Lecture 4:

Last time: given a smooth proper A_∞ category $\mathcal{C}$, we constructed a polarized VSHS $(HC_\bullet^-(\mathcal{C}), \nabla^{GGM}, (\cdot,\cdot))$.

We defined $\mathrm{Fuk}(X, D)$ and constructed a map

$$OC : HC_\bullet^-(\mathrm{Fuk}(X,D)) \longrightarrow H^{\bullet+n}(X; \mathbb{K}_A)[u].$$

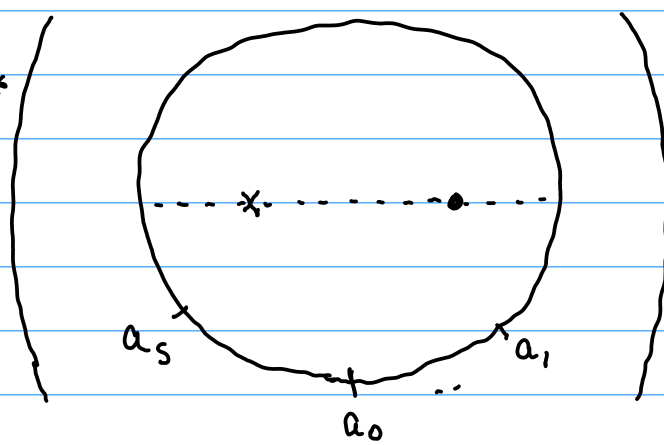
Now we want to show it respects ∇, $(\circ,\circ)$. Versions of these results are implicit in [Cos07], see also [FOOO12] and [FOOO, in progress] regarding the pairing.

<u>Lem</u>: $OC \circ \nabla^{GGM} = \nabla^A \circ OC$.

<u>Pf</u>: We define

$$H : CC_\bullet^-(\mathrm{Fuk}(X,D)) \longrightarrow H^*(X; \mathbb{K}_A)[[u]]$$

$H(a_0[\ldots]) := ev_*$



disc parametrized so that a_0 lies at $-i$, $\bullet$ at $+t$, $\times$ at $-t$, where $t \in [0,1]$.

$\times$ = point constrained to lie on D

We prove:

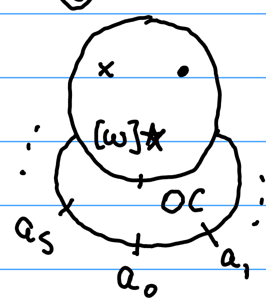
$$\nabla^{A}_{Q\partial_Q} \circ OC - OC \circ \nabla^{GGM}_{Q\partial_Q} = u^{-1}\Big(H \circ (b + uB) + \partial \circ H\Big)$$

$$\Leftrightarrow \underbrace{Q\partial_Q OC(-)}_{1} - \underbrace{u^{-1}[\omega] \star_Q OC(-)}_{2}$$

$$- OC\Big(\underbrace{Q\partial_Q(-)}_{3} - \underbrace{u^{-1} b^1(Q\partial_Q(m^*), -)}_{4} - \underbrace{B^1(Q\partial_Q(m^*), -)}_{5}$$

$$= u^{-1} H\Big(\underbrace{b(-)}_{6} + \underbrace{uB(-)}_{7}\Big)$$

u^{-1} terms on LHS



(2)

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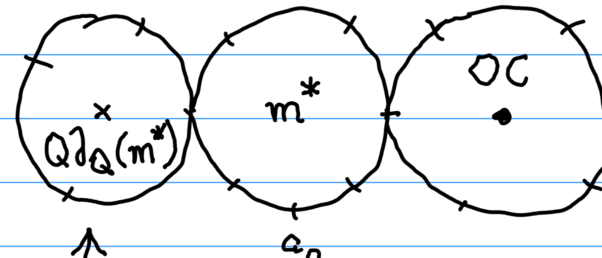

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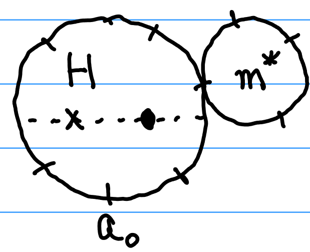


(6)

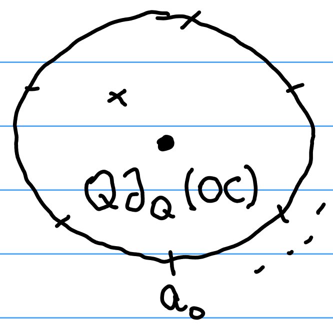
discs in m^* are weighted by $Q^{u \cdot D}$, so $Q\partial_Q(m^*)$ counts discs weighted by $u \cdot D\, Q^{u \cdot D}$, which is equivalent to counting discs with an internal marked point x lying on D (cf. divisor axiom in GW theory).

