.4), it follows that G' has an exhaustive vital $(p + 4q)$-linkage, and so by (1), Γ' has at most $\lambda(p + 4q, q, r + 2)$ Σ'-jumps. But these Σ'-jumps are in 1-1 correspondence with the Σ-jumps of Γ, and so Γ has at most $\lambda(p + 4q, q, r + 2) \leq \lambda$ Σ-jumps, as required. This proves (2).

(3) $c(\Sigma) = 1$.

Subproof. Since $r \geq 4$, it follows that $c(\Sigma) \geq 1$. If $c(\Sigma) \geq 2$ then there is a region of Γ incident with vertices in two different cuffs, and so there is a proper line F with ends vertices of Γ in different cuffs, and with no internal point in $U(\Gamma)$. But then F does not separate Σ, contrary to (2). This proves (3).

(4) *We may assume that for every proper line $F \subseteq \Sigma$ with ends in $V(\Gamma)$ and with no internal point in $U(\Gamma)$, there is a closed disc $\Delta \subseteq \Sigma$ with $F \subseteq bd(\Delta) \subseteq F \cup bd(\Sigma)$.*

Subproof. By (2), F separates Σ. Let $\Sigma_1, \Sigma_2 \subseteq \Sigma$ be surfaces with $\Sigma_1 \cup \Sigma_2 = \Sigma$ and $\Sigma_1 \cap \Sigma_2 = F$. For $i = 1, 2$ let $\Gamma_i = \Gamma \cap \Sigma_i$, and let $G_i = \Gamma \cup \mu(bd(\Sigma) \cap \Sigma_i)$. Now for $i = 1, 2$, there is a presentation of G_i in Σ_i of depth $\leq q$ and defect $\leq r + 1$, using Γ_i; and $L \cap G_i$ is an exhaustive vital $(p + 2q)$-linkage. If both Σ_1 and Σ_2 are not discs, then by (4.2)(i) and (4.2)(ii), either $\hat{\Sigma}_i$ is simpler than $\hat{\Sigma}$, or $\hat{\Sigma}_i$ is homeomorphic to $\hat{\Sigma}$ and $c(\Sigma_i) < c(\Sigma)$ for $i = 1, 2$. But then from (1), there are $\leq \lambda(p + 2q, q, r + 1)$ Σ_i-jumps in Γ_i for $i = 1, 2$, and hence $\leq 2\lambda(p + 2q, q, r + 1) \leq \lambda$ Σ-jumps in Γ, as required. Consequently we may assume that one of Σ_1, Σ_2 is a disc. This proves (4).

From (4), it follows in particular that every Σ-jump J is homotopic in $\hat{\Sigma}$ to the lines in $bd(\Sigma)$ joining the ends of J. Let R be the closure in $\hat{\Sigma}$ of $\hat{\Sigma} \setminus \Sigma$; thus $R \cap \Sigma$ is the unique cuff of Σ. It follows that every O-arc in $R \cup U(\Gamma)$ is null-homotopic in $\hat{\Sigma}$. By theorem (11.10) of [5] there is a closed disc $\Delta \subseteq \hat{\Sigma}$ with $R \cup U(\Gamma) \subseteq \Delta$. By extending Δ to a sphere and removing $R \setminus bd(R)$ from it, we deduce that G has a presentation in a disc Σ', with depth $\leq q$ and defect $\leq r$. Then the result follows from (1), since $\lambda \geq \lambda(p + 4q, q, r + 2) \geq \lambda(p, q, r)$ and Σ is not a disc. ■

7 Presentations and tree-width

If $\mathcal{T}$ is a tangle in a graph G, a subgraph A of G is *small* (with respect to $\mathcal{T}$) if there exists $B \subseteq G$ such that $(A, B) \in \mathcal{T}$. An easy consequence of theorem (2.9) of [6] is the following (we omit the proof).

7.1 *If $\mathcal{T}$ is a tangle in G, and $A \subseteq G$, then the following are equivalent:*

(i) A is small with respect to $\mathcal{T}$;

(ii) if X denotes the set of vertices of A incident with edges of G not A, then $|X| < ord(\mathcal{T})$, and $(A, B) \in \mathcal{T}$ where $V(B) = (V(G) \setminus V(A)) \cup X$ and $E(B) = E(G) \setminus E(A)$;

(iii) there is a separation (A, B) of order $< ord(\mathcal{T})$, and $\mathcal{T}$ contains every such separation.

If $\Gamma, F_1^*, \ldots, F_r^*, H_1, \ldots, H_r, \mu$ is a presentation of G in Σ, and $\mathcal{T}$ is a tangle in G, we say that the presentation *surrounds* $\mathcal{T}$ if $\mu(s)$ is small for every space $s \subseteq F_1^* \cup \ldots \cup F_r^*$.

If Γ is a drawing in a surface Σ such that $|V(\Gamma) \cap C| \geq 2$ for every cuff C, we define Γ^+ to be the drawing in $\hat{\Sigma}$ with $U(\Gamma^+) = U(\Gamma) \cup bd(\Sigma)$ and $V(\Gamma^+) = V(\Gamma)$.

7.2 *Let $\theta \geq 1$, and let $\Gamma, F_1^*, \ldots, F_r^*, H_1, \ldots, H_r, \mu$ be a presentation of a graph G in a surface Σ, of depth $\leq q$, and surrounding a tangle of order $\geq q\theta$. Then Γ^+ has a tangle of order $\geq \theta$.*

Proof. Γ^+ has the same vertex set as Γ, and its edges are the edges and spaces of Γ. Let $S(\Gamma)$ be the set of spaces of Γ. For any subgraph A of Γ^+ we define $\sigma(A)$ to be the union of $A \cap \Gamma$ with all the graphs $\mu(v)$ $(v \in V(A) \cap bd(\Sigma))$ and $\mu(s)$ $(s \in E(A) \cap S(\Gamma))$.

(1) *If (A, B) is a separation of Γ^+ then $(\sigma(A), \sigma(B))$ is a separation of G of order $\leq q|V(A \cap B)|$.*

Subproof. Clearly $\sigma(A) \cup \sigma(B) = G$. No edge of G belongs to both $\sigma(A)$ and $\sigma(B)$, by (V1) and (V4), and so $(\sigma(A), \sigma(B))$ is a separation of G. We claim that

$$V(\sigma(A) \cap \sigma(B)) \subseteq V(A \cap B) \cup \bigcup(\mu(u) : u \in V(A \cap B) \cap bd(\Sigma)).$$

For let $v \in V(\sigma(A) \cap \sigma(B)) \setminus V(A \cap B)$. Since $\sigma(A) \cap \Gamma = A$ and $\sigma(B) \cap \Gamma = B$, it follows that $v \notin V(\Gamma)$; let $v \in V(H_1)$ say. There exists x such that $v \in V(\mu(x))$, and either $x \in V(A) \cap F_1^*$ or $x \in E(A) \cap S(\Gamma)$ and $x \subseteq F_1^*$; and similarly there exists y (with B instead of A). Choose x and y as close together in F_1^* as possible, in the natural sense. If z lies in F_1^* between x and y, and $z \in V(\Gamma) \cup S(\Gamma)$, and $z \neq x, y$, then z belongs to one of A, B, and $v \in V(\mu(z))$ by (V3) and (V4), contrary to our choice of x and y. If $x \in V(\Gamma)$ and $y \in S(\Gamma)$ with one end x, then $x \in V(A \cap B)$ and $v \in V(\mu(x))$ as required. Finally, if $x = y \in V(\Gamma)$ then again $x \in V(A \cap B)$ and $v \in V(\mu(x))$; while if $x = y \in S(\Gamma)$ then $u \in V(A \cap B)$ and $v \in V(\mu(u))$, where u is one end of x. This proves our claim that

$$V(\sigma(A) \cap \sigma(B)) \subseteq V(A \cap B) \cup \bigcup(\mu(u) : u \in V(A \cap B) \cap bd(\Sigma)).$$

Consequently, $|V(\sigma(A) \cap \sigma(B))| \leq q|V(A \cap B)|$. This proves (1).

Let $\mathcal{T}$ be a tangle in G of order $\geq q\theta$, surrounded by the presentation. Let $\mathcal{T}'$ be the set of all separations (A, B) of Γ^+ of order $< \theta$ such that $(\sigma(A), \sigma(B)) \in \mathcal{T}$. We claim that $\mathcal{T}'$ is a tangle in Γ^+ of order θ. Let us verify the three axioms for a tangle.

For the first axiom, let (A, B) be a separation of Γ^+ of order $< \theta$. Then $(\sigma(A), \sigma(B))$ has order $< q\theta \leq ord(\mathcal{T})$ by (1), and so one of $(\sigma(A), \sigma(B)), (\sigma(B), \sigma(A))$ belongs to $\mathcal{T}$. Hence one of $(A, B), (B, A)$ belongs to $\mathcal{T}'$. This verifies the first axiom.

For the second, let $(A_i, B_i) \in \mathcal{T}'$ $(i = 1, 2, 3)$. Then $(\sigma(A_i), \sigma(B_i)) \in \mathcal{T}$ $(i = 1, 2, 3)$, and so $\sigma(A_1) \cup \sigma(A_2) \cup \sigma(A_3) \neq G$. But

$$\sigma(A_1) \cup \sigma(A_2) \cup \sigma(A_3) = \sigma(A_1 \cup A_2 \cup A_3)$$

and $\sigma(\Gamma^+) = G$, and so $A_1 \cup A_2 \cup A_3 \neq \Gamma^+$. This verifies the second axiom.

For the third, it suffices by [3, theorem (2.7)] to show that if $e \in E(\Gamma^+)$ and K_e denotes the graph consisting of e and its ends, then $(\Gamma^+ \setminus e, K_e) \notin \mathcal{T}'$, that is, $(\sigma(\Gamma^+ \setminus e), \sigma(K_e)) \notin \mathcal{T}$. If $e \in E(\Gamma)$, or if $e \in S(\Gamma)$ and $e \not\subseteq F_1^* \cup \ldots \cup F_r^*$ then $V(\sigma(\Gamma^+ \setminus e)) = V(G)$, and so $(\sigma(\Gamma^+ \setminus e), \sigma(K_e)) \in \mathcal{T}$, since $\mathcal{T}$ satisfies the third axiom. We assume then that $e \in S(\Gamma)$, and $e \subseteq F_1^* \cup \ldots \cup F_r^*$. Then $\sigma(K_e) = \mu(e)$, and so $(\sigma(\Gamma^+ \setminus e), \sigma(K_e)) \notin \mathcal{T}$ by (7.1) since $\mu(e)$ is small. This verifies the third axiom.

Hence $\mathcal{T}'$ is a tangle in Γ^+ of order θ, as required. ■

The main result of this section is the following.

