Lecture 3:

Examples of A_∞ categories we will consider:

E.g. Let $D \subset X$ be an ample divisor, ω a Kähler form on X with $\omega|_{X \setminus D} = d\alpha$.

Define an A_∞ cat. $\mathrm{Fuk}(X, D)$ over $\mathbb{K}_A := \mathbb{C}((Q))$

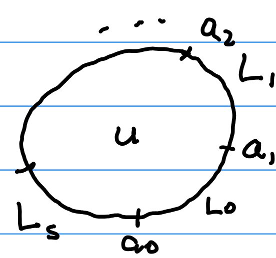
- Obj = closed exact Lagrangians $L \subset X \setminus D$.

  closed: $\partial L = \phi$ exact: $\theta|_L = dh$

  (also equipped with grading and spin structure).

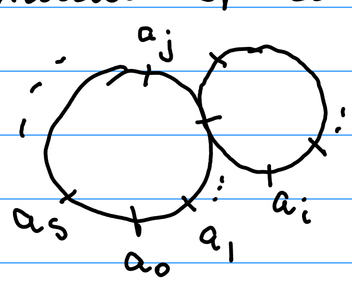
- Mor $= \mathcal{C}(L_0, L_1) := \mathbb{K}_A \langle L_0 \cap L_1 \rangle$

- Structure maps: coefficient of $Q^d \cdot a_0$ in $m^s(a_1, \dots, a_s)$ is

  $\#\Big($ holomorphic discs  with $u \cdot D = d \Big)$

'#' means we count the points in the 0-dimensional component of the moduli space of such curves.

The A_∞ relations hold because the boundary points of the 1-dimensional component of this moduli space are configurations



$\xleftrightarrow{1:1}$ terms in A_∞ relations and $\#$(boundary points of compact 1-mfld $= 0$)

Note: $|L_0 \cap L_1| < \infty \Rightarrow \mathrm{Fuk}(X, D)$ is proper.

Note: Use intersection points with D to stabilize domain, as in John's talks.

(Note: $\mathrm{Fuk}(X)$ defined in same way, but weight discs u by $Q^{\omega(u)} \in \mathbb{C}((Q^{\mathbb{R}}))$. This would force us to work over this field in $V^A(X)$, $V^B(Y)$. I don't know how to define monodromy weight filtration in this world).

E.g. Let $Y =$ smooth projective $/ M_B \supset$ formal punctured disc $\mathrm{Spec}\, \mathbb{K}_B$.

Define an A_∞ category $D^b\mathrm{Coh}(Y)$ over $\mathbb{K}_B$.

Objects = bounded below complexes of injective quasi-coherent sheaves whose cohomology sheaves are bounded and coherent.

Morphisms $= \mathcal{C}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) := \mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$

$$m^1(f) = d_{\mathcal{F}^\bullet} \circ f \pm f \circ d_{\mathcal{E}^\bullet},$$

$$m^2(f, g) = g \circ f, \quad m^{\geq 3} = 0.$$

This is an A_∞ category (in fact a DG category, because $m^{\geq 3} = 0$). It also is proper.

<u>Goal</u>: Given an A_∞ cat. $\mathcal{C}$ over $\mathbb{K} \cong \operatorname{Spec} \mathbb{C}((q))$, define a polarized pre-VSHS

$$V(\mathcal{C}) = (HC_\bullet(\mathcal{C}), \nabla, (\cdot,\cdot))$$

over $\mathcal{M} := \operatorname{Spec} \mathbb{K}$.

Prove $V(\mathrm{Fuk}(X,D)) \cong V^A(X)$

$$V(D^b\mathrm{Coh}(Y)) \cong V^B(Y).$$

<u>Defn</u>: Let $\mathcal{C} = A_\infty$ cat. Define

$$CC_\bullet(\mathcal{C}) := \bigoplus_{L_i} \mathcal{C}(L_0, L_1, \ldots, L_s, L_0)$$

$$b : CC_\bullet(\mathcal{C}) \circlearrowleft$$

$$b(a_0[a_1|\ldots|a_s]) := \sum_{cyc} a_0[\ldots|m^*(\ldots)|\ldots|a_s] + \sum_{cyc} m^*(\ldots, a_s, a_0, \ldots)[\ldots].$$

<u>Lem</u>: $b^2 = 0$.

