OPEN PROBLEMS

1 Notation

Throughout, $v(G)$ and $e(G)$ mean the number of vertices and edges of a graph G, and $\omega(G)$ and $\chi(G)$ denote the maximum cardinality of a clique of G and the chromatic number of G.

2 Sergey Norin

Let H be a fixed graph. The notation $H \leq G$ means H is a minor of G. We define

$$c(H) = \sup_{H \not\leq G} e(G)/v(G)$$

and

$$c_\infty(H) = \lim_{n\to\infty} \sup_{H \not\leq G; v(G) \geq n} e(G)/v(G).$$

If H is connected, then $c(H) = c_\infty(H)$ because any extremal example can be replicated to include arbitrarily many vertices. However, $c(H)$ need not equal $c_\infty(H)$ if H is not connected. For example, using the notation sH to denote s disjoint copies of H, we have

$$c(sK_{1,r}) = s(r+1)/2 - 1$$

but

$$c_\infty(sK_{1,r}) = (r-1)/2 + s - 1.$$

2.1. Question: *If H has no isolated vertices, is $c_\infty(H) = \Omega(\sqrt{c(H)})$?*

Also, Reed and Wood proposed

2.2. Conjecture: *If H is 2-regular, then $c(H) \leq 2v(H)/3 - 1$.*

Norin and Qian proved:

2.3. Theorem: *Let H_1 and H_2 be graphs, and $H_1 \sqcup H_2$ be their disjoint union. Then*

$$c_\infty(H_1 \sqcup H_2) \leq c_\infty(H_1) + c_\infty(H_2) + 1.$$

Consequently, if H is 2-regular, then $c_\infty(H) \leq v(H)/2 + c(H)/2 - 1$.

2.4. Question: *With H_1, H_2 as before, is it true that*

$$c(H_1 \sqcup H_2) \leq c(H_1) + c(H_2) + 1?$$

3 Peter Keevash

A *triangle decomposition* of G means a partition of $E(G)$ into triangles. Nash-Williams proposed:

3.1. Conjecture: *Let G be a graph with all vertices of even degree, $3 \mid e(G)$, and $\delta(G) \geq 3v(G)/4$. Then G has a triangle decomposition.*

Notice that $3v(G)/4$ is tight because if G_1 and G_2 are graphs with n vertices and maximum degree less than $n/2$, the complete join of G_1 and G_2 does not have a triangle decomposition.

The conjecture can be adapted to edge weights:

3.2. Question: *Let G be a graph with weight function $w\colon E(G) \to \mathbb{R}$ (weights may be negative) such that $\sum_{e \in E(G)} w(e) > 0$. When must there be a triangle T with positive total weight?*

There are some clear examples answering the question affirmatively and negatively. If $G = K_n$, then the positive overall weight condition implies that the average weight of triangles is positive, so a positive-weight triangle exists. Conversely, if G is triangle-free, then there are certainly no positive-weight triangles. Analogous to the Nash-Williams conjecture, maybe a bound on the minimum degree is sufficient to guarantee a positive-weight triangle:

3.3. Question: *For what values of c does $\delta(G) \geq cn$ guarantee a positive-weight triangle for every weight function w giving G total positive weight?*

Karaschuk showed that $c \leq 22/23$, and this was improved by Dross to $c \leq 0.913$; and the extremal example from the Nash-Williams conjecture can be adapted to show $c \geq 3/4$.

4 Sang-Il Oum

A *quiver* is a digraph with no loops and no directed cycles of length 2. Given a quiver D, a *mutation about* $v \in V(D)$ creates a new quiver D' as follows:

- for every x adjacent to v and y adjacent from v, add $m(x,v)m(v,y)$ arcs from x to y;
- remove any cycles of length 2 formed by adding these arcs (the opposite arcs cancel);
- reverse all arcs incident to and from v.

Notice that applying mutation about v twice returns the original quiver.

The *mutation class* of a quiver is the equivalence class of all quivers that can be reached by a sequence of mutations. This equivalence class may contain infinitely many quivers.