7.3 *For every surface Σ and all integers $p, r \geq 0$ and $q \geq 1$, there exists $\theta \geq 1$ such that, if a graph G has a presentation in Σ of depth $\leq q$, and defect $\leq r$ which surrounds a tangle of order $\geq \theta$, then G has no vital p-linkage.*

Proof. Choose λ so that (6.3) holds. Choose θ_1 so that (5.2) holds with Σ, p, θ replaced by $\hat{\Sigma}, 2(p+\lambda), \theta_1$. Choose θ_2 so that (4.6) holds, with $\Sigma, \theta', c, \theta$ replaced by $\hat{\Sigma}, \theta_1, c(\Sigma), \theta_2$. Let $\theta = q\theta_2$. We claim that θ satisfies (7.3). For let $\Gamma, F_1^*, \ldots, F_r^*, H_1, \ldots, H_r, \mu$ be a presentation of G in Σ, of depth $\leq q$ (we may assume its defect is exactly r), surrounding a tangle of order $\geq \theta$. Suppose that L is a vital p-linkage in G. Let Γ' be the union of all paths in $L \cap \Gamma$ with both ends in $bd(\Sigma)$ and no internal vertex in $bd(\Sigma)$ (including one-vertex paths). Then $\Gamma', F_1^*, \ldots, F_r^*, H_1, \ldots, H_r, \mu$ is a presentation of $G' = \Gamma' \cup H_1 \cup \ldots \cup H_r$ in Σ, of depth $\leq q$. Moreover, $L \cap G'$ is a vital linkage in G', by (2.3), and it is a p-linkage since each component of L includes at most one component of $L \cap G'$, and it is exhaustive. From (6.3), there are at most λ Σ-jumps in Γ'. Let X be the set of all $v \in V(\Gamma) \cap bd(\Sigma)$ such that $\{v\}$ is the vertex set of a component of $L \cap \Gamma$. Every other component of $L \cap \Gamma$ either is a Σ-jump in Γ', or has an end some vertex $v \in V(\Gamma) \setminus bd(\Sigma)$ which is a terminal of L, by (2.2). Consequently, $L \cap (\Gamma \setminus X)$ has at most $\lambda + p$ components, and so is a vital $2(\lambda + p)$-linkage in $\Gamma \setminus X$. From (5.2), $\Gamma \setminus X$ has no tangle of order $\geq \theta_1$. Now $\Gamma \setminus X$ is obtained from Γ^+ by deleting some vertices and edges of Γ^+, all incident with one of $c(\Sigma)$ regions of Γ^+ in $\hat{\Sigma}$. From (4.6), Γ^+ has no tangle of order $\geq \theta_2$. From (7.2), the presentation of G captures no tangle of order $\geq q\theta_2 = \theta$, a contradiction. Thus L is not a a vital p-linkage in G, as required. ■

8 Pseudo-presentations

In a presentation, the graphs $H_1, \ldots, H_r$ are disjoint. Now we want to consider a slightly more general object, in which the "last" vertices of each H_i may equal some of the "first" vertices of the next one of $H_1, \ldots, H_r$ on the same cuff. More precisely, a *pseudo-presentation* of a graph G in a surface Σ is defined as follows. Let $\Gamma \subseteq G$ be a drawing in Σ such that $|V(\Gamma) \cap C| \geq 2$ for each cuff C. Let $F_1^*, \ldots, F_r^*$ be mutually disjoint supports of Γ in Σ with $V(\Gamma) \cap bd(\Sigma) \subseteq F_1^* \cup \ldots \cup F_r^*$. For $1 \leq j \leq r$ let $q_j \geq 1$ be an integer; for each $u \in V(\Gamma) \cap F_j^*$ let $\mu(u)$ be an edge-less subgraph of G with q_j vertices and with $V(\mu(u) \cap \Gamma) = \{u\}$; and for each space s with $s \subseteq F_j^*$ let $\mu(s) \subseteq G$, satisfying the following. (A *unit* is either a vertex in $V(\Gamma) \cap bd(\Sigma)$ or a space in $F_1^* \cup \ldots \cup F_r^*$.)

(P1) *$G = \Gamma \cup \bigcup(\mu(F_j^*) : 1 \leq j \leq r)$; Γ and all the graphs $\mu(s)$ for $s \in S(\Gamma)$ and $s \subseteq F_1^* \cup \ldots \cup F_r^*$ are mutually edge-disjoint; and if x_1, x_2 are units in different cuffs then $\mu(x_1)$ and $\mu(x_2)$ are mutually vertex-disjoint.*

(P2) *For $1 \leq j \leq r$, if $s \subseteq F_j^*$ is a space with ends u_1, u_2 then $V(\mu(s) \cap \Gamma) = \{u_1, u_2\}$ and $\mu(u_1), \mu(u_2) \subseteq \mu(s)$, and there are q_j mutually vertex-disjoint paths of $\mu(s)$ between $V(\mu(u_1))$ and $V(\mu(u_2))$.*

(P3) *For each cuff C, if x_1 and x_2 are units in C and $u_1, u_2 \in C \cap V(\Gamma)$ are such that x_1, u_1, x_2, u_2 occur in order in C, then $\mu(x_1) \cap \mu(x_2) \subseteq \mu(u_1) \cup \mu(u_2)$.*

(P4) *For each cuff $C, \Cap(\mu(u) : u \in V(\Gamma) \cap C)$ is null.*

Then we say that $\Gamma, F_1^*, \ldots, F_4^*, \mu$ is a *pseudo-presentation* of G in Σ. Its *depth* is $\max(q_j : 1 \leq j \leq r)$, or 0 if $r = 0$, and its *defect* is r. We say that the pseudo-presentation is *disjointed* if for each space s with ends u_1, u_2 such that $s \not\subseteq F_1^* \cup \ldots \cup F_r^*$, the graphs $\mu(u_1)$ and $\mu(u_2)$ are vertex-disjoint.

8.1 *Let $\Gamma, F_1^*, \ldots, F_r^*, \mu$ be a disjointed pseudo-presentation of a graph G in a surface Σ. Then $\Gamma, F_1^*, \ldots, F_r^*,\ \mu(F_1^*), \ldots, \mu(F_r^*), \mu$ is a presentation of G in Σ.*

Proof. Let $H_j = \mu(F_j^*)$ $(1 \le j \le r)$. We must check that (V1)–(V4) hold (and that $\mu(s) \subseteq H_j$ if $s \subseteq F_j^*$ is a space, which is obvious). For (V1),

$$G = \Gamma \cup \bigcup(\mu(F_j^*) : 1 \le j \le r) = \Gamma \cup H_1 \cup \ldots \cup H_r$$

from (P1). To see that $H_1, \ldots, H_r$ are mutually disjoint, suppose that $v \in V(H_{j_1} \cap H_{j_2})$ say. For $i = 1, 2$ let x_i be a unit in $F_{j_i}^*$ with $v \in V(\mu(x_i))$. From (P1), x_1 and x_2 belong to the same cuff C. We may assume that $k \le r$, and that for $1 \le j \le r, F_j^* \subseteq C$ if and only if $j \le k$, and that $F_1^*, \ldots, F_k^*$ occur in order around C. Let $s_1^*, \ldots, s_k^*$ be the spaces in C not in $F_1^* \cup \ldots \cup F_r^*$, numbered so that if $k > 1$ then s_j^* has one end in F_j^* and one in F_{j+1}^* for $1 \le j < k$, and s_k^* has one end in F_k^* and one in F_1^*. For $1 \le j \le k$, let F_j^* have ends a_j, b_j, where a_j is an end of s_j^* and b_j is an end of s_{j-1}^* (or of s_k^*, if $j = 1$). Let us write $b_{k+1} = b_1$ and $a_0 = a_k$. Now since the pseudo-presentation is disjointed, v does not belong to both $\mu(a_{j_1 - 1})$ and $\mu(b_{j_1})$; choose $u_1 \in \{a_{j_1-1}, b_{j_1}\}$ so that $v \notin V(\mu(u_1))$. Similarly choose $u_2 \in \{a_{j_1}, b_{j_1+1}\}$ with $v \notin V(\mu(u_2))$. Now x_1, u_1, x_2, u_2 occur in C in order, contrary to (P3), since $v \in V(\mu(x_1) \cap \mu(x_2))$ and $v \notin V(\mu(u_1) \cup \mu(u_2))$.

This proves that $H_1, \ldots, H_r$ are disjoint. To complete the proof of (V1), we must check that for $1 \le j \le r, V(H_j \cap \Gamma) = V(\Gamma) \cap F_j^*$, and $E(H_j \cap \Gamma) = \emptyset$. Now $V(\Gamma) \cap F_j^* \subseteq V(\mu(F_j^*)) = V(H_j)$, because $u \in V(\mu(u))$ for each $u \in V(\Gamma) \cap F_j^*$. Conversely, let $u \in V(H_j \cap \Gamma)$. Let x be a unit in F_j^* with $u \in V(\mu(x))$. Since $V(\mu(x) \cap \Gamma) = \{x\}$ if x is a vertex, and $V(\mu(x) \cap \Gamma)$ is the set of ends of x by (P2) if x is a space, it follows that either $v \in V(\Gamma) \cap F_j^*$ or v is an end of some space in F_j^*; and so in either case, $v \in V(\Gamma) \cap F_j^*$. This proves that $V(H_j \cap \Gamma) = V(\Gamma) \cap F_j^*$. Finally, $E(H_j \cap \Gamma) = \emptyset$ because $E(\Gamma \cap \mu(S)) = \emptyset$ for each space $s \subseteq F_j^*$, by (P1). This proves (V1).

Now (V2) holds by definition, and (V3) is the same as (P2), and so it only remains to verify (V4). Let $1 \le j \le r$, and let $s_1, s_2 \subseteq F_j^*$ be spaces, and let $u \in F_j^* \cap V(\Gamma)$ lie in F_j^* between s_1 and s_2. Let s_0 be a space with $s_0 \not\subseteq F_1^* \cup \ldots \cup F_r^*$, in the same cuff C as F_j^*. (This exists since $|V(\Gamma) \cap C| \ge 2$.) Let s_0 have ends a_0 and b_0. Now a_0, s_1, u, s_2 occur in order in C, and so by (P3), $\mu(s_1) \cap \mu(s_2) \subseteq \mu(a_0) \cup \mu(s_2)$. Similarly, $\mu(s_1) \cap \mu(s_2) \subseteq \mu(b_0) \cup \mu(s_2)$, and so

$$\mu(s_1) \cap \mu(s_2) \subseteq (\mu(a_0) \cap \mu(b_0)) \cap \mu(s_2) = \mu(s_2).$$

This proves (V4), as required. ∎

Secondly, we wish to discuss splitting vertices in a pseudo-presentation, in order to make it disjointed. More precisely, let $\Gamma, F_1^*, \ldots, F_r^*, \mu$ be a pseudo-presentation of G in Σ. Let s^* be a space with $s^* \not\subseteq F_1^* \cup \ldots \cup F_r^*$, with ends u_1, u_2 say. Let C be the cuff including s^*. Let $v \in V(\mu(u_1) \cap \mu(u_2))$.