These 3 terms are boundary components of moduli space defining H, so add up to $u^{-1}\partial \circ H$.

u^0 terms on LHS: first, $Q\partial_Q(\underset{①}{OC}(-)) - \underset{③}{OC}(Q\partial_Q(-))$

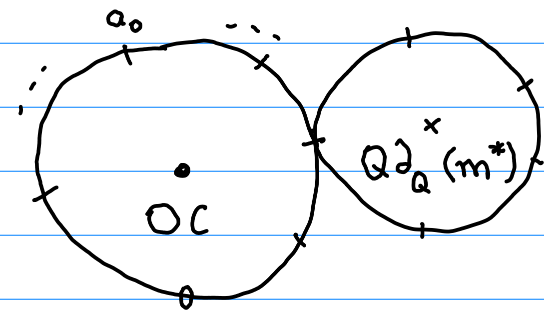
$$= Q\partial_Q(OC)(-)$$

counts discs with marked point $\times$ constrained to lie on D.



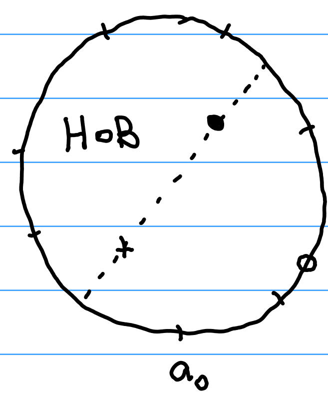
①-③

$+$



⑤

$+$



⑦

⑤ "vanishes" for the same reason $OC \circ B$ does; and ①-③ $=$ ⑦ cancel because the moduli spaces are isomorphic (the positions of $\bullet$ and $\times$ tell you the position of $\circ$ uniquely). $\square$

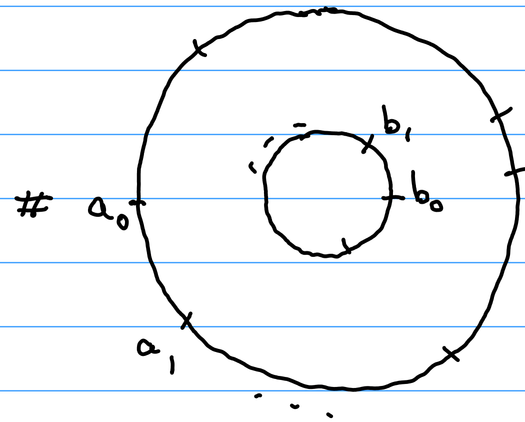
Thus, OC respects connections.

Rmk: If we used $Fuk(X)$, where discs u are weighted by $Q^{\omega(u)}$, then we would need to use a de Rham model for $H^\bullet(X)$ and $\int_{\mathcal{M}(\bigcirc)} ev^*\omega = \omega(\bigcirc)$.

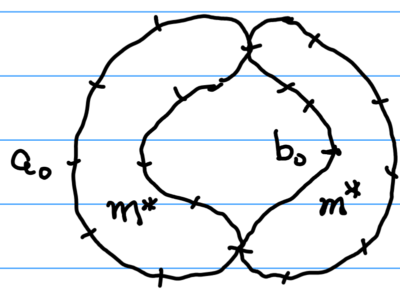
Lem: $\big(OC(a_0[\dots]),\ OC(b_0[\dots])\big) = \big(a_0[\dots],\ b_0[\dots]\big)$
(the *Cardy relation*).

Pf: Define $H : CC_\bullet^-(Fuk(X,D)) \otimes CC_\bullet^-(Fuk(X,D)) \to \mathbb{K}_A[\![u]\!]$ by

$H(a_0[\dots], b_0[\dots]) :=$

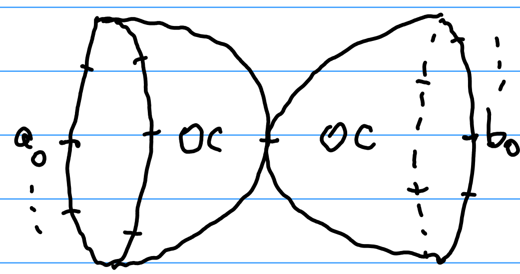


The boundary of this moduli space is



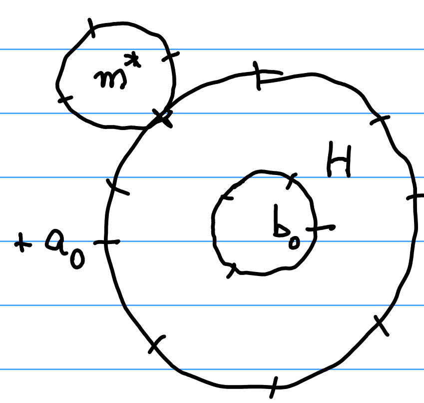
$\big(a_0[\dots], b_0[\dots]\big)$

$+$



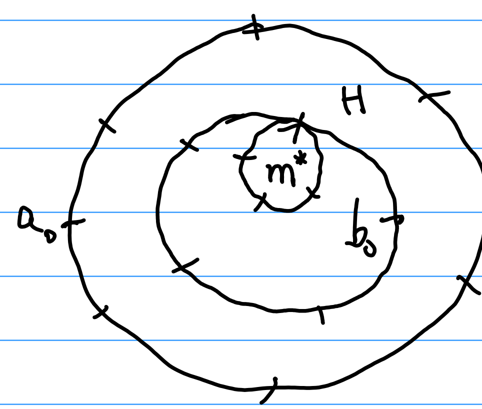
$\big(OC(a_0[\dots]),\ OC(b_0[\dots])\big)$

$+$



$H\big(b(a_0[\dots]),\ b_0[\dots]\big)$

$+$



$H\big(a_0[\dots],\ b(b_0[\dots])\big)$

so the sum of the first two terms is nullhomotopic. $\square$

Thus, we have proved

<u>Thm</u> (Ganatra-Perutz-S.): OC is a morphism of polarized pre-VSHS.