<u>Defn</u>: $HH_\bullet(\mathcal{C}) := H^\bullet(CC_\bullet(\mathcal{C}), b)$.

Now we define a new A_∞ category $\mathcal{C}_e$:

$$\mathcal{C}_e(L_0, L_1) := \begin{cases} \mathcal{C}(L_0, L_1) & \text{for } L_0 \neq L_1 \\ \mathcal{C}(L_0, L_1) \oplus \mathbb{K}\langle e\rangle & \text{for } L_0 = L_1. \end{cases}$$

$m_2(e,a) = m_2(a,e) = a \quad \forall a$
$m_i(\ldots, e, \ldots) = 0 \quad \forall i \neq 0.$

The inclusion $CC_\bullet(\mathcal{C}) \hookrightarrow CC_\bullet(\mathcal{C}_e)$ is a quasi-iso.

$D_\bullet := \langle a_0[\ldots|e|\ldots] \rangle$ is an acyclic subcomplex of $CC_\bullet(\mathcal{C}_e)$; henceforth we quotient by it on chain level.

Defn: $B : CC_\bullet(\mathcal{C}_e) \circlearrowleft$ Connes B-operator

$$B(a_0[a_1|\ldots|a_s]) := \sum_{cyc} e[a_i|\ldots|a_0|\ldots].$$

Lem: $bB + Bb = 0, \quad B^2 = 0.$

Defn: $CC_\bullet^-(\mathcal{C}) := \big(\overbrace{CC_\bullet(\mathcal{C}_e) \otimes \mathbb{K}[u]}^{\text{complete w.r.t. } u\text{-adic filtration.}},\ \underbrace{b + uB}_{\text{Connes-Tsygan differential}}\big)$

$HC_\bullet^-(\mathcal{C}) := H^\bullet(CC_\bullet^-(\mathcal{C})).$ (Kevin's V^{S^1})
'negative cyclic homology'

$CP_\bullet(\mathcal{C}) := \big(\overbrace{CC_\bullet(\mathcal{C}_e) \otimes \mathbb{K}[u^{\pm 1}]}, b + uB\big)$

$HP_\bullet(\mathcal{C}) := H^\bullet(CP_\bullet(\mathcal{C})).$ (Kevin's V_{Tate}
'periodic cyclic homology'.

The u-adic filtration on $CC_\bullet^-(\mathcal{C})$ is complete by construction. The corresponding spectral sequence

$$HH_\bullet(\mathcal{C}) \otimes \mathbb{K}[u] \Rightarrow HC_\bullet^-(\mathcal{C})$$

is the <u>Hodge-de Rham spectral sequence</u>. If $\mathcal{C}$ is smooth and proper/compact, it degenerates at E_2 page by Kaledin's proof of Kontsevich-Soibelman's conjecture [Kal 16] (we assume $\mathcal{C}$ is $\mathbb{Z}$-graded). Compare Tony's talk.

$HC_\bullet^-(\mathcal{C})$ is an $\mathcal{O}_M[u]$-module, where $M = \operatorname{Spec} \mathbb{K}$. This is the $\mathcal{E}$ in our pre-VSHS.

Next we define the connection, following Getzler [Get 93]. It has the form

$$\nabla : TM \otimes CC_\bullet^-(\mathcal{C}) \longrightarrow u^{-1} CC_\bullet^-(\mathcal{C})$$

$$\nabla_v (\alpha) = v(\alpha) - u^{-1} b^1(v(m^\cdot), \alpha) - B^1(v(m^\cdot), \alpha).$$

To define $v(\alpha)$ we need a $\mathbb{K}$-basis for $CC_\bullet^-$. We obtain one by choosing a $\mathbb{K}$-basis for all $\mathcal{C}(L_0, L_1)$.

