Turán's Theorem for random graphs

Write $t_r(G)$ (resp. $b_r(G)$) for the maximum size of a K_r-free (resp. $(r-1)$-partite) subgraph of a graph G. Of course $t_r(G) \geq b_r(G)$ for any G, while Turán's Theorem (or Mantel's Theorem if $r = 3$) says that equality holds if $G = K_n$. We are interested in a question first considered by Babai, Simonovits and Spencer about 25 years ago: for what $p = p(n)$ is it true that

$$t_r(G_{n,p}) = b_r(G_{n,p}) \quad \textit{w.h.p.?} \tag{1}$$

(Here $G_{n,p}$ is the usual "binomial" random graph and an event holds *with high probability* (w.h.p.) if its probability tends to 1 as $n \to \infty$.)

Less precise versions of (1) have been the subject of intense efforts over the last decade or two and, more recently, of some spectacular successes, though this work seems largely orthogonal to the problem under discussion.

The theorem of the title proves the natural guess that the "threshold" for (1) is the same as that for the property that every edge lies in a copy of K_r:

Theorem. *For each fixed r there is a C such that if*

$$p > Cn^{-\frac{2}{r+1}} \log^{\frac{2}{(r+1)(r-2)}} n,$$

then w.h.p. every maximum K_r-free subgraph of $G_{n,p}$ is $(r-1)$-partite.

I will say a little about some of these developments and *maybe* about some of the difficulties presented by the above theorem.

Joint with Bobby DeMarco.