From (P4), there exists $u^* \in V(\Gamma) \cap C$ with $v \notin V(\mu(u^*))$. Let F_i be the support in C with ends u_i and u^* which does not include s^* ($i = 1, 2$). For $i = 1, 2$ let δ_i be the set of all edges e incident with v such that $e \in E(\mu(s))$ for some space $s \subseteq F_i \cap (F_1^* \cup \ldots \cup F_r^*)$. Then $\delta_1 \cap \delta_2 = \emptyset$, for by (P2) there is only one space $s \subseteq C$ with $e \in E(\mu(s))$; and $\delta_1 \cup \delta_2$ is the set of all edges of G incident with v, because by (P1) e is an edge of $\mu(s)$ for some space $s \subseteq C \cap (F_1^* \cup \ldots \cup F_r^*)$. Let G' be obtained from G by splitting v according to δ_1, δ_2, and let the two new vertices of G' be v_1, v_2 where the edges in δ_i are incident with v_i ($i = 1, 2$). For $i = 1, 2$, and for each subgraph A of G with $v \in V(A)$, let $\sigma_i(A)$ be the graph with the same set of edges as A, and with vertex set $(V(A) \setminus \{v\}) \cup \{v_i\}$, with the same incidence relation as A except that the edges of A that are incident with v in A are incident with v_i in $\sigma_i(A)$. Then $\sigma_i(A)$ is a subgraph of G' provided that δ_i contains all edges of A incident with v. Define μ' by:

$\mu'(u) = \mu(u)$ if $v \in V(\Gamma) \cap bd(\Sigma)$ and $v \notin V(\mu(u))$

$\mu'(u) = \sigma_i(\mu(u))$ if $v \in V(\Gamma) \cap F_i$ and $v \in V(\mu(u))$ $(i = 1, 2)$

$\mu'(s) = \mu(s)$ if s is a space with $s \subseteq F_1^* \cup \ldots \cup F_r^*$ and $v \notin V(\mu(s))$

$\mu'(s) = \sigma_i(\mu(s))$ if s is a space with $s \subseteq (F_1^* \cup \ldots \cup F_r^*) \cap F_i$ and $v \in V(\mu(s))$ $(i = 1, 2)$.

8.2 $\Sigma, F_1^*, \ldots, F_r^*, \mu'$ *is a pseudo-presentation of* G'.

Proof. Since $v \in V(\mu(u_1) \cap \mu(u_2))$ it follows that $v \neq u_1, u_2$, and so $v \notin V(\Gamma)$. Consequently $\Gamma \subseteq G'$. Let us check that $\mu'(x) \subseteq G'$, for each unit x. If x is a vertex this is clear, so let x be a space with $x \subseteq F_1^* \cup \ldots \cup F_r^*$. If $v \notin V(\mu(x))$ then $\mu'(x) = \mu(x) \subseteq G'$ as required. We assume then that $v \in V(\mu(x))$, and $x \subseteq F_1$ say. Then $\mu'(x) = \sigma_1(\mu(x))$, and every edge of $\mu(x)$ incident with v belongs to δ_1, by definition of δ_1. Consequently $\mu'(x) \subseteq G'$, as required.

We must verify (P1)-(P4). For (P1), we observe that

$$E(G') = E(G) = E(\Gamma \cup \bigcup(\mu(F_j^*) : 1 \le j \le r)) = E(\Gamma \cup \bigcup(\mu'(F_j^*) : 1 \le j \le r))$$

and

$$\begin{aligned} V(G') &= (V(G) \setminus \{v\}) \cup \{v_1, v_2\} = (V(\Gamma \cup \bigcup(\mu(F_j^*) : 1 \le j \le r)) \setminus \{v\}) \cup \{v_1, v_2\} \\ &\subseteq V(\Gamma) \cup \bigcup(\mu'(F_j^*) : 1 \le j \le r) \end{aligned}$$

since $v_1 \in \mu'(F_1^*)$ and $v_2 \in \mu'(F_2^*)$. The remainder of (P1) is clear and so (P1) holds.

For (P2), let $s \subseteq F_j^*$ be a space with ends a, b say. Certainly $V(\mu'(s) \cap \Gamma) = V(\mu(s) \cap \Gamma) = \{a, b\}$. To see that $\mu'(a) \subseteq \mu'(s)$ we argue as follows. If $v \notin V(\mu(a))$ then

$$\mu'(a) = \mu(a) \subseteq \mu(s) \setminus v = \mu'(s) \setminus v \subseteq \mu'(s)$$

as required. We assume then that $v \in V(\mu(a))$, and hence $v \in V(\mu(s))$. Thus $s \subseteq F_1$ or $s \subseteq F_2$, say $s \subseteq F_1$; and so $a \in F_1$. Hence

$$\mu'(a) = \sigma_1(\mu(a)) \subseteq \sigma_1(\mu(s)) = \mu'(s)$$

as required. This verifies the second assertion of (P2). The third assertion follows because there is an isomorphism from $\mu(s)$ to $\mu'(s)$ mapping $\mu(a)$ to $\mu'(a)$ and $\mu(b)$ to $\mu'(b)$.

For (P3), let x_1, a, x_2, b be units of a cuff C in order, where $a, b \in V(\Gamma)$. We must show that

$$\mu'(x_1) \cap \mu'(x_2) \subseteq \mu'(a) \cup \mu'(b).$$

Since

$$E(\mu'(x_1) \cap \mu'(x_2)) = E(\mu(x_1) \cap \mu(x_2)) \subseteq E(\mu(a) \cup \mu(b)) = E(\mu'(a) \cup \mu'(b)),$$

it suffices to show that every vertex of $\mu'(x_1) \cap \mu'(x_2)$ is a vertex of $\mu'(a) \cup \mu'(b)$. Let $w \in V(\mu'(x_1) \cap \mu'(x_2))$. If $w \neq v_1, v_2$ then

$$w \in V(\mu(x_1) \cap \mu(x_2)) \subseteq V(\mu(a) \cup \mu(b))$$

and since $w \neq v$ it follows that $w \in V(\mu'(a) \cup \mu'(b))$ as required. We may assume then that $w = v_1$ say. Thus $v_1 \in V(\mu'(x_1) \cap \mu'(x_2))$. From the definition of $\mu'(x_1)$ and $\mu'(x_2)$ it follows that $v \in \mu(x_1) \cap \mu(x_2)$ and x_1, x_2 belong to F_1. Since x_1, a, x_2, b occur in order in C we may assume that $a \in F_1$ and lies in F_1 between x_1 and x_2. Since x_1, a, x_2, u^* occur in order in C, it follows that

$$v \in V(\mu(x_1) \cap \mu(x_2)) \subseteq V(\mu(a) \cup \mu(u^*))$$

and since $v \notin V(\mu(u^*))$ we deduce that $v \in V(\mu(a))$. But $a \in F_1$, and $\mu'(a) = \sigma_1(\mu(a))$, and so $v_1 \in V(\mu'(a))$, as required. This proves (P3).

For (P4), suppose that $w \in V(\mu'(u))$ for every $u \in V(\Gamma) \cap C'$ for some cuff C'. If $w \neq v_1, v_2$ then $w \in \mu(a)$ for all $u \in V(\Gamma) \cap C'$, and so $w \in \bigcap(\mu(u) : u \in V(\Gamma) \cap C')$, contrary to the truth of (P4) for μ. Thus we may assume that $w = v_1$ say, and hence $C' = C$. But $v_1 \notin V(\mu(u^*)) = V(\mu'(u^*))$, a contradiction. This proves (P4), and therefore completes the proof of (8.2). ■

Now let $\Gamma, F_1^*, \ldots, F_r^*, \mu$ be a pseudo-presentation of G, and let $\mathcal{T}$ be a tangle in G. We say the pseudo-presentation *surrounds* $\mathcal{T}$ if $\mu(s)$ is small for every space $s \in F_1^* \cup \ldots \cup F_r^*$.

8.3 *With notation as in* (8.2), *suppose that* $\Gamma, F_1^*, \ldots, F_r^*, \mu$ *is a pseudo-presentation of depth* $\leq q$, *and surrounds a tangle* $\mathcal{T}$ *in* G *of order* $\theta \geq 2q+3$. *Then* $\Gamma, F_1^*, \ldots, F_r^*, \mu'$ *surrounds a tangle in* G' *of order* $\theta - 1$.

Proof. Let $\mathcal{T}'$ be the tangle of order $\theta - 1$ in G' induced by $\mathcal{T} \setminus \{v\}$. (This exists, from (3.2) and (3.4).) Let $s \subseteq F_1^* \cup \ldots \cup F_r^*$ be a space, and choose $B \subseteq G$ minimal such that $(\mu(s), B)$ is a separation of G. By (7.1), $(\mu(s), B) \in \mathcal{T}$ since $\mu(s)$ is small. Let $w \in V(\mu(s) \cap B)$; we claim that $w \in V(\mu(a) \cup \mu(b))$, where s has ends a and b. For if $w = a$ or b this is true, and we assume not. By (P2), $w \notin V(\Gamma)$. Since $u \in V(B)$ it follows from the minimality of B that some edge e of G not in $\mu(s)$ is incident with w, and $e \notin E(\Gamma)$ since $w \notin V(\Gamma)$. Choose a space s' with $e \in E(\mu(s'))$. Then $s' \neq s$, and so s, a, s', b are in order in the cuff containing them (this exists, by (P1)). By (P3),

$$w \in V(\mu(s) \cap \mu(s')) \subseteq V(\mu(a) \cup \mu(b))$$

as claimed. We have proved then that $V(\mu(s) \cap B) \subseteq V(\mu(a) \cup \mu(b))$. Hence $(\mu(s), B)$ has order $\leq 2q \leq \theta - 2$.

Let A_1 be the subgraph of G with vertex set $V(\mu(s)) \cup \{v\}$ and edge set $E(\mu(s))$; and let B_1 have vertex set $V(B) \cup \{v\}$ and edge set $E(B)$. By theorem (2.9)(iii) of [6], $(A_1, B_1) \in \mathcal{T}$, since $(\mu(s), B)$ has order $\leq \theta - 2$. Hence $(A_1 \setminus v, B_1 \setminus v) \in \mathcal{T} \setminus v$. Let $A_2 = \mu'(s)$, and let $B_2 \subseteq G'$ be minimal such that (A_2, B_2) is a separation of G'. Then (A_2, B_2) has order $\leq 2q$. But $A_2 \cap (G \setminus v) = A_1 \setminus v$, and so $(A_2 \cap (G \setminus v), B_2 \cap (G \setminus v)) \in \mathcal{T} \setminus v$ by theorem (2.9)(iii) of [6], since $(A_1 \setminus v, B_1 \setminus v) \in \mathcal{T} \setminus v$ and $\mathcal{T} \setminus v$ has order ≥ 2. From the definition of $\mathcal{T}'$ it follows that $(A_2, B_2) \in \mathcal{T}'$, and so $\mu'(s)$ is small with respect to $\mathcal{T}'$, as required. ■

The main result of this section is the following.

