

## NOTE

## A Note on Exponential Sums

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## INTRODUCTION AND STATEMENT OF RESULTS

In this note, we will be concerned with estimating two sorts of exponential sums, especially in characteristic two. The interest of these estimates in characteristic two is that, when combined with some recent work of Evans [4], they prove that Terras's "upper half plane over a finite field" [9] is a "Ramanujan graph" in characteristic two (a result already known in odd characteristic). I thank Ron Evans for telling me about this problem.

In what follows,  $k$  is a finite field of characteristic  $p$  and cardinality  $q$ ,  $\psi: \mathbb{F}_p \rightarrow \mathbb{C}$  is a nontrivial additive character of the prime field  $\mathbb{F}_p$ , and  $\psi_k: k \rightarrow \mathbb{C}$  denotes the nontrivial additive character of  $k$  given by

$$\psi_k(x) := \psi(\text{Trace}_{k/\mathbb{F}_p}(x)).$$

**THEOREM 1.** *Let  $\chi: k^\times \rightarrow \mathbb{C}$  be any nontrivial multiplicative character of  $k$ . For any  $b$  in  $k$  with  $b \neq 0$ ,  $b \neq 1$ ,*

$$\begin{aligned} |\sum_{x \in k^\times, x^2+1 \neq 0} \chi(x) \psi_k(bx/(x^2+1))| &\leq 4 \text{ Sqrt}(q) && \text{if } p \text{ odd} \\ &\leq 2 \text{ Sqrt}(q) && \text{if } p = 2. \end{aligned}$$

To state the second result, denote by  $K/k$  the quadratic extension of  $k$ , and by

$$N_{K/k}: K \rightarrow k, \quad \text{Tr}_{K/k}: K \rightarrow k,$$

the norm and trace maps. Denote by  $U \subset K^\times$  the subgroup consisting of those elements  $x$  in  $K^\times$  with  $N_{K/k}(x) = 1$ .

**THEOREM 2.** *Let  $\chi: U \rightarrow \mathbb{C}$  be any nontrivial character of  $U$ . For any  $b \in k$  with  $b \neq 0, b \neq 1$ ,*

$$\begin{aligned} |\sum_{x \in U, \text{Tr}_{K/k}(x) \neq 0} \chi(x) \psi_k(b/(\text{Tr}_{K/k}(x)))| &\leq 4 \text{Sqrt}(q) && \text{if } p \text{ odd} \\ &\leq 2 \text{Sqrt}(q) && \text{if } p = 2. \end{aligned}$$

*Remark.* In Theorem 2 above, note that for  $x$  in  $K$  of norm 1 to  $k$ , its trace to  $k$  is  $x + 1/x = (x^2 + 1)/x$ . Thus Theorem 2 bears a formal resemblance to Theorem 1. We will see that in fact Theorem 2 results from (the proof of) Theorem 1.

*Proof of Theorem 1.* Fix a prime number  $l \neq p$ , an isomorphism of  $\overline{\mathbb{Q}}_l$  with  $\mathbb{C}$ , and consider the lisse  $\overline{\mathbb{Q}}_l$ -sheaf

$$\mathcal{F} := \mathcal{L}_{\chi(x)} \otimes \mathcal{L}_{\psi(bx/(x^2+1))}$$

on the open set  $\mathbb{A}_k^1 \setminus \{1/x(x^2 + 1)\}$  of  $\mathbb{A}_k^1$ , where  $x(x^2 + 1)$  is invertible. The sheaf  $\mathcal{F}$  is lisse of rank one and pure of weight zero (being of finite order). By the Lefschetz trace formula [5],

$$\begin{aligned} \sum_{x \in k^\times, x^2+1 \neq 0} \chi(x) \psi_k(bx/(x^2 + 1)) \\ = \sum_i (-1)^i \text{Trace}(F_k | H_c^i(\mathbb{A}_k^1 \setminus \{1/x(x^2 + 1)\} \otimes_k \bar{k}, \mathcal{F})). \end{aligned}$$