4.1. Theorem: *The mutation class of a quiver D is finite if and only if*

- *$v(D) \leq 2$, or*
- *D is a triangulation of a Riemann surface, or*
- *D is one of eleven exceptions.*

4.2. Question: *Is there an equivalent graph-theoretic statement of theorem 4.1?*

4.3. Question: *Is the problem of deciding whether two quivers are equivalent in P?*

A mutation about v is *balanced* if there exist x adjacent to v and y adjacent from v with $|m'(x,y)| > |m(x,y)|$ if and only if there exist x' adjacent to v and y' adjacent from v with $|m'(x',y')| < |m(x',y')|$. (Here m is the net multiplicity in the original quiver and m' the net multiplicity in the new quiver.) We define "balanced mutation classes" similarly. Lee proposed:

4.4. Conjecture: *Is the balanced mutation class of a quiver always finite?*

Lee and his student answered affirmatively for quivers with at most four vertices.

5 Maria Chudnovsky

We consider assigning colours to the vertices of a graph G so that certain cliques are not monochromatic. The Hoàng-McDiarmid conjecture says:

5.1. Conjecture: *If G has no odd holes and $\omega(G) > 1$, then $V(G)$ can be 2-coloured such that no maximum clique is monochromatic.*

A corollary would be that if G has no odd holes, then $\chi(G) \leq 2^{\omega(G)}$.

What happens if we replace "maximum" (under cardinality) by "maximal" (under inclusion)? Let $\chi_c(G)$ be the minimum number of colours in a colouring of $V(G)$ such that no maximal clique of size at least 2 is monochromatic.

5.2. Question: *Is there a constant c such that $\chi_c(G) \leq c$ for every perfect graph? Does $c = 3$ work?*

Notice that $c = 2$ does not work. The graph obtained from a cycle of length 9 by making vertices $3, 6$ and 9 adjacent has no odd hole, and cannot be 2-clique-coloured. Here are some known results:

5.3. Theorem (Penev): *If G is perfect and has no balanced skew partition, then $\chi_c(G) \leq 2$.*

5.4. Theorem (Chudnovsky, Lo): *If G is odd-hole-free and diamond-free (no K_4 minus an edge), then $\chi_c(G) \leq 3$.*

5.5. Theorem (Chudnovsky, Gauthier, Seymour): *If G is the complement of a comparability graph then $\chi_c(G) \leq 3$.*

Bruce Reed noted the applicability of strongly perfect graphs to the question. A strongly perfect graph is a graph G for which every induced subgraph H has an independent set S meeting all maximal cliques. It is immediate that a strongly perfect graph G satisfies $\chi_c(G) \leq 2$.

Colin McDiarmid noted that almost all perfect graphs are generalized split graphs, and almost all generalised split graphs have $\chi_c \leq 2$, so almost all perfect graphs have $\chi_c \leq 2$.

5.6. Question: *Is there an induced subgraph characterization of strongly perfect graphs?*

6 Maya Stein

Suppose the edges of K_n have been coloured with r colours (not necessarily a proper colouring). We want to cover all vertices with a small number of disjoint monochromatic cycles (which, for the purpose of this problem, include edges and single vertices). When $r = 2$, this can be done with 2 cycles; for higher r, we can do it with $O(r \log r)$ cycles, though this may not be best possible. (The true answer might be linear in r, as the best lower bound known is $r + 1$.)

We can instead consider r-local colouring, in which we have arbitrarily many colours, but each vertex is incident with edges of at most r colours. Again, when $r = 2$, we can cover the vertices with two disjoint cycles; when r is greater, we can do it with $O(r^2)$ cycles, but again perhaps the true answer is linear.

The question arises from considering r-mean colouring, where the average number of colours each vertex sees is at most r. Again, when $r = 2$, two cycles suffice.