8.4 *For every surface* Σ *and all integers* $p, r \geq 0$ *and* $q \geq 1$, *there exists* $\theta \geq 1$ *such that, if a graph* G *has a pseudo-presentation in* Σ *of depth* $\leq q$ *and defect* $\leq r$ *which surrounds a tangle of order* $\geq \theta$, *then* G *has no vital* p*-linkage.*

Proof. Choose $\theta' \geq 2q+2$ so that (7.3) holds, with Σ, p, q, r replaced by $\Sigma, p + 2qr, q, r$. Let $\theta = \theta' + qr$. We claim that θ satisfies (8.4). For suppose that G has a pseudo-presentation in Σ of depth $\leq q$ and defect $\leq r$ which surrounds a tangle of order $\geq \theta$. For each space $s \not\subseteq F_1^* \cup \ldots \cup F_r^*$ with ends u_1, u_2 and for each $v \in V(\mu(u_1) \cap \mu(u_2))$, split v as discussed earlier in this section. After at most qr splittings we obtain a graph G' with a disjointed pseudo-presentation in Σ of depth $\leq q$ and defect $\leq r$, which by (8.3) surrounds a tangle of order $\geq \theta - qr = \theta'$, since $\theta' + 1 \geq 2q + 3$. By (8.1), G' has a presentation in Σ of depth $\leq q$ and defect $\leq r$ which surrounds a tangle of order $\geq \theta'$. By (7.3), G' has no vital $(p + 2qr)$-linkage, from the choice of θ'. But G' is obtained from G by splitting $\leq qr$ vertices, and so by (2.4) G has no vital p-linkage, as required. ■

9 Paintings and portraits

In this section we derive a modification of (8.4) appropriate for applying the results of [9, 10]. A *painting* Γ in a surface Σ is a pair (U, V), where $U \subseteq \Sigma$ is closed and $V \subseteq U$ is finite, satisfying

(i) $U \setminus V$ has only finitely many connected components, called *cells*

(ii) for each cell c, its closure $\overline{c}$ is a closed disc, and $\tilde{c} \subseteq bd(\overline{c})$ (where $\tilde{c}$ denotes $\overline{c} \setminus c$) and $|\tilde{c}| = 2$ or 3

(iii) $bd(\Sigma) \subseteq U$

(iv) for each cell c, if $c \cap bd(\Sigma) \neq \emptyset$ (that is, c is a *border* cell) then $|\tilde{c}| = 2$ and $\overline{c} \cap bd(\Sigma)$ is a line with ends the two members of $\tilde{c}$.

(This differs very slightly from the definition in [10]. In that paper it was convenient to allow cells c with $|\tilde{c}| = 0$ or 1, but we no longer need such cells.) The set of cells of Γ is denoted by $C(\Gamma)$, and we write $U = U(\Gamma), V = V(\Gamma)$. A cell which is not a border cell is called an *internal* cell.

Let G be a graph. We say that a pair Γ, α is a *portrait* of G in Σ (this is closely connected with the "portrayal" of [10]) if

(R1) *Γ is a painting in Σ and $V(\Gamma) \subseteq V(G)$, and α is a function with domain $(V(\Gamma) \cap bd(\Sigma)) \cup C(\Gamma)$.*

(R2) *For each c of $\Gamma, \alpha(c) \subseteq G$ and $V(\alpha(c)) \cap V(\Gamma) = \tilde{c}$.*

(R3) *$G = \bigcup(\alpha(c) : c \in C(\Gamma))$, and $E(\alpha(c_1) \cap \alpha(c_2)) = \emptyset$ for all distinct cells c_1, c_2.*

(R4) *For each $v \in V(\Gamma) \cap bd(\Sigma), \alpha(v)$ is an edge-less subgraph of G and $V(\alpha(v) \cap V(\Gamma) = \{v\}$.*

(R5) *For each cuff C there is an integer $q(C) \geq 1$ such that $|V(\alpha(v))| = q(C)$ for all $v \in V(\Gamma) \cap C$.*

(R6) *For each border cell c with $\tilde{c} = \{u, v\}, \alpha(u)$ and $\alpha(v)$ are subgraphs of $\alpha(c)$, and either*

(i) *there are $|V(\alpha(u))|$ mutually vertex-disjoint paths of $\alpha(c)$ between $V(\alpha(u))$ and $V(\alpha(v))$, that is, c is "linked", or*

(ii) *there is a cell c' with $|\tilde{c}'| = 3$ and $u, v \in \tilde{c}'$.*

(R7) *If $c_1, c_2 \in C(\Gamma)$ are distinct, then $V(\alpha(c) \cap \alpha(c')) = \tilde{c}_1 \cap \tilde{c}_2$ unless c_1 and c_2 border the same cuff (that is, unless $c_1 \cap C \neq \emptyset \neq c_2 \cap C$ for some cuff C).*

(R8) *If $c_1, c_2 \in C(\Gamma)$ border a cuff C, and $u_1, u_2 \in V(\Gamma) \cap C$ and c_1, u_1, c_2, u_2 are in order around C, then $\alpha(c_1) \cap \alpha(c_2) \subseteq \alpha(u_1) \cup \alpha(u_2)$.*

(R9) *If $c \in C(\Gamma)$ and $|\tilde{c}| = 3$, then $\alpha(c)$ cannot be drawn in a disc so that the three members of $\tilde{c}$ are drawn in the boundary of the disc.*

(R10) *For each cuff $C, \bigcap(\alpha(u) : u \in V(\Gamma) \cap C)$ is null.*

Its *depth* is the maximum of $q(C)$, taken over all cuffs C (or 0 if $bd(\Sigma) = \emptyset$).)

9.1 *If Γ, α is a portrait in a surface Σ of a graph G, and L is a vital linkage in G, then for every cell c of Γ which is not a border cell, either $V(\alpha(c)) \setminus \tilde{c}$ contains a terminal of L, or $\alpha(c)$ can be drawn in a disc with the members of $\tilde{c}$ drawn in the boundary.*

Proof. Let $c \in C(\Gamma)$ be an internal cell such that $V(\alpha(c)) \setminus \tilde{c}$ contains no terminal of L. We must prove that $\alpha(c)$ can be drawn in a disc with $\tilde{c}$ in the boundary. If $V(\alpha(c)) = \tilde{c}$ this is clear, and we therefore may assume that $V(\alpha(c)) \neq \tilde{c}$. Since by (R2) $\tilde{c} \subseteq V(\alpha(c))$, there is a vertex in $V(\alpha(c)) \setminus \tilde{c}$. Since $V(L) = V(G)$, there is a component P of L with $V(P \cap (\alpha(c))) \not\subseteq \tilde{c}$. Since $V(\alpha(c)) \setminus \tilde{c}$ contains no end of P, and c is internal, it follows from (R7) that $|V(P) \cap \tilde{c}| \geq 2$; and so P is the only component of L with a vertex in $V(\alpha(c)) \setminus \tilde{c}$, since $|\tilde{c}| \leq 3$. Consequently, $V(\alpha(c)) \setminus \tilde{c} \subseteq V(P)$. But L is vital, and so any two vertices of $P \cap \alpha(c)$ that are adjacent in $\alpha(c)$ are also adjacent in P. Hence either $\alpha(c)$ is a path with both ends in $\tilde{c}$, or $\alpha(c) \setminus v$ is a path with both ends in $\tilde{c}$, for some $v \in \tilde{c}$. In either case $\alpha(c)$ can be drawn in a disc with all the members of $\tilde{c}$ drawn in the boundary, as required. ■

If $\mathcal{T}$ is a tangle in G, a portrait Γ, α of G in Σ *surrounds* $\mathcal{T}$ if $\alpha(c)$ is small for each $c \in C(\Gamma)$.

9.2 *For every surface Σ, and all integers $p \geq 0$ and $q \geq 1$, there exists $\theta \geq 1$ such that, if a graph G has a portrait in Σ of depth $\leq q$ which surrounds a tangle of order $\geq \theta$, then G has no vital p-linkage.*

Proof. Let $n = 3p + 2q(3p + c(\Sigma))$. Choose $\theta' > 2q$ so that (8.4) holds, with Σ, p, q, r, θ replaced by $\Sigma, p + n, q, r, \theta'$. Let $\theta = \theta' + n$. We claim that θ satisfies (9.2). For let Γ, α be a portrait in Σ of a graph G, with depth $\leq q$, surrounding a tangle $\mathcal{T}$ of order $\geq \theta$. Suppose that L is a vital p-linkage in G. Let $\mathcal{A}_1$ be the set of all internal cells c such that $V(\alpha(c)) \setminus \tilde{c}$ contains a terminal of L. Since the sets $V(\alpha(c)) \setminus \tilde{c}$ $(c \in \mathcal{A}_1)$ are mutually disjoint, by (R7), it follows that $|\mathcal{A}_1| \leq p$. By (9.1) and (R9), $\mathcal{A}_1$ contains every cell of Γ with $|\tilde{c}| = 3$. By (R6), there are $\leq 3|\mathcal{A}_1|$ border cells that are not linked. Let $\mathcal{A}_2 \subseteq C(\Gamma)$ be minimal such that $\mathcal{A}_2$ contains all border cells that are not linked, and contains at least one cell bordering each cuff. Thus, $|\mathcal{A}_2| \leq 3p + c(\Sigma)$.

For each internal cell $c \in C(\Gamma) \setminus \mathcal{A}_1$, there is a drawing of $\alpha(c)$ in the closed disc $\bar{c}$ with the vertices in $\tilde{c}$ representing themselves, by (9.1), and this drawing can be chosen so that it meets $bd(\bar{c})$ only in $\tilde{c}$. We may therefore assume, to simplify the notation, that $\alpha(c)$ is such a drawing in $\bar{c}$ for each internal cell $c \in C(\Gamma) \setminus \mathcal{A}_1$. Let Γ' be the drawing formed by the vertices in $V(\Gamma)$ and the union of the drawings $\alpha(c)$ over all internal $c \in C(\Gamma) \setminus \mathcal{A}_1$. Then Γ' is a drawing in Σ, and there is a 1-1 correspondence between the spaces of Γ' and the border cells of Γ.

Let G' be the subgraph of G formed by the vertices in $V(\Gamma)$, the graphs $\alpha(u)$ $(u \in V(\Gamma') \cap bd(\Sigma))$ and the graphs $\alpha(c)$ $(c \in C(\Gamma) \setminus (\mathcal{A}_1 \cup \mathcal{A}_2))$. Then $\Gamma' \subseteq G'$. Now $V(\Gamma') \cap bd(\Sigma) = V(\Gamma) \cap bd(\Sigma)$; define $\mu(u) = \alpha(u)$ for each $u \in V(\Gamma') \cap bd(\Sigma)$. Let $F_1^*, \ldots, F_r^*$ be the connected components of $bd(\Sigma) \setminus \bigcup(c : c \in \mathcal{A}_2)$. Then $F_1^*, \ldots, F_r^*$ are all supports of Γ' since $\mathcal{A}_2$ contains at least one cell bordering each cuff; they are mutually disjoint, and $V(\Gamma') \cap bd(\Sigma) \subseteq F_1^* \cup \ldots \cup F_r^*$. Moreover, $r = |\mathcal{A}_2| \leq 3p + c(\Sigma)$. For each space s of Γ' with $s \subseteq F_1^* \cup \ldots \cup F_r^*$, let $\mu(s) = \alpha(c)$, where c is the cell of Γ with $c \cap bd(\Sigma) = s$.