We define

$$b^1(v(m^\cdot), a_0[a_1|\dots|a_s]) := \sum_{cyc} m^*(\dots, v(m^*)(\dots), \dots, a_0, \dots)[\dots|a_i]$$

Note: again we need our choice of $\mathbb{K}$-basis in morphism spaces, in order to define

$$v(m^*)(a_1, \dots, a_s) := v(m^*(a_1, \dots, a_s)) - \sum m^*(\dots, v(a_i), \dots)$$

(note: $b^1(-,-): CC^\cdot(\mathcal{C}) \otimes CC_\cdot(\mathcal{C}) \to CC_\cdot(\mathcal{C})$ induces module structure of $HH_\cdot(\mathcal{C})$ over $HH^\cdot(\mathcal{C})$).

$$B^1(v(m^\cdot), a_0[\dots|a_s]) := \sum_{cyc} e[\dots|v(m^*)(\dots)|\dots|a_0|\dots].$$

Getzler proves that

- $[\nabla_v, b + uB] = 0 \Rightarrow$ well-defined on homology
- ∇ is flat on $HP_\cdot(\mathcal{C})$ (automatic in our case since base is 1-dim'l).

The induced connection on $HC_\cdot^-(\mathcal{C})$ is the Getzler-Gauss-Manin connection. It depends on the choice of $\mathbb{K}$-basis on the chain level, but not on the homology level. This is the connection ∇ in our pre-VSHS.

Finally we introduce the polarization $(\cdot,\cdot)$, following Costello, Kontsevich-Soibelman, Shklyarov [Shk 07].
If $\mathcal{C}(L_0,L_1)$ is finite-dimensional for all L_i (which is stronger than properness = $H^\bullet\mathcal{C}(L_0,L_1)$ f.d.), we can define

$$\left(a_0[a_1|\ldots|a_s],\ b_0[b_1|\ldots|b_t]\right)$$

$$:= \sum Tr\left(m^*(a_i,\ldots,a_0,\ldots,m^*(a_j,\ldots,-,b_k,\ldots,b_0,\ldots),b_\ell,\ldots\right).$$

This induces a pairing

$$HC_\bullet^-(\mathcal{C}) \times HC_\bullet^-(\mathcal{C}) \longrightarrow \mathbb{K}[\![u]\!].$$

One can show it is covariantly constant for Getzler's connection.

The same formula defines a pairing on

$$HH_\bullet(\mathcal{C}) \underset{\substack{\uparrow \\ \text{if H-dR degen. holds.}}}{\cong} HC_\bullet^-(\mathcal{C})/u\cdot HC_\bullet^-(\mathcal{C}).$$

Shklyarov proves that, if $\mathcal{C}$ is smooth and proper, $HH_\bullet(\mathcal{C})$ is finite-dimensional and the pairing is nondegenerate.

Putting all of these ingredients together, we have defined a pre-VSHS $(HC_\bullet^-(\mathcal{C}), \nabla)$. If $\mathcal{C}$ is proper we have a polarization. If $\mathcal{C}$ is furthermore smooth, it is a polarized pre-VSHS (although we don't need Kaledin's theorem for our main result).

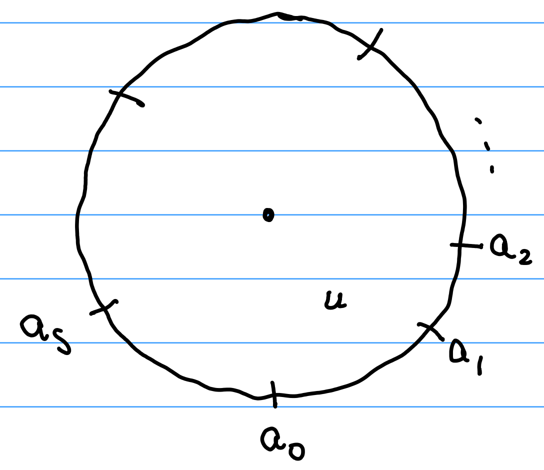
III. Open-closed map

Now we want to compare $HC_\bullet^-(Fuk)$ with V^A.

