Lecture 2:

Let $V := (\mathcal{E}, \nabla, (\cdot,\cdot))$ be a polarized VSHS over a formal punctured disc $\mathcal{M}$.

Defn: V has a regular singular point if there exists a basis for $\mathcal{V}$ w.r.t. which the connection matrix has a pole of order 1.

Ex: Check that $V^A(x)$ has this property.

For such a VSHS we can define the monodromy map

$$T : \mathcal{V} \longrightarrow \mathcal{V}$$

by 'parallel translating once anticlockwise about $q=0$'.

Ex: Satisfy yourself that monodromy makes sense even though we don't assume convergence in q (see e.g. [Sabbah, §II.2]).

Defn: V is unipotent if $(T - \mathrm{id})^k = 0$ for some k.

Ex: Check that $V^A(x)$ is unipotent. [Cox-Katz §8.5].

For such a VSHS we can define $N : \mathcal{V} \to \mathcal{V}$

$$N = \log T := \sum_i \frac{1}{i} (\mathrm{id} - T)^i.$$

As T is unipotent, N is nilpotent.

Defn: Suppose $N^{k+1} = 0$, $N^k \neq 0$ (call k the order of nilpotence). Define the weight filtration:

$$0 \subset W_{\leq -\frac{k}{2}} \subset \cdots \subset W_{\leq k/2} = \mathcal{V}$$

to be the unique $\frac{1}{2}\mathbb{Z}$-filtration satisfying

- $N : W_{\leq i} \longrightarrow W_{\leq i-1}$
- $N^i : W_{\leq i/2} / W_{\leq \frac{i-1}{2}} \xrightarrow{\sim} W_{\leq -\frac{i}{2}} / W_{\leq \frac{-i-1}{2}}$.

Ex: Show that for $V^A(X)$,

- $N : H^{\geq i}(X; \mathbb{C}((Q))) \longrightarrow H^{\geq i+2}(X; \mathbb{C}((Q)))$
- $N(\alpha) = 2\pi i [\omega] \cup \alpha + O(Q)$ (for $\alpha \in H^\bullet(X; \mathbb{C})$).

Hence show that

$$W_{\leq i/2} = H^{\bullet \geq n-i}(X; \mathbb{C}((Q))).$$

(the point is that $\cup [\omega]^i : H^{n-i} \xrightarrow{\sim} H^{n+i}$ by Hard Lefschetz - see [Cox-Katz § 8.5]).

Defn: Say V is Hodge-Tate if the filtrations $W_{\leq \bullet}$, $F^{\geq \bullet}$ are opposed: i.e., the natural map

$$\bigoplus_i W_{\leq \frac{i}{2}} \cap F^{\geq \frac{i}{2}} \longrightarrow \mathcal{V}$$

is an iso. This means we have

$$
\begin{tikzcd}
\mathcal{V} \cong \bigoplus_i W_{\leq \frac{i}{2}} \cap F^{\geq \frac{i}{2}} =: \bigoplus_i \mathcal{V}_{-i} \arrow[d, "\sim"'] \arrow[dr, "\sim"] & \\
\mathrm{Gr}_W \mathcal{V} := \bigoplus_i W_{\leq \frac{i}{2}} / W_{\leq \frac{i-1}{2}} & \mathrm{Gr}_F \mathcal{V} := \bigoplus_i F^{\geq \frac{i}{2}} / F^{\geq \frac{i+1}{2}}.
\end{tikzcd}
$$

Say: "the monodromy weight filtration defines a splitting of the Hodge filtration".

Ex: Show that for V^A, $\mathcal{V}_i = H^{n+i}(X)$.

In this situation, ∇ induces a trivial connection on $\mathrm{Gr}_W \mathcal{V}$. Thus, we can choose a basis of $\mathcal{V}_i$ consisting of flat sections, and the connection matrix w.r.t. this basis will be a linear map $TM \otimes \mathcal{V}_i \longrightarrow \mathcal{V}_{i+2}$.