(1) $\Gamma', F_1^*, \ldots, F_r^*, \mu$ *is a pseudo-presentation of* G'.

Subproof. Certainly $|V(\Gamma') \cap C| \geq 2$ for each cuff C, because Γ is a painting and $V(\Gamma) \subseteq V(\Gamma')$. We must verify (P1)–(P4). First let us verify (P1). Now

$$\begin{aligned} G' &= \Gamma' \cup \bigcup(\alpha(c) : c \in C(\Gamma) \setminus \mathcal{A}_2 \text{ and } c \text{ is a border cell }) \cup \bigcup(\alpha(u) : u \in V(\Gamma') \cap bd(\Sigma)) \\ &= \Gamma' \cup \mu(F_1^*) \cup \ldots \cup \mu(F_r^*). \end{aligned}$$

By (R3), $E(\mu(s) \cap \mu(s')) = \emptyset$ if s and s' are distinct spaces, since each border cell of Γ includes only one space of Γ'. Since each edge of Γ' is an edge of $\alpha(c)$ for some internal cell c, it follows from (R3) again that $E(\Gamma' \cap \mu(s)) = \emptyset$ for each space $s \subseteq F_1^* \cup \ldots \cup F_r^*$. Finally, let x_1, x_2 be units of Γ in different cuffs. Then there are border cells c_1, c_2 of Γ bordering different cuffs with $\mu(x_i) \subseteq \alpha(c_i)$, by (R6) and condition (iii) in the definition of a painting. By condition (iv) in the definition of a painting, $\tilde{c} \cap \tilde{c}' = \emptyset$, and so by (R7), $\alpha(c_1) \cap \alpha(c_2)$ is null. Consequently, $\mu(x_1) \cap \mu(x_2)$ is null. This proves (P1).

For (P2), let $1 \leq j \leq r$ and let $s \subseteq F_j^*$ be a space of Γ', with ends u_1, u_2. Let c be the cell of Γ with $c \cap bd(\Sigma) = s$; then $\tilde{c} = \{u_1, u_2\}$. Since $s \subseteq F_j^*$ it follows that $c \notin \mathcal{A}_2$, and so c is linked. Consequently, (P2) holds, taking $q_j = q(C)$ where C is the cuff including F_j^* and (see (R5)) $q(C)$ is the common cardinality of all the sets $V(\alpha(v))$ ($v \in V(\Gamma) \cap C$).

For (P3), let C be a cuff, let x_1, x_2 be units of Γ' in C, and let $u_1, u_2 \in C \cap V(\Gamma)$ so that x_1, u_1, x_2, u_2 are in order in C. We must show that $\mu(x_1) \cap \mu(x_2) \subseteq \mu(u_1) \cup \mu(u_2)$. If one of x_1, x_2 equals one of u_1, u_2 then the inclusion is trivial, and so we assume that $x_1, x_2 \neq u_1, u_2$. For $i = 1, 2$, let c_i be a border cell of Γ with $c_i \cap bd(\Sigma) = x_i$ if x_i is a space of Γ', and $x_i \in \tilde{c}_i$ if $x_i \in V(\Gamma')$. By (R6),

$$\mu(x_i) = \alpha(x_i) \subseteq \alpha(c_i) \ (i = 1, 2)$$

and so $\mu(x_1) \cap \mu(x_2) \subseteq \alpha(c_1) \cap \alpha(c_2)$. But c_1, u_1, c_2, u_2 are in order, and so by (R8),

$$\alpha(c_1) \cap \alpha(c_2) \subseteq \alpha(u_1) \cup \alpha(u_2) = \mu(u_1) \cup \mu(u_2).$$

This proves (P3).

But (P4) is immediate from (R10). This completes the proof of (1).

For each internal $c \in C(\Gamma)$, let $B(c) \subseteq G$ be such that $(\alpha(c), B(c))$ is a separation of G and $V(\alpha(c) \cap B(c)) = \tilde{c}$. (This is possible by (R7).) For each border cell c, let $B(c) \subseteq G$ be such that $(\alpha(c), B(c))$ is a separation of G and $\alpha(c) \cap B(c) = \mu(u_1) \cup \mu(u_2)$, where $\tilde{c} = \{u_1, u_2\}$. (This is possible by (R7) and (R8).)

(2) $G' = G \cap \bigcap(B(c) : c \in \mathcal{A}_1 \cup \mathcal{A}_2)$.

Subproof. Certainly $G' \subseteq B(c)$ for each $c \in \mathcal{A}_1 \cup \mathcal{A}_2$. For the converse inclusion, if $e \in E(G)$ is an edge of $B(c)$ for all $c \in \mathcal{A}_1 \cup \mathcal{A}_2$, then by (R3) e is an edge of $\alpha(c)$ for some $c \in C(\Gamma) \setminus (\mathcal{A}_1 \cup \mathcal{A}_2)$, and so $e \in E(G')$. If $v \in V(G)$ is a vertex of $B(c)$ for all $c \in \mathcal{A}_1 \cup \mathcal{A}_2$, choose $c \in C(\Gamma)$ with $v \in V(\alpha(c))$. If $c \in C(\Gamma) \setminus \mathcal{A}_1 \cup \mathcal{A}_2$ then $v \in V(G')$ as required. If $c \in \mathcal{A}_1$ then $v \in \tilde{c}$ because $v \in V(B(c) \cap \alpha(c)) = \tilde{c}$, and so $v \in V(\Gamma) \subseteq V(G')$ as required. If $c \in \mathcal{A}_2$, then $v \in V(\mu(u_1) \cup \mu(u_2))$ where $\tilde{c} = \{u_1, u_2\}$ for the same reason, and so again $v \in V(G')$. This proves (2).

Let $A_0 = \bigcup(\alpha(c) : c \in \mathcal{A}_1 \cup \mathcal{A}_2)$. From (2), (A_0, G') is a separation of G. Since every vertex of $V(A_0 \cap G')$ belongs to $V(\alpha(c) \cap B(c))$ for some $c \in \mathcal{A}_1 \cup \mathcal{A}_2$, it follows that (A_0, G') has order

$$\leq 3|\mathcal{A}_1| + 2q|\mathcal{A}_2| \leq 3p + 2q(3p + c(\Sigma)) = n.$$

We deduce from (3.5) that

(3) *There is a tangle $\mathcal{T}'$ in G' of order $\geq \theta - n = \theta'$, such that $(A \cap G', B \cap G') \in \mathcal{T}'$ for every $(A, B) \in \mathcal{T}$ of order $< \theta'$.*

(4) *The pseudo-presentation $\Gamma', F_1^*, \ldots, F_r^*, \mu$ of G' surrounds $\mathcal{T}'$.*

Subproof. Let s be a space of Γ' in Σ with $s \subseteq F_1^* \cup \ldots \cup F_r^*$, and let $c \in C(\Gamma)$ with $c \cap bd(\Sigma) = s$. Then c is a border cell, and $c \notin \mathcal{A}_2$ by definition of $F_1^* \cup \ldots \cup F_r^*$. Now $(\alpha(c), B(c)) \in \mathcal{T}$ by (7.1) since it has order $\leq 2q < ord(\mathcal{T})$ and $\alpha(c)$ is small with respect to $\mathcal{T}$. By (3), $(\alpha(c) \cap G', B(c) \cap G') \in \mathcal{T}'$ since $(\alpha(c), B(c))$ has order $\leq 2q < \theta'$. Hence $\alpha(c) \cap G'$ is small with respect to $\mathcal{T}'$. But $\alpha(c) \subseteq G'$ since $c \in C(\Gamma) \setminus (\mathcal{A}_1 \cup \mathcal{A}_2)$, and $\alpha(c) = \mu(s)$. Consequently $\mu(s)$ is small with respect to $\mathcal{T}'$. This proves (4).

(5) *There is no vital $(p + n)$-linkage in G'.*

Subproof. This follows from (1), (4) and (8.4), because of the choice of θ'.

Since (A_0, G') is a separation of G of order $\leq n$, it follows from (5) and (2.4) that there is no vital p-linkage in G, as required. ■

10 The main proof

Finally we are able to apply the theorems of [10]. Theorems (8.2), (8.4), (9.8) and particularly (13.4) of [10] imply the following. (A *K_n-minor* of G is a minor isomorphic to K_n.)

10.1 *For every integer $n \geq 0$ there exist $\rho, \xi \geq 0, q \geq 1$ and $\theta > \xi$ such that, if a graph G has no K_n-minor, and $\mathcal{T}$ is a tangle in G of order $\geq \theta$, then there exists a surface Σ such that $c(\Sigma) \leq \rho$ and K_n cannot be drawn in Σ, and there exists $Z \subseteq V(G)$ with $|Z| \leq \xi$ such that there is a portrait of $G \setminus Z$ in Σ with depth $\leq q$, surrounding $\mathcal{T} \setminus Z$.*

We deduce

10.2 *For all integers $n, p \geq 0$ there exists $\theta \geq 1$ such that, if a graph G has a tangle of order $\geq \theta$ and has no K_n-minor then G has no vital p-linkage.*

Proof. Choose $\rho, \xi \geq 0, q \geq 1$ and $\theta_1 > \xi$ so that (10.1) holds with θ replaced by θ_1. Choose $\theta_2 \geq 1$ so that for all surfaces Σ with $c(\Sigma) \leq \rho$ in which K_n cannot be drawn, (9.2) holds, with Σ, p, q, θ replaced by $\Sigma, p + 2\zeta, q, \theta_2$. Let $\theta = \max(\theta_1, \xi + \theta_2)$. We claim that θ satisfies (10.2). For let G have a tangle $\mathcal{T}$ of order $\geq \theta$, and have no K_n-minor. By (10.1), since $\theta \geq \theta_1$, there is a surface Σ with $c(\Sigma) \leq \rho$ in which K_n cannot be drawn, and there exists $Z \subseteq V(G)$ with $|Z| \leq \xi$ such that there is a portrait of $G \setminus Z$ in Σ with depth $\leq q$, surrounding $\mathcal{T} \setminus Z$. Since $\mathcal{T} \setminus Z$ has order $\geq \theta - |Z| \geq \theta_2$, it follows from (9.2) that $G \setminus Z$ has no vital $(p + 2\xi)$-linkage. Hence by (2.1), G has no vital p-linkage, as required. ■

10.3 *For every integer $p \geq 0$, no graph with a K_n-minor has a vital p-linkage, where $n = \lfloor \frac{5}{2}p \rfloor + 1$.*

Proof. Suppose that L is a vital p-linkage in a graph G with a K_n-minor. Let Z be the set of terminals of L. By theorem (6.1) of [8], there exists $v \in V(G) \setminus Z$ such that, in the language of [8], G and $G \setminus v$ have the same 0-folio relative to Z. In particular, there is a linkage in $G \setminus v$ with the same pattern as L, a contradiction since L is vital. The result follows. ■

From (10.2) and (10.3) we deduce the following, which in view of (3.1) implies our main result (1.1).

