Lecture 1:

I. Introduction:

$X \subset \mathbb{CP}^4$ smooth quintic threefold

$H \in H^2(X)$ hyperplane class

$N_d :=$ "#" deg-d hol. maps $u : \mathbb{CP}^1 \to X$
with $u(0) \in H$, $u(1) \in H$, $u(\infty) \in H$.

Mirror symmetry gives a prediction for N_d in terms of periods of a mirror variety Y, in a way we will describe later [Candelas-de la Ossa-Green-Parkes]. Result:

$$N_0 = 5, \quad N_1 = 2875, \quad N_2 = 4876875, \ldots$$

Thm A [Givental, Lian-Liu-Yau]: These predictions are correct.

Thm B [S.]: Homological MS holds for (X, Y): i.e.,

$$D^{\pi}\mathrm{Fuk}(X) \cong D^b\mathrm{Coh}(Y).$$

Thm C [Ganatra-Perutz-S. in prep, building on Barannikov, Costello, Katzarkov, Kontsevich, Pantev...]

Thm B $\Rightarrow$ Thm A.

Our aim: outline proof of Thm C.

Outline: Define A-VHS, $V^A(X)$
B-VHS, $V^B(Y)$

Say <u>Hodge MS holds for (X,Y)</u> if
$$V^A(X) \cong V^B(Y).$$

Step 1: There's a procedure for extracting N_d from $V^A(X)$.
The MS predictions for N_d are obtained by applying this procedure to $V^B(Y)$.

Step 2: Show there are isos

'cyclic homology'

$$\begin{tikzcd}
HP_\bullet(D^\pi Fuk(X)) \arrow[d, leftrightarrow, "2"] \arrow[r, leftrightarrow, dashed, "HMS", "\sim"'] & HP_\bullet(D^b Coh(Y)) \arrow[d, leftrightarrow, "2"] \\
V^A(X) & V^B(Y)
\end{tikzcd}$$

our main focus

So HMS $\Rightarrow$ Hodge MS
$\Rightarrow$ enumerative predictions correct. $\square$

Plan:

II. Hodge-theoretic MS

III. VHS from A_∞ categories

IV. Open-closed maps.

II. Hodge-theoretic MS

Defn: Let R = complete DVR with max. ideal $\mathfrak{m}$, residue field $R/\mathfrak{m} \simeq \mathbb{C}$, field of fractions $\mathbb{K}$. Denote $\mathcal{M} := \operatorname{Spec} \mathbb{K}$, call such $\mathcal{M}$ a formal punctured disc.

A coordinate $q \in \mathbb{K}$ is an element of valuation 1: a choice of coordinate determines isomorphisms

$$\begin{array}{ccc} \mathbb{K} & \cong & \mathbb{C}((q)) \\ \cup & & \cup \\ R & \cong & \mathbb{C}[[q]] \\ \cup & & \cup \\ \mathfrak{m} & \cong & q\,\mathbb{C}[[q]]. \end{array}$$

u will be a formal variable of degree 2.

(perhaps $\mathbb{C}$-VHS better in this context?)

Defn: A polarized pre-VSHS over $\mathcal{M}$ consists of data $(\mathcal{E}, \nabla, (\cdot,\cdot))$, where

- $\mathcal{E}$ is a graded $\underset{\overset{\|}{\mathbb{K}}}{\mathcal{O}_{\mathcal{M}}}[u]$-module;
- $u\nabla : \underset{\overset{\|}{\operatorname{Der}_{\mathbb{C}} \mathbb{K}}}{T\mathcal{M}} \otimes \mathcal{E} \longrightarrow \mathcal{E}$ is a degree-2 map
- $u\nabla_X (f \cdot s) = u \cdot X(f) \cdot s + u \cdot f \cdot \nabla_X (s)$
  (i.e. ∇ is a connection)

- $[u\nabla_X, u\nabla_Y] = u^2 \nabla_{[X,Y]}$ (i.e. ∇ is flat)
- $(\cdot,\cdot) : \mathcal{E} \times \mathcal{E} \to \mathcal{O}_M[u]$ has degree 0 and is $\mathcal{O}_M[u]$-sesquilinear, in the sense that

$$(f_1 \cdot s_1, f_2 \cdot s_2) = f_1 \cdot f_2^* (s_1, s_2)$$

for $f_1, f_2 \in \mathcal{O}_M[u]$, where $f^*(u) := f(-u)$.