Ex: Show that for V^A, the connection matrix sends

$$Q\partial_Q \otimes \alpha \longmapsto [\omega] \star_Q \alpha.$$

Defn: Suppose the bottom graded piece $\mathcal{V}_{-n}$ is 1-dim'l. A flat section $\Omega \in \mathcal{V}_{-n}$ is called a normalized volume form. A coordinate $q \in \mathcal{O}_M$ is called canonical or flat if the connection matrix sends $q\partial_q \otimes \Omega$ to a flat section of $\mathcal{V}_{-n+2}$ (K. Saito).

Ex: Show that a canonical coordinate is unique up to scaling by a non-zero complex number.

Ex: For $V^A(X)$, show that

- the identity $e \in H^0(X; \mathbb{C})$ is a normalized volume form;

- the Novikov parameter Q is a canonical coordinate.

(the second uses the fact that quantum cup product with the fundamental class is undeformed: $e \star_Q \alpha = e \cup \alpha = \alpha$).

The connection matrix, w.r.t. a basis of flat sections, expanded as a Taylor series in a canonical coordinate q, contains info. about GW invts of X.

E.g. For $X = CY3$, we have

$$\left([\omega] \star_Q [\omega] \star_Q e,\ [\omega] \star_Q e \right) = \left([\omega] \star_Q [\omega],\ [\omega] \right)$$

$$= 5 + \sum_{d=1}^{\infty} N_d \, d^3 Q^d .$$

connection matrix

normalized volume form

can. coordinate

So up to the ambiguity of two complex scalars (in choosing our normalized volume form, and in choosing our canonical coordinate), we can extract the numbers N_d from $V^A(X)$.

We can fix the ambiguity using the leading term 5 and the next term 2875 (see [Cox-Katz, §2]).

In particular, if X and Y are Hodge-theoretically mirror, then we can extract N_d from $V^B(Y)$ by the same procedure (up to some finite ambiguity). Also note that the mirror map Ψ is uniquely determined up to a complex scalar, because it has to send canonical coordinates to canonical coordinates.

<u>Ex</u>*: Compute the numbers N_d for the quintic in this way (see [Cox-Katz]).

II. VSHS from categories

Defn: An A_∞ cat. $\mathcal{C}$ over $\mathbb{K}$ consists of

- Objects L
- Morphisms $\mathcal{C}(L_0, L_1)$ = graded $\mathbb{K}$-vector spaces
- $\mathbb{K}$-linear Structure maps

$$m^s : \mathcal{C}(L_0, \dots, L_s) \longrightarrow \mathcal{C}(L_0, L_s) \qquad (s \geq 1)$$

$$\mathcal{C}(L_0, \dots, L_s) = \mathcal{C}(L_0, L_1) \otimes \dots \otimes \mathcal{C}(L_{s-1}, L_s) \qquad \deg = 2-s$$

satisfying A_∞ relns:

$$\sum \pm m^*(a_1, \dots m^*(a_i, \dots), a_j \dots a_s) = 0.$$

In particular, $m^1(m^1(a)) = 0 \Rightarrow m^1$ a differential on $\mathcal{C}(L_0, L_1) \Rightarrow$ can define cohomology $H^\bullet \mathcal{C}(L_0, L_1)$

$$m^2(m^1(a_1), a_2) \pm m^2(a_1, m^1(a_2)) = m^1(m^2(a_1, a_2))$$

$\Rightarrow m^2$ descends to a composition

$$H^\bullet \mathcal{C}(L_0, L_1) \otimes H^\bullet \mathcal{C}(L_1, L_2) \longrightarrow H^\bullet \mathcal{C}(L_0, L_2)$$

$$m^2(m^2(a_1, a_2), a_3) = m^2(a_1, m^2(a_2, a_3)) + [m^1, m^3]$$

$\Rightarrow m^2$ associative. So $H^\bullet \mathcal{C}(L_0, L_1)$, m^2 define an ordinary category.

If $H^\bullet \mathcal{C}(L_0, L_1)$ is finite-dimensional for all L_0, L_1, we say $\mathcal{C}$ is proper.