10.4 *For every integer $p \geq 0$ there exists $\theta \geq 1$ such that, if a graph G has a tangle of order $\geq \theta$ then G has no vital p-linkage.*

Proof. Let n be as in (10.3), and choose θ so that (10.2) is satisfied. Let G be a graph with a tangle of order $\geq \theta$. If G has no K_n-minor then by (10.2) G has no vital p-linkage. If G has a K_n-minor then by (10.3) G has no vital p-linkage. The result follows. ■

11 Intertwinings

The "intertwining conjecture", of Lovász [2] and of Milgram and Ungar [3], states that for every two graphs G_1 and G_2, there is a finite list $H_1, \ldots, H_n$ of graphs, such that a graph G topologically contains both G_1 and G_2 if and only if it topologically contains one of $H_1, \ldots, H_n$. (G *topologically contains* H if some subgraph of G is isomorphic to a subdivision of H.) This conjecture was proved in [11] by well-quasi-ordering methods. Our object here is to give a different proof of the intertwining conjecture, one that is "constructive" in the sense that it yields an algorithm which, given G_1 and G_2, computes $H_1, \ldots, H_n$ as above.

Poljak and Turzik [4] showed that the intertwining conjecture is (constructively) implied by another conjecture, (11.1) below, and we shall give a constructive proof of (11.1). Let L_1 and L_2 be linkages in a graph G. We say that G is an *intertwining* of L_1 and L_2 if $L_1' \cup L_2' = G$ for all linkages L_1', L_2' in G such that L_i' has the same pattern as L_i $(i = 1, 2)$. Let us say a graph G has *rank*

$$|E(G)| - |V(G)| + \kappa(G),$$

where $\kappa(G)$ denotes the number of components of G. Poljak and Turzik essentially reduced proving the intertwining conjecture to proving the following.

11.1 *For every integer $p \geq 0$ there exists $r(p)$ such that every intertwining of two p-linkages has rank $\leq r(p)$.*

We shall derive (11.1) from (1.1), and the proof will be constructive in the sense that it yields an algorithm to compute a value for $r(p)$ given a numerical value for p. Our thanks to A. Gupta and R. Impagliazzo, who proved (11.1) constructively for planar intertwinings (unpublished) and brought the general problem to our attention. Thanks also to R. Thomas, who collaborated with us in discovering a constructive proof of (11.1) (not exactly the one given here).

Let (T, W) be a tree-decomposition of a graph G. If S is a subtree of T, we define $W(S) = \bigcup(W(t) : t \in V(S))$. If e is an edge of T, and T_1, T_2 are the two components of $T \setminus e$ and t_i is the end of e in T_i $(i = 1, 2)$, then $(W(T_1), W(T_2))$ is a separation of G and $W(T_1) \cap W(T_2) = W(t_1) \cap W(t_2)$. We define $W(e) = V(W(T_1) \cap W(T_2))$.

We say that a tree-decomposition (T, W) of G is *linked* if for all $e_1, e_2 \in E(T)$ such that $|W(e_1)| = |W(e_2)| = k$ say, and $|W(e)| \geq k$ for all edges e of the path of T between e_1 and e_2, there are k mutually vertex-disjoint paths of G between $W(e_1)$ and $W(e_2)$. We need the following lemma.

11.2 *Let G be a graph of tree-width $\leq w$. Then there is a linked tree-decomposition (T, W) of G, with width $\leq w$, such that every vertex of T has degree ≤ 3.*

Proof. By the main result of [14], there is a linked tree-decomposition (T, W) of G, with width $\leq w$. Choose it such that $\Sigma(d(t) - 3)$ is minimum, where $d(t)$ is the degree of t in T and the sum is over all $t \in V(T)$ with degree ≥ 3. Suppose that some $t_0 \in V(T)$ has degree ≥ 4. Then we may assume that there is a tree T' with an edge e_0 with ends t_1, t_2, such that T is obtained from T' by contracting e_0, and t_0 results from identifying t_1 and t_2, and such that t_1 and t_2 both have degree ≥ 3. Let $W'(t) = W(t)$ if $t \in V(T') \setminus \{t_1, t_2\}$, and let $W'(t_1), W'(t_2)$ be graphs both with vertex set $W(t_0)$, so that $W'(t_1) \cup W'(t_2) = W(t_0)$ and $E(W'(t_1) \cap W'(t_2)) = \emptyset$. Then (T', W') is a tree-decomposition of G with width $\leq w$, and $W'(e_0) = V(W(t_0))$, and if $e \in E(T') \setminus \{e_0\}$ then $e \in E(T)$ and $W'(e) = W(e)$. Let $e_1, e_2 \in E(T')$ with $|W'(e_1)| = |W'(e_2)| = k$ say, and $|W'(e)| \geq k$ for all edges e of the path of T' between e_1 and e_2. We claim that there are k mutually vertex-disjoint paths between $W'(e_1)$ and $W'(e_2)$. If $e_1, e_2 \neq e_0$, then $e_1, e_2 \in E(T)$ and the claim therefore holds, since $W'(e_2) = W(e_i)$ $(i = 1, 2)$ and (T, W) is linked. If $e_1 = e_2 = e_0$ the claim is trivial. If $e_1 = e_0$ and $e_2 \neq e_0$, let e_3 be the edge of T different from e_1, between e_1 and e_2 with a common end t_1 say with e_1. Then $|W(e_3)| \geq k$, but

$$W(e_3) \subseteq V(W(t_1)) = W(e_0)$$

and $|W(e_0)| = k$; and so $W(e_0) = W(e_3)$. Since there are k mutually disjoint paths of G between $W(e_3) = W(e_0)$ and $W(e_2)$, the claim follows. Consequently (T', W') is linked, contrary to the choice of (T, W). We deduce that there is no such t_0, and so (i) holds, as required. ■

Secondly, we need the following.

11.3 *Let (T, W) be a tree-decomposition of G, and let $e_1, e_2 \in E(T)$ be distinct. Let T_0, T_1, T_2 be the three components of $T \setminus e_1, e_2$, where e_i has an end in $V(T_0)$ and in $V(T_i)$ $(i = 1, 2)$. Suppose that $W(T_0) \subseteq W(T_1) \cap W(T_2)$. Let T' be the tree obtained from $T_1 \cup T_2$ by adding an edge joining the end of e_1 in T_1 to the end of e_2 in T_2; and let W' be the restriction of W to $V(T')$. Then (T', W') is a tree-decomposition of G, and if (T, W) is linked then so is (T', W').*

Proof. It is straightforward to check that (T', W') is a tree-decomposition of G. Let $X = V(W(T_1) \cap W(T_2))$. Since $W(T_0) \subseteq W(T_1) \cap W(T_2)$ it follows that $W(e) = X$ for every edge e of the path of T between e_1 and e_2, and now the second claim follows easily. ■

Proof of (11.1). Choose $w \geq 0$ so that (1.1) is satisfied. Choose $n \geq 0$ so that (2.7) is satisfied with k replaced by $w + 1$, and let $r(p) = 2^{n-1}w(w + 1)$. We claim that (11.1) holds. For we prove by induction on $|V(G)|+|E(G)|$ that if G is an intertwining of two p-linkages L_1 and L_2 then G has rank $\leq r(p)$.

(1) *We may assume that G has no isolated vertices.*

Subproof. Suppose that $v \in V(G)$ has degree 0. For $i = 1, 2$, let $L'_i = L_i$ if $v \notin V(L_i)$, and $L'_i = L_i \setminus v$ if $v \in V(L_i)$. Then $G \setminus v$ is an intertwining of the p-linkages L'_i and L'_2, and from our inductive hypothesis $G \setminus v$ has rank $\leq r(p)$. But G and $G \setminus v$ have the same rank, and the result follows. This proves (1).

(2) *We may assume that $V(L_1) = V(L_2) = V(G)$.*

Subproof. Suppose that $v \in V(G) \setminus V(L_2)$ say. By (1) there is an edge e of G incident with v, and since $L_1 \cup L_2 = G$ it follows that $e \in E(L_1)$. The graph G/e (that is, obtained from G by contracting e) is an intertwining of L_1/e and L_2, if we regard L_2 as a subgraph of G/e in the obvious way; and L_1/e and L_2 are both p-linkages. From the inductive hypothesis, G/e has rank $\leq r(p)$; but it has the same rank as G (for e is not a loop since $e \in E(L_1)$) and so G has rank $\leq r(p)$) as required. This proves (2).

(3) *We may assume that L_1 and L_2 are vital, and $E(L_1 \cap L_2) = \emptyset$.*

Subproof. Suppose that there is a linkage L'_1 in G with the same pattern as G, with $E(L'_1 \cap L_2) \neq \emptyset$. Let $e \in E(L'_1 \cap L_2)$; then G/e is an intertwining of the p-linkages L'_1/e and L_2/e, and the result follows from the inductive hypothesis. We may therefore assume that there is no such L'_1. Since $L'_1 \cup L_2 = G$ for every linkage L'_1 with the same pattern as L'_1, it follows that $E(L'_1) = E(G) \setminus E(L_2)$, and in particular $E(L_1) = E(G) \setminus E(L_2)$. Since $E(L'_1) = E(L_1)$ and hence $L'_1 = L_1$ for every linkage L'_1 with the same pattern as L_1, it follows from (2) that L_1 is vital, and similarly we may assume that L_2 is vital. This proves (3).

From (1.1) and (3), G has tree-width $\leq w$. From (11.2) there is a linked tree-decomposition (T, W) of G of width $\leq w$, such that every vertex of T has degree ≤ 3. From (11.3) we deduce

(4) *If* $e_1, e_2 \in E(T)$ *are distinct, and* T_0, T_1, T_2 *are the three components of* $T \setminus e_1, e_2$ *as in (11.3), then* $W(T_0) \not\subseteq W(T_1) \cap W(T_2)$.

(5) *Every path of* T *has* $< n$ *edges.*

Subproof. Otherwise there is one with exactly n edges, say $e_1, \ldots, e_n$ in order. For $1 \leq i \leq n$, let e_i have ends t_{i-1}, t_i, and let the two components of $T \setminus e_i$ be S_i, T_i, where $t_{i-1} \in V(S_i)$ and $t_i \in V(T_i)$. Let $A_i = W(S_i)$ and $B_i = W(T_i)$. Since (T, W) is linked, it follows that there are linkages $M_{ii'}$ as in (2.7), with k replaced by $w+1$. (Each (A_i, B_i) has order $\leq w+1$ since each $|V(W(t_i))| \leq w+1$.) By (2.7), there exist i, i' with $1 \leq i < i' \leq n$, such that $|V(A_i \cap B_i)| = |V(A_{i'} \cap B_{i'})| = k$ say, and $|V(A_j \cap B_j)| > k$ for $i < j < i'$, and

$$L_1 \cap B_i \cap A_{i'} = L_2 \cap B_i \cap A_{i'} = M$$

say, where M is a linkage with k components, each with one end in $V(A_i \cap B_i)$ and the other in $V(A_{i'} \cap B_{i'})$. From (3), $E(L_1 \cap L_2) = \emptyset$, and so $E(M) = \emptyset$, and consequently $V(M) = V(A_i \cap B_i) = V(A_{i'} \cap B_{i'})$. Hence $M \subseteq W(S_i) \cap W(T_{i'})$. But every vertex and edge of $W(T_i \cap S_{i'})$ belongs to one of L_1, L_2 since $L_1 \cup L_2 = G$, and hence belongs to M. Consequently,

$$W(T_i \cap S_{i'}) = M \subseteq W(S_i) \cap W(T_{i'}).$$

But this contradicts (4). Hence there is no such path, and (5) holds.