We define a map

$$OC : CC_\bullet(Fuk(X,D)) \longrightarrow H^*(X; \mathbb{K}_A)$$
$$\uparrow$$
$$\mathbb{C}((Q))$$

$OC(a_0[a_1|\dots|a_s])$ is defined by considering the moduli space of holomorphic discs:

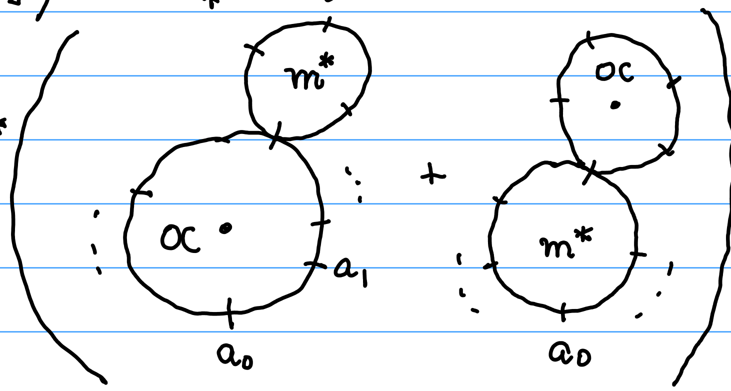


This is some finite-dimensional moduli space $\mathcal{M}$; evaluation at • defines a cycle $ev_* \mathcal{M}$; the contribution to $OC(a_0[\dots])$ is

$$Q^{u \cdot D} \cdot ev_* \mathcal{M}.$$

Lem: $\partial \circ OC = OC \circ b$.

Pf: $\partial \circ OC(a_0[\dots]) = ev_* \, \partial \mathcal{M}$

$= ev_*$ 

$$= OC\Big(\sum a_0[\dots m^*(\dots)\dots]$$

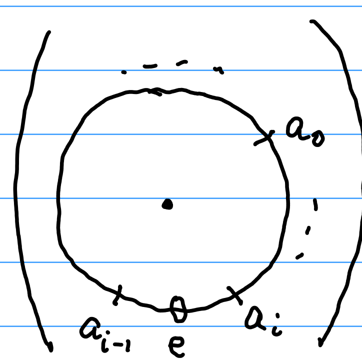
$$+ \sum m^*(\dots a_0 \dots)[\dots]\Big)$$

$$= OC \circ b.(a_0[\dots]) \qquad \square$$

$\Rightarrow OC$ defines a map $HH_\bullet(Fuk(X,D)) \to H^*(X)$.

We extend to $OC: CC_\bullet(\mathcal{C}_e) \to H^*(X)$ by regarding e as a marked point —o— on the boundary of our disc, at which there is no constraint.

"**Lem:**" $OC \circ B = 0$

"**Pf:**" $OC(B(a_0[\dots])) = \sum ev_*$ 

this chain factors through a space

of lower dimension, by forgetting θ, so it is degenerate. $\square$

Thus we have a map

$$OC : HC_\bullet^-(\mathrm{Fuk}(X,D)) \longrightarrow H^*(X;\mathbb{K}_A)[[u]].$$

$$H^*(X;\mathbb{K}_A)[[u]] = V^A(X)$$

<u>Rmk</u>: In [FOOO10] the authors construct this map by proving $OC \circ B$. In our technical setup, it is difficult to arrange for this to hold on the nose, so we actually construct

$$\widetilde{OC} = OC + u \cdot OC_1 + u^2 OC_2 + \dots$$

with $\widetilde{OC} \circ (b + uB) = \partial \circ \widetilde{OC}$.