We also have:

<u>Thm</u> [GPS15] If $Fuk(X,D)$ is smooth, OC is an isomorphism.

<u>Pf</u>: $OC : HH_\bullet(Fuk) \to H^{n+\bullet}(X)$ preserves pairings, and the pairing on $HH_\bullet$ is non-degenerate by Shklarov's result, so OC is injective.
Next we need some notions we haven't discussed (see [GPS15] for details): $Fuk(X,D)$ is n-Calabi-Yau, which means

$$\underset{\substack{\neq \\ 0}}{HH^0(Fuk)} \cong HH_n(Fuk)^\vee$$

$\Rightarrow HH_{-n}(Fuk)$ by Shklyarov's result, as $HH_n \otimes HH_{-n} \to \mathbb{K}$ non-degenerate.

$\Rightarrow OC : HH_{-n}(Fuk) \to H^0(X)$ must be surjective, as it is injective and both have rank 1.

Now we use a result of Ganatra: OC is a $QH^*(X)$-module homomorphism; since $e \in QH^0(X)$ is in the image, all of $QH^0(X)$ is.

Thus OC is an iso. It follows that $OC_- : HC_-(Fuk) \to V^A(X)$ is, by a comparison argument for the Hodge-de Rham spectral sequence. $\square$

<u>Cor</u>: If $Fuk(X,D)$ is smooth, then we have an iso of polarized VSHS

$$OC : HC_\bullet^-(Fuk(X,D)) \xrightarrow{\sim} V^A(X).$$

On the B-side, there is an iso

$$HKR : HC_\bullet^-(D^b_{dg}Coh(Y)) \cong V^B(Y)$$

as $\mathcal{O}_M[u]$-modules [Kel 98, Wei 97]

<u>Conj</u>: $HKR \wedge td^{1/2}(Y)$ intertwines connections and polarizations.

<u>Rmk</u>: [CFW 11] should imply compatibility with connections. [Mar 08, Ram 08] establish the compatibility with polarizations, mod u.

We believe the remaining steps are 'well-known to experts' in both cases, but aren't expert enough to carry them out ourselves.

Assuming this conjecture, we put it all together:

Let X = quintic in $\mathbb{CP}^4$, $D = \{z_1 \dots z_5 = 0\} \subset X$.
We consider $\mathrm{Fuk}(X,D)$, an A_∞ cat. $/ \mathbb{K}_A = \mathbb{C}((Q))$.
Let Y = 'mirror quintic', a variety $/ \mathbb{K}_B = \mathbb{C}((q))$.

<u>Thm</u> [S]: There exists $\psi^* : \mathbb{K}_B \xrightarrow{\sim} \mathbb{K}_A$, and a quasi-equivalence of A_∞ categories $/ \mathbb{K}_A$

$$D^\pi \mathrm{Fuk}(X,D) \cong \psi^* D^b_{dg} \mathrm{Coh}(Y).$$

Furthermore, $\psi^*(q) = Q + O(Q)$
leading coefficient 1.

<u>Cor</u>:
$$\begin{tikzcd}
HC^-_\bullet(D^\pi \mathrm{Fuk}(X,D)) \arrow[r, phantom, "\cong"] \arrow[d, leftrightarrow, "\cong"] & \psi^* HC^-_\bullet(D^b_{dg} \mathrm{Coh}(Y)) \arrow[d, leftrightarrow, "\cong"] \\
V^A(X) & \psi^* V^B(Y)
\end{tikzcd}$$

D^π doesn't change $HC^-_\bullet$; and
D^π Fuk is smooth because
D^b_{dg} Coh is, because Y is;
so $\mathcal{OC}$ is an iso.

$\Rightarrow$ X and Y are Hodge mirror, and the leading coefficient in the mirror map is 1.

Recall Hodge MS allowed us to determine

$$5 + \sum_d N_d d^3 Q^d$$

up to ambiguity of 2 complex constants; one is determined by the leading coefficient of the mirror map, and the other is determined by the leading term 5. Thus we can determine all N_d in this way.

Rmk: HMS is proved (following Seidel's proof for the quartic K3) by showing the two categories are 'versal' deformations in Q-direction, hence are matched up by some ψ^*. This ψ^* is undetermined a priori (other than the leading term), but we just showed it is the mirror map realizing Hodge MS, hence sends flat coordinates to flat coordinates. This, together with the leading term, determines it uniquely.