From (5) and since T has maximum degree ≤ 3, it follows that $|V(T)| \leq 2^n$. But from (3), G has no parallel edges, for if e_1, e_2 are parallel and $e_1 \in E(L_1)$ then L_1 is not vital. Consequently, each W_t has $\leq \frac{1}{2}w(w+1)$ edges, since $|V(W_t)| \leq w+1$. Since $G = W(T)$, it follows that

$$|E(G)| \leq 2^n \cdot \frac{1}{2}w(w+1) = r(p),$$

and hence the rank of G is also at most $r(p)$, as required. ■

There is some question of whether this proof is really constructive. Certainly it is, if our proof of (1.1) is constructive; but that proof uses several complicated results from earlier papers in this series, and it is necessary to check back through all these proofs and verify that they are indeed constructive in the sense we require. But they are.

12 Path-width

We recall that a *path-decomposition* of G is a tree-decomposition (T, W) of G such that T is a path, and the *path-width* of G is the minimum width of all path-decompositions of G. We saw in (1.2) that for $p \leq 5$, every graph with a vital p-linkage has path-width $\leq p$, and our next objective is to show that (1.1) holds in general with tree-width replaced by path-width. This result is not needed for anything, and is included only as a curiosity.

First, we need the following lemma.

12.1 *Let $(A, B), (A', B')$ be separations of a graph, both of order k and with $A \subseteq A'$ and $B' \subseteq B$. Suppose that in $B \cap A'$ there is a unique set of k mutually vertex-disjoint paths between $V(A \cap B)$ and $V(A' \cap B')$, and every vertex of $B \cap A'$ belongs to one of these paths. Then there is a path-decomposition (P, W) of G where $|V(P)| \geq 2$ with the following properties:*

(i) P has ends s, s' where $W(s) = A, W(P \setminus s) = B, W(P \setminus s') = A'$ and $W(s') = B'$

(ii) $|W(e)| = k$ for every edge e of P

(iii) for each $t \in V(P)$ with $t \neq s, s'$, $|E(W(t))| = 1$ and if e, e' are the two edges of P incident with t, then either

(a) $W(e) = W(e') = V(W(t))$, or

(b) $W(e) \cap W(e') = X$ say where $|X| = k - 1$, and the unique edge of $W(t)$ has ends v, v' say, where $W(e) = X \cup \{v\}$ and $W(e') = X \cup \{v'\}$.

Proof. We proceed by induction on $|E(B \cap A')|$. Let $V(A \cap B) = Z$ and $V(A' \cap B') = Z'$.

(1) *We may assume that no edge of $B \cap A'$ has both ends in Z.*

Subproof. If some edge e of $B \cap A'$ has both ends in Z, let A_1 be obtained from A by adding the edge e, and let $B_1 = B \setminus e$. Then $A_1 \subseteq A'$ and $B' \subseteq B_1$, and in $B \cap A'$ there is a unique set of k disjoint paths between $V(A_1 \cap B_1)$ and $V(A' \cap B')$, so from the inductive hypothesis there is a path-decomposition (P_1, W_1) of $B \cap A_1$ as in the theorem. Let P_1 have ends s_1, s_1' where $W_1(s_1) = A_1$; let P be obtained from P_1 by adding a new vertex s of degree 1, adjacent to s_1; and define $W(s) = A, V(W(s_1)) = V(A \cap B), E(W(s_1)) = \{e\}$, and $W(t) = W'(t)$ for $t \in V(P) \setminus \{s, s_1\}$.

Then (P, W) satisfies the theorem, as required. Consequently (1) holds.

(2) *We may assume that every $v \in Z \setminus Z'$ has degree $\neq 1$ in $B \cap A'$.*

Subproof. If some $v \in Z \setminus Z'$ has degree 1 in $B \cap A'$, let e be the unique edge of $B \cap A'$ incident with v, and let e have ends v, v_1. Let A_1 be obtained from A by adding v_1 and e, and let $B_1 = B \setminus v$. Again the result follows from the inductive hypothesis applied to (A_1, B_1) and (A', B'). Consequently (2) holds.

Let $P_1, \ldots, P_k$ be mutually vertex-disjoint paths of $B \cap A'$ between Z and Z'. From the hypothesis,

$$V(B \cap A') = V(P_1 \cup \ldots \cup P_k).$$

Let P_i have ends $z_i \in Z$ and $z'_i \in Z'$ $(1 \leq i \leq k)$. Let $z_i \neq z'_i$ for $1 \leq i \leq h$ and $z_i = z'_i$ for $h < i \leq k$, where $0 \leq h \leq k$. Let H be the directed graph with vertex set $\{1, \ldots, h\}$, where there is a directed edge from i to j if some edge e of $B \cap A'$ has one end z_i and the other end a vertex of $P_j \setminus z_j$, and $e \notin E(P_i)$.

(3) *Every vertex of H has out-degree ≥ 1.*

Subproof. Let $1 \leq i \leq h$; we claim that i has out-degree ≥ 1 in H. Since $i \leq h$ it follows that $z_i \neq z'_i$, and by (2) there is an edge e of $B \cap A'$ incident with z_i with $e \notin E(P_i)$. Since $V(B \cap A') = V(P_1, \ldots, P_k)$, there exists j with $1 \leq j \leq k$ such that $v \in V(P_j)$, where e has ends z_i, v. By (1), $v \notin Z$, and in particularly $z_j \neq z'_j$, and hence $j \leq h$. Consequently i is adjacent to j in H. This proves (3).

(4) *H has no directed circuit.*

Subproof. Suppose that $\{1, 2, \ldots, r\}$ is the vertex set of a directed circuit of H, and there are edges of H from i to $i+1$ for $1 \leq i < r$ and from r to 1. For $1 \leq i < r$, let e_i be an edge of $B \cap A'$ with one end z_i and the other end in $P_{i+1} \setminus z_{i+1}$, and let $e_r \in E(B \cap A')$ with one end z_r and the other end in $P_1 \setminus z_1$. For $1 \leq i < r$, let P'_i be the path consisting of z_i, e_i and the subpath of P_{i+1} from the end of e_i to z'_{i+1}, and define P'_r similarly. Then $\{P'_1, \ldots, P'_r, P_{r+1}, \ldots, P_k\}$ is a set of k mutually disjoint paths of G between Z and Z', different from $\{P_1, \ldots, P_p\}$, since e_1 is an edge of one of them. This contradicts our hypothesis, and hence (4) follows.

From (3) and (4) it follows that $V(H) = \emptyset$, and so $z_i = z'_i$ for $1 \leq i \leq k$. Hence

$$V(B \cap A') = V(P_1 \cup \ldots \cup P_k) = Z$$

and so $E(B \cap A') = \emptyset$, from (1). The theorem is therefore satisfied by a 2-vertex path P. ■

The converse of (12.1) is obvious, that if there is a path-decomposition satisfying (i), (ii) and (iii), then there is a unique set of k paths as in the hypothesis.

Secondly we need the following.

12.2 *Let (T, W) be a linked tree-decomposition of a graph G, with width $\leq w$. Let $f_1, f_2 \in E(T)$ be distinct, and let T_1, T_2, T_0 be the three components of $T \setminus \{f_1, f_2\}$ where f_i has ends in T_i and T_0 ($i = 1, 2$). Let $|W(f_1)| = |W(f_2)| = k$, and let (P, W') be a path-decomposition of G satisfying statements (i)–(iii) of (12.1), with W, A, B, A', B' replaced by $W', W(T_1), W(T_0) \cup W(T_2), W(T_0) \cup W(T_1), W(T_2)$. Let P have ends s_1, s_2, where s_i is the end of f_i in $V(T_i)$ ($i = 1, 2$), and otherwise let P be disjoint from T. Let T^* be the tree $T_1 \cup T_2 \cup P$, and define $W^*(t) = W(t)$ if $t \in V(T_1) \cup V(T_2)$ and $W^*(t) = W'(t)$ if $t \in V(P) \setminus \{s, s'\}$. Then (T^*, W^*) is a linked tree-decomposition of G, with width $\leq w$.*

Proof. It is easy to see that (T^*, W^*) is a tree-decomposition of G. We shall check its width, and check that it is linked. Since (P, W') satisfies (12.1)(iii), it follows that

(1) *There are k mutually vertex-disjoint paths of G between $W(f_1)$ and $W(f_2)$.* Let Q be the path of T with first and last edges f_1 and f_2. From (1), we have

(2) *$|W(e)| \geq k$ for each edge e of Q.*

(3) *(T^*, W^*) has width $\leq w$.*

Subproof. If $t \in V(T_1) \cup V(T_2)$ then $|V(W^*(t))| = |V(W(t))| \leq w + 1$, and if $t \in V(P) \setminus \{s_1, s_2\}$, then

$$|V(W^*))| = |V(W'(t))| \leq k + 1 = |W(f_1)| + 1 \leq w + 2.$$

If $k \leq w$ then (3) holds, and so we assume that $k = w + 1$. From (2), $|W(e)| \geq w + 1$ for each $e \in E(Q)$, and so $V(W(t_1)) = V(W(t_2))$ if $e \in E(Q)$ has ends t_1, t_2, since

$$W(e) \subseteq V(W(t_1)) \cap V(W(t_2))$$

and $|V(W(t_1))|, |V(W(t_2))| \leq w + 1$. Consequently, $V(W(t)) = W(f_1)$ for all $t \in V(Q)$, and in particular $W(f_1) = W(f_2)$. Hence $W'(f) = W(f_1)$ for all $f \in E(P)$, and so $|V(W^*(t))| \leq k = w+1$ for each $t \in V(P) \setminus \{s_1, s_2\}$, by (12.1)(ii). This proves (3).