- $X(s_1, s_2) = (\nabla_X s_1, s_2) + (s_1, \nabla_X s_2)$

(i.e. $(\cdot,\cdot)$ is covariantly constant

- $(s_1, s_2) = (-1)^{n+|s_1|\cdot|s_2|} (s_2, s_1)^*$.

If $\mathcal{E}$ is finitely-generated and free, and the induced pairing of $\mathcal{O}_M$-vector spaces

$$(\cdot,\cdot) : \mathcal{E}/u\mathcal{E} \otimes \mathcal{E}/u\mathcal{E} \longrightarrow \mathcal{O}_M$$

is nondegenerate, then we call it a polarized VSHS.

References: [Barannikov '01], [Coates-Iritani-Tseng '09], [Gross '11]. This notion is originally due to K. Saito.

Rmk: An integral/rational/real lattice of flat sections is not part of the structure. We don't know how to get that from the categories, although see [Iritani 09], [KKP 08].

<u>Lem</u>: A polarized VSHS is equivalent to the data $(\mathcal{V}, \nabla, \mathcal{F}^{\geq \bullet}, (\cdot,\cdot))$ where

- $\mathcal{V} = \mathcal{V}_{ev} \oplus \mathcal{V}_{odd}$ is a $\mathbb{Z}/2$-graded finite-dimensional $\mathcal{O}_M$-vector space

- ∇ is a flat connection on $\mathcal{V}$ preserving grading

- $\cdots \supset F^{\geq p} \mathcal{V}_{ev} \supset F^{\geq p+1} \mathcal{V}_{ev} \supset \cdots$

  $\cdots \supset F^{\geq p-\frac{1}{2}} \mathcal{V}_{odd} \supset F^{\geq p+\frac{1}{2}} \mathcal{V}_{odd} \supset \cdots$

  decreasing filtrations, satisfying <u>Griffiths transversality</u>:

  $$\nabla F^{\geq p} \subset F^{\geq p-1}.$$

- $(\cdot,\cdot) : \mathcal{V}_\sigma \otimes \mathcal{V}_\sigma \to \mathcal{O}_M$ is a covariantly constant bilinear pairing for $\sigma =$ ev, odd

- $(\alpha, \beta) = (-1)^n (\beta, \alpha)$

- $(F^{\geq p}, F^{\geq q}) = 0$ for $p+q > 0$

- $(\cdot,\cdot) : F^{\geq p}/F^{\geq p+1} \otimes F^{\geq -p}/F^{\geq -p+1} \longrightarrow \mathcal{O}_M$

  is non-degenerate $\forall$ p.

<u>Pf</u>: Note that we have isos

$$\left(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}}[u^{\pm 1}]\right)_k \xrightarrow{\cdot u} \left(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}}[u^{\pm 1}]\right)_{k+2}$$

so we can identify all of the graded pieces of a fixed parity. We define

$$F^{\geq p - k/2} := \left(u^{\geq p} \cdot \mathcal{E}\right)_k .$$

We leave the rest as an exercise (see [GPS, Lemma 2.7]. $\square$

<u>Defn</u>: Let (X, ω) be a Calabi-Yau integral symplectic manifold (i.e., one for which $c_1(TX) = 0$ and $[\omega] \in H^2(X; \mathbb{Z})$). Let

$\mathcal{M}_A := \operatorname{Spec} \mathbb{K}_A$, $\mathbb{K}_A := \mathbb{C}(\!(Q)\!)$. Then we define