6.1. Question: *How many cycles are needed for r-mean colouring when $r \geq 3$?*

7 Stéphan Thomassé

Harutyunyan, McDiarmid, and Scott proposed:

7.1. Conjecture: *There exists $\epsilon > 0$ such that for every digraph D with no directed 3-cycle, there is a subset $S \subseteq V(D)$ of size at least $v(D)^\epsilon$ such that $D[S]$ has no directed cycles.*

The conjecture is reminiscent of the Erdős-Hajnal conjecture. One question is to prove this conjecture. There are easier subsidiary questions:

7.2. Question: *Does there exist $\epsilon > 0$ such that for every digraph D with no directed 3-cycle, there is a subset $S \subseteq V(D)$ of size at least $v(D)^\epsilon$ such that $D[S]$ has no directed cycles of length 4?*

7.3. Question: *Does there exist $\epsilon > 0$ such that for every digraph D with no directed 3-cycle, there is a subset $S \subseteq V(D)$ of size at least $v(D)^\epsilon$ such that $D[S]$ has no directed cycles of length less than 100?*

7.4. Question: *Does there exist $\epsilon > 0$ such that for every digraph D with no directed cycle of length less than 100, there is a subset $S \subseteq V(D)$ of size at least $v(D)^\epsilon$ such that $D[S]$ has no directed cycles?*

We denote by $\lambda(G)$ the edge-connectivity of G. Borát and Thomassen proposed:

7.5. Conjecture: *For every tree T there exists c_T such for every G with $\lambda(G) \geq c_T$ and $e(T) \mid e(G)$, $E(G)$ decomposes into copies of T.*

Botlen, Nota, Oshivo, and Wakubayashi showed that the conjecture holds for paths with up to six vertices. The following theorem also holds:

7.6. Theorem: *There exists a function f such that for every graph G, if $\lambda \geq 148$, $\delta \geq f(\ell)$, and $(\ell - 1) \mid e(G)$, then $E(G)$ decomposes into copies of P_ℓ (the ℓ-vertex path).*

7.7. Question: *Can the number 148 in the statement of theorem 7.6 be reduced to 2?*

7.8. Question: *Does there exist a function f such that for all trees T and all graphs G, if $\lambda(G) \geq f(\Delta_T)$, $\delta(G) \geq f(v(T))$, and $e(T) \mid e(G)$, then $E(G)$ decomposes into copies of T?*

Stéphan thinks finding a counterexample may be in order.

8 Colin McDiarmid

A graph G is a *unit disc graph* if there is a function $f\colon V \to \mathbb{R}^2$ such that for all $u \neq v$, $u \sim v$ if and only if $d(u,v) < 1$ (that is, the unit diameter discs centered at u and v intersect). We consider $\chi(G)$ versus $\omega(G)$ for unit disc graphs.

Around 1990, it was shown that $\chi(G) \leq 3\omega(G) - 2$; this is shown by taking the leftmost vertex v in the plane and noticing that the neighbours of v in each $\pi/3$ sector are a clique. But it is believed that χ/ω is at most some number less than 3.

8.1. Question: *If G is a unit disc graph must $\chi(G) \leq 3\omega(G)/2$?*

Since C_5 is a unit disc graph with $\chi = 3$ and $\omega = 2$, this would be best possible. Notice that if G is a unit disc graph with $\omega(G) = 2$ then G is a planar triangle-free graph and is therefore 3-colourable; so $\chi(G) \leq 3\omega(G)/2$ in this case.

Let $\chi_f(G)$ denote the fractional chromatic number of G.

8.2. Question: *Is $\chi_f \leq 5\omega/4 + o(1)$ for unit disc graphs?*

During the workshop, Zdenek Dvorak and McDiarmid disproved this; for every $\omega \geq 2$, the $(\omega-1)$th power of $C_{3\omega-1}$ is a unit disc graph, and $\chi_f = 3\omega/2 - 1/2$. (And then Colin noticed the same construction in a 2001 paper due to Gerke and himself.)

8.3. Question: *Is there a unit disc graph G with $\omega(G) = 3$ that is not 4-colourable?*

It is worth noting that minimum degree approaches do not work. Although it is true that $\delta(G) \leq 3\omega(G) - 3$ for unit disc graphs G, this bound is tight; a ladder-type construction produces $(3k-3)$-regular unit disc graphs with clique number k for all $k \geq 2$. (Colin remarks that this observation seems to be new.)