Let $e_1, e_2 \in E(T^*)$ with $|W^*(e_1)| = |W^*(e_2)| = k'$ say, such that $|W^*(e)| \geq k'$ for all edges e of the path R of T^* with first edge e_1 and last edge e_2. We must show that there are k' mutually vertex-disjoint paths of G between $W^*(e_1)$ and $W^*(e_2)$. If $e_1, e_2 \in E(P)$ this is clear, and so we may assume that $e_1 \in E(T_1)$. If $e_1, e_2 \in E(T_1)$ the claim follows since (T, W) is linked, and so we may assume that $e_2 \in E(P)$ or $e_2 \in E(T_2)$. If $e_2 \in E(T_2)$ then $k' \leq k$ since $P \subseteq R$, and so $|W(e)| \geq k'$ for every edge of e of the path of T between e_1 and e_2, by (2); and the claim follows since (T, W) is linked. We assume then that $e_2 \in E(P)$, and so $k' = k$. Since (T, W) is linked, there are k mutually disjoint paths of G between $W(e_1)$ and $W(f_1)$, and these are paths of $A = W(T_1)$. Since (P, W') satisfies (12.1), there are k mutually disjoint paths of G between $W(f_1)$ and $W'(e_2)$, and these are paths of $B = W(T_0) \cup W(T_2)$. Since (A, B) is a separation and $V(A \cap B) = W(f_1)$, we can pair these paths to obtain k mutually disjoint paths of G between $W^*(e_1) = W(e_1)$ and $W^*(e_2) = W'(e_2)$, as required. Consequently, (T^*, W^*) is linked. ■

We use these lemmas to prove the main result of this section, the following.

12.3 *For every integer $p \geq 0$ there exists $w \geq 0$ such that every graph with a vital p-linkage has path-width $\leq w$.*

Proof. Choose w' so that (1.1) holds with w replaced by w'. Choose n so that (2.6) holds with k replaced by $w' + 1$. Let $w = w' + 3(w' + 1)2^{n-1}$. We claim that w satisfies (12.3). For let G be a graph with a vital p-linkage L. By (1.1), G has tree-width $\leq w'$. By (11.2), there is a linked tree-decomposition (T, W) of G, with width $\leq w'$, such that every vertex of T has degree at most 3. Let $N(T)$ be the set of vertices of T with degree 3 in T, and let us choose (T, W) with $|N(T)|$ minimum.

(1) *Let $f_1, f_2 \in E(T)$ with $|W(f_1)| = |W(f_2)| = k$ say, and let T_0, T_1, T_2 be the three components of*

$T\setminus\{f_1,f_2\}$ *where* f_i *has ends in* T_0 *and* T_i $(i=1,2)$. *Suppose that there is a unique set of* k *mutually vertex-disjoint paths of* $W(T_0)$ *between* $W(f_1)$ *and* $W(f_2)$ *and every vertex of* $W(T_0)$ *belongs to one of these paths. Then* $V(T_0)\cap N(T)=\emptyset$.

Subproof. Let $A=W(T_1), B=W(T_0)\cup W(T_2), A'=W(T_0)\cup W(T_1), B'=W(T_2)$. Then $B\cap A'=W(T_0)$. Choose a path-decomposition (P,W') as in (12.1) (with W replaced by W'). We may assume that P has ends s_1,s_2, where s_i is the end of f_i in T_i $(i=1,2)$, and otherwise P is disjoint from T. Let (T^*,W^*) be as in (12.2). Then $N(T^*)\subseteq N(T)$, and so equality holds, from the choice of (T,W); but $V(T_0)\cap N(T^*)=\emptyset$, and so $V(T_0)\cap N(T)=\emptyset$. This proves (1).

(2) $|V(P)\cap N(T)|\le n$ *for every path* P *of* T.

Subproof. Suppose not; then there is a path P of T with both ends in $N(T)$ and with $|V(P)\cap N(T)|=n+1$. Let $V(P)\cap N(T)=\{t_0,t_1,\dots,t_n\}$, in order on P. For $1\le i\le n$, let $f_i\in E(P)$, chosen so that f_i is between t_{i-1} and t_i, and of all such edges $|W(f_i)|$ is minimum; let S_i,T_i be the two components of $T\setminus e_i$ where $t_0\in V(S_i)$, and let $A_i=W(S_i), B_i=W(T_i)$. Then for $1\le i<j\le n, A_i\subseteq A_j$ and $B_j\subseteq B_i$. Suppose that $1\le i<i'\le n$, and (A_i,B_i) and $(A_{i'},B_{i'})$ have the same order k say, and (A_j,B_j) has order $>k$ for $i<j<i'$. In other words, $|W(f_i)|=|W(f_{i'})|=k$, and $|W(f_j)|>k$ for $i<j<i'$. From the definition of f_j $(1\le j\le n)$, it follows that $|W(e)|\ge k$ for all edges e of the path of T between f_i and $f_{i'}$. Since (W,T) is linked, there are k mutually vertex-disjoint paths of G between $W(f_i)$ and $W(f_{i'})$, and so there is a linkage $M_{ii'}$ in $B_i\cap A_{i'}$ with k components, each with one end in $V(A_i\cap B_i)$ and the other in $V(A_{i'}\cap B_{i'})$. By (1), since there is a member of $N(T)$ between e_i and $e_{i'}$, it follows that either $V(M_{ii'})\neq V(B_i\cap A_{i'})$, or there is more than one choice for $M_{ii'}$. In either case, we may choose $M_{ii'}$ so that $M_{ii'}\neq L\cap B_i\cap A_{i'}$, since $V(L\cap B_i\cap A_{i'})=V(B_i\cap A_{i'})$. But this contradicts (2.6), since each (A_j,B_j) has order $\le w+1=k$. Hence there is no such path P, and so (2) holds.

Since T has maximum degree ≤ 3, it follows from (2) that $|N(T)|\le 2^n$. Let $Z=\bigcup(V(W_t):t\in N(T))$; then $|Z|\le 2^n(w'+1)$. Now every component of $T\setminus N(T)$ is a path, and so we may add edges to $T\setminus N(T)$ to obtain a path P. For each $t\in V(P)$, let $W'(t)=W(t)\cap(G\setminus Z)$.

(3) (P,W') *is a path-decomposition of* $G\setminus Z$.

Subproof. Certainly

$$\bigcup(W'(t) : t \in V(P)) = (G \setminus Z) \cap \bigcup(W(t) : t \in V(T \setminus N(T))) = G \setminus Z$$

and the graphs $W'(t)$ ($t \in V(P)$) are mutually edge-disjoint. Let $t, t', t'' \in V(P)$, with t' between t and t''. We must show that $W'(t) \cap W'(t'') \subseteq W'(t')$, and may therefore assume that $W'(t) \cap W'(t'')$ is non-null. If t and t'' lie in different components of $T \setminus N(T)$, choose $s \in N(T)$ between them; then

$$W(t) \cap W(t'') \subseteq W(s) \subseteq Z$$

and so $W'(t) \cap W'(t'')$ is null, a contradiction. Hence t and t'' lie in the same component of $T \setminus N(T)$, and t' therefore also lies in this component, between t and t''. Thus $W(t) \cap W(t'') \subseteq W(t')$, and hence $W'(t) \cap W'(t'') \subseteq W'(t')$, as required. This proves (3).

Now (P, W') has width $\leq w'$, and so $G \setminus Z$ has path-width $\leq w'$. Since $|Z| \leq 2^n(w'+1)$, it follows that G has path-width $\leq w' + 2^n(w'+1) = w$, as required. ■

Actually, we could repeat the same kind of argument to get even more. Let us say that a graph G is a *p-chain* if there exist $Y, Z \subseteq V(G)$ with $|Y| = |Z| \leq p$, such that there is a unique set of $|X|$ mutually vertex-disjoint paths of G between Y and Z, and every vertex of G belongs to one of these paths. (Note that $Y \cap Z$ may be nonempty.) It follows from (12.1) that every p-chain has a particularly nice path-decomposition of width $\leq p$, and so the following is a strengthening of (12.3).

12.4 *For every integer $p \geq 0$ there exists $p' \geq 0$ such that every graph with a vital p-linkage is a p'-chain.*

Proof. We only sketch the proof, since we shall not use the result. Let w be as in (12.3), let n satisfy (2.6) with k replaced by $w+1$, and let $p' = (w+1)(2n+1)$. Let G have a vital p-linkage L. By (12.3), G has path-width $\leq w$. By a variation of (11.2) (proved in the same way as (11.2), but somewhat easier) there is a linked path-decomposition (P, W) of G with width $\leq w$. Let $t \in V(P)$ have degree 2, and be incident with e_1 and e_2 say. We say that t is *bad* unless $|W(e_1)| = |W(e_2)|$ and t satisfies (12.1)(iii). Let us choose (P, W) to minimize the number of bad vertices, and let B be the set of bad vertices. An argument similar to step (2) in the proof of (12.3) implies that $|B| \leq n$. Let $Z = \bigcup(W(t) : t \in B)$; then $|Z| \leq (w+1)n$. For each component C of $P \setminus B$, the graph

$$\bigcup(W(t) \cap (G \setminus Z) : t \in V(C))$$

is a $(w+1)$-chain, since no vertex of C is bad. Hence $G \setminus Z$ is a $(w+1)(n+1)$-chain, since $P \setminus B$ has $\leq n+1$ components, and so G is a $((w+1)n + (w+1)(n+1))$-chain, as required. ■

References

[1] D. B. A. Epstein, "Curves on 2-manifolds and isotopies", *Acta Math.* 115 (1966), 83–107.

[2] L. Lovász, in *Combinatorics* (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II, p. 1208, North-Holland, Amsterdam, 1978.

[3] M. Milgram and P. Ungar, *Amer. Math. Monthly,* 85 (1978), no. 8, 664–666.

[4] S. Poljak and D. Turzik, "Some equivalent formulations of the intersection problem of finitely generated classes of graphs," *Acta. Sci. Math.* (Szeged) 41 (1979), No. 3–4, 357–364.

[5] N. Robertson and P. D. Seymour, "Graph minors. VII. Disjoint paths on a surface", *J. Combinatorial Theory, Ser. B,* 45 (1988), 212–254.

[6] N. Robertson and P. D. Seymour, "Graph minors. X. Obstructions to tree-decomposition", *J. Combinatorial Theory, Ser. B,* 52 (1991), 153–190.

[7] N. Robertson and P. D. Seymour, "Graph minors. XII. Distance on a surface", *J. Combinatorial Theory, Ser. B*, 64 (1995), 240–272.

[8] N. Robertson and P. D. Seymour, "Graph Minors. XIII. The disjoint paths problem", *J. Combinatorial Theory, Ser. B*, 63 (1995), 65–110.

[9] N. Robertson and P. D. Seymour, "Graph minors. XVI. Excluding a non-planar graph", *J. Combinatorial Theory, Ser. B*, 89 (2003), 43–76.

[10] N. Robertson and P. D. Seymour, "Graph minors. XVII. Taming a vortex", *J. Combinatorial Theory, Ser. B*, 77 (1999), 162–210.

[11] N. Robertson and P. D. Seymour, "Graph minors. XX. Wagner's conjecture", *J. Combinatorial Theory, Ser. B*, 92 (2004), 325–357.

[12] N. Robertson, P. D. Seymour and R. Thomas, "Quickly excluding a planar graph", *J. Combinatorial Theory, Ser. B,* 62 (1994), 323–348.

[13] R. Thomas, "A Menger-like property of tree-width. The finite case," *J. Combinatorial Theory, Ser. B,* 48 (1990), 67–76.