$$V^A(X) = (\mathcal{E}^A, \nabla^A, (\cdot,\cdot))$$

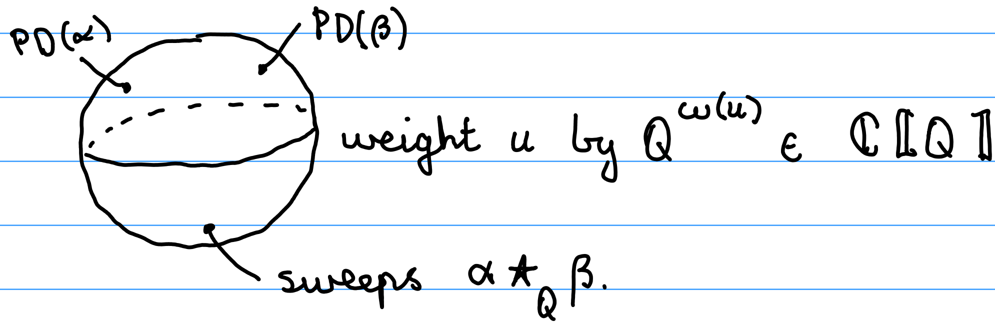
$$\mathcal{E}^A := H^\bullet(X; \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{M}_A}[u][n]$$

shift degrees by $n = \dim X$.

$$\nabla^A_{Q\partial_Q}(\alpha) := Q\partial_Q(\alpha) - u^{-1}[\omega] \star_Q \alpha$$

$$(\alpha, \beta) := (-1)^{n(n+1)/2} \int_X \alpha \cup \beta^*$$

Here, '$\star_Q$' denotes quantum cup product:



The quantum cup product $\star_Q$ is associative, and equal to the classical cup product at $Q = 0$ (see [Clay, §26.5.1]).

Ex: Check this is a polarized VSHS.

Ex: Show it corresponds, under our lemma, to

$$\mathcal{V} = H^\bullet(X; \mathbb{C}(\!(q)\!)), \text{ with filtration}$$

$$F^{\geq \frac{k}{2}} \mathcal{V} = \bigoplus_{\bullet \leq n-k} H^\bullet(X; \mathbb{C}(\!(q)\!)).$$

E.g. For a CY 3: Hodge filtration looks like

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{1,2} & & h^{1,2} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

$F^{\geq 3/2}$ $\quad \cdots \quad$ $F^{\geq -3/2}$

<u>Defn</u>: Let M_B be a formal punctured disc and $Y \to M_B$ a smooth projective connected n-dim'l variety$/M_B$. We define

$$V^B(Y) = (\mathcal{E}^B, \nabla^B, (\cdot,\cdot))$$

It is easiest to define the corresponding data $(\mathcal{V}, \nabla, \mathcal{F}^{\geq \bullet}, (\cdot,\cdot))$ from our lemma:

$$\mathcal{V} := H^{\bullet}_{dR}(Y/M_B)$$

$$\nabla^B := \text{Gauss-Manin connection}$$

$$F^{\geq s}\mathcal{V} := \bigoplus_p H^p(\Omega^{\geq p+2s}_{Y/M_B}, d)$$

$$(\alpha,\beta) := \int_Y \alpha^\vee \wedge \beta \qquad \alpha^\vee := i^{|\alpha|}\cdot \alpha.$$

<u>Ex</u>: Check this is a polarized VSHS. (see e.g. [Voisin '02] for analytic theory of VHS, see [Katz-Oda '68] for how to define G-M connection algebraically).

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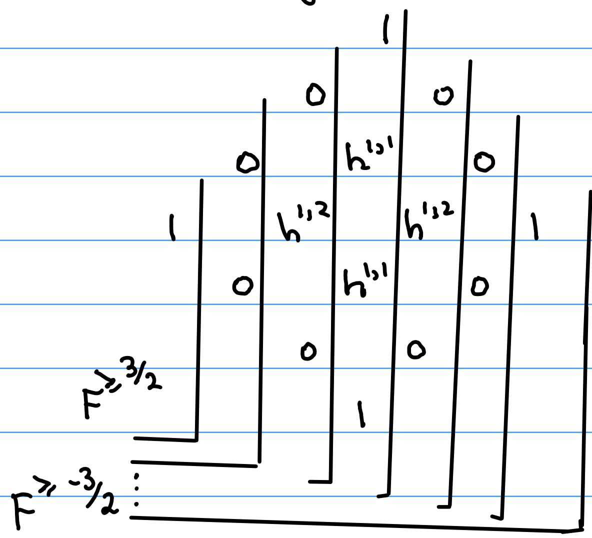

<u>Defn</u>: X and Y are <u>Hodge-theoretically mirror</u> if there exists

$$\psi : \mathcal{M}_A \xrightarrow{\sim} \mathcal{M}_B$$

$$\psi^* V^B(Y) \cong V^A(X).$$