9 Zhentao Li

9.1. Question: *Is there an equivalent of Kempe chains for 3-colouring?*

Kempe chains give information about the 4-colourings of the vertices of a face of a planar graph that can be extended to a 4-colouring of the entire graph. Paul Seymour answered this in the negative, because of the following:

9.2. Theorem (Devos, Seymour): *For every cycle F, and for every set S of 3-colourings of F closed under permuting colours, there exists a planar graph G containing F as a face such that the colourings in S are precisely the 3-colourings of F that can be extended to a 3-colouring of G.*

This means it is impossible to prove Kempe-chain-like theorems for 3-colouring planar graphs.

10 Frederic Havet

Consider the "F-subdivision decision problem" for digraphs: we have a fixed digraph F and are given a digraph D, and we want to decide if D contains a subdivision of F.

For many F, the problem is NP-complete because the k-linkage problem for directed graphs can be reduced to F-subdivision. However, the following is a consequence of the recent directed grid theorem proved by Kreutzer and Kawabayashi. A *big vertex* in a digraph D is a vertex v such that $\delta^+(v) \geq 3$ or $\delta^-(v) \geq 3$ or $\delta^+(v) = \delta^-(v) = 2$.

10.1. Theorem: *Let F be a planar digraph with no big vertices. Then the F-subdivision problem is in P.*

10.2. Conjecture: *If F is not planar then the F-subdivision problem is NP-complete.*

It is unknown whether this conjecture holds for $K_{3,3}$ oriented by making the edges of a perfect matching point upwards and all other edges downwards.

10.3. Question: *For which digraphs F can we solve the k-linkage problem in F-subdivision-free graphs in polynomial time?*

11 Paul Seymour

It is known that for all $t \geq 1$ there is a constant $c(t)$ with the following property:

- if G is a simple graph with a bipartition (A, B), such that every vertex in B has degree t, and $|B| \geq c(t)|A|$, then G has a K_{t+1} minor.

(Note that if we only ask that the vertices in B have degree $t-1$ then there is no such $c(t)$.)

11.1. Question: *How large must $c(t)$ be to have this property?*

Seymour pointed out that $c(t)$ likely must be at least $O(\log(t)^{1/2})$, and $c(t) = O(t\log(t)^{1/2})$ is large enough. During the workshop, Colin McDiarmid proved that $c(t)$ must be at least $O(\log(t))$; Sergey Norin proved that $c(t) = O(t\log(t)^{1/4})$ is large enough; and Zdenek Dvorak improved the latter, showing that $c(t) = O(t\log\log(t))$ is large enough.

12 Katherine Edwards

It is proved by Chekuri and Chuzhoy with a probabilistic argument that if G is a simple graph with maximum degree Δ and at least $O(r^3\Delta)$ edges, then there are r disjoint subsets of $V(G)$ such that for each of them, say X, the number of edges with exactly one end in X is at most $O(r)$ times the number with both ends in X. In joint work with Paul Seymour, we improved this to the same conclusion, with a deterministic argument, just assuming that G has at least $O(r^2\Delta)$ edges.

12.1. Question: *Is the same true if we just assume that G has at least $O(r\Delta)$ edges?*

In the course of the meeting, Alex Scott proved the result if G has at least $O(r\Delta\log r)$ edges, provided that Δ is much larger than r^3.

On a different topic:

12.2. Question: *Let $s, t \geq 0$ be integers, and let G be a simple graph with average degree at least $s+t+2$. Can $V(G)$ always be partitioned into two sets A, B, such that the average degree of $G[A]$ is at least s and the average degree of $G[B]$ is at least t?*

This would imply a yes answer to question 2.4. When G is regular, the claim follows from a theorem of Stiebitz, and indeed in that case we can choose A, B such that the minimum degrees in $G[A], G[B]$ are $s, t+1$ respectively. Luke Postle proved during the meeting that 12.2 is true if $|V(G)|$ is sufficiently large in terms of s, t.

Colin McDiarmid asks whether there is any c such that the conclusion of 12.2 holds if we assume instead that G has average degree at least $s + t + c$. He proved with a probabilistic argument that the result holds for $c = 3$ if $s = t$, and also that if $s/t = a/b$ where a, b are integers between 1 and k, then the result holds with $c = k(k + 3)$.