Since we are on a curve, the only possibly nonzero  $H_c^i$  have  $0 \leq i \leq 2$ . Because  $\mathcal{F}$  is lisse and the curve is open,  $H_c^0 = 0$ . What about  $H_c^2$ ? The sheaf  $\mathcal{F}$  is ramified at  $x = 0$ ; indeed  $\mathcal{L}_{\chi(x)}$  is ramified at  $x = 0$  because  $\chi$  is nontrivial, while  $\mathcal{L}_{\psi(bx/(x^2+1))}$  is lisse at  $x = 0$ . Therefore  $\mathcal{F}$  is not geometrically constant and, hence, being of rank one, has  $H_c^2 = 0$ . Thus our exponential sum is equal to

$$-\text{Trace}(F_k | H_c^1(\mathbb{A}_k^1 \setminus \{1/x(x^2 + 1)\} \otimes_k \bar{k}, \mathcal{F})),$$

and

$$\dim H_c^1(\mathbb{A}_k^1 \setminus \{1/x(x^2 + 1)\} \otimes_k \bar{k}, \mathcal{F}) = -\chi_c(\mathbb{A}_k^1 \setminus \{1/x(x^2 + 1)\} \otimes_k \bar{k}, \mathcal{F}).$$

By Weil [10], cf. also [3], we know that this  $H_c^1$  is mixed of weight  $\leq 1$ , so it suffices to show that

$$\begin{aligned}\chi_c(\mathbb{A}_k^1[1/x(x^2 + 1)] \otimes_k \bar{k}, \mathcal{F}) &= -4, & \text{if } p \text{ is odd,} \\ &= -2, & \text{if } p = 2.\end{aligned}$$

For  $p$  odd, the sheaf  $\mathcal{F}$  is tame at  $x = 0$ , while at  $\pm i$  it has Swan conductor 1 [write  $bx/(x^2 + 1) = ((b/2)(1/(x + i) + 1/(x - i))]$ , so the formula  $\chi_c = -4$  is immediate from the Euler–Poincaré formula [7].

For  $p = 2$ , it will be more convenient to write  $b$  as  $\beta^2$  with  $\beta$  in  $k$  not 0 or 1. Then

$$\begin{aligned}bx/(x^2 + 1) &= \beta^2 x/(x + 1)^2 = \beta^2(x + 1)/(x + 1)^2 - \beta^2/(x + 1)^2 \\ &= \beta^2/(x + 1) - \beta^2/(x + 1)^2 \\ &= (\beta^2 - \beta)/(x + 1) + [(\beta/(x + 1)) - \beta/(x + 1)^2].\end{aligned}$$

The expression in square brackets above is of the form  $w - w^p$ , with  $p = 2$ , so by Artin–Schreier theory,

$$\begin{aligned}\mathcal{L}_{\psi(bx/(x^2+1))} &\approx \mathcal{L}_{\psi((\beta^2 - \beta)/(x + 1))}, \\ \mathcal{F} &\approx \mathcal{L}_{\chi(x)} \otimes \mathcal{L}_{\psi((\beta^2 - \beta)/(x + 1))}.\end{aligned}$$

Thus in characteristic 2, we see that  $\mathcal{F}$  is tame at  $x = 0$ , and has Swan conductor 1 at  $x = 1$ . Again, the formula  $\chi_c = -2$  is immediate from the Euler–Poincaré formula [7]. Q.E.D.

*Proof of Theorem 2.* Fix a prime number  $l \neq p$ , and an isomorphism of  $\bar{\mathbb{Q}}_l$  with  $\mathbb{C}$ . Exactly as explained in [6] we consider the set of points  $x$  in  $U$  where  $\text{Tr}_{K/k}(x) \neq 0$  as the  $k$ -points of an open smooth curve  $U[1/\text{Tr}]$  over  $k$ , and our exponential sum as the alternating sum of traces of Frobenius on the groups

$$H_c^i(U[1/\text{Tr}] \otimes_k \bar{k}, \mathcal{F}), \quad \mathcal{F} := \mathcal{L}_\chi \otimes \mathcal{L}_{\psi(b/\text{Tr})}.$$

Just as above, the only possibly nonzero groups are the  $H_c^i$  with  $0 < i \leq 2$ . The sheaf  $\mathcal{F}$  is again lisse of rank one, and pure of weight zero. As above,  $H_c^0 = 0$  ( $\mathcal{F}$  is lisse and  $U[1/\text{Tr}]$  is open). By [10],  $H_c^1$  is mixed of weight  $\leq 1$ , so it suffices to show that  $H_c^2 = 0$  and that  $\chi_c = -4$  if  $p$  is odd, while  $\chi_c = -2$  if  $p$  is 2.

These are both geometric statements. But already over  $K$ , the group-scheme  $U$  splits:  $U \otimes_k K \pm \mathbb{G}_{m,K}$ , and  $\text{Tr}$  becomes the function  $x \mapsto x + 1/x$  on  $\mathbb{G}_{m,K}$ . The sheaf  $\mathcal{L}_\chi$  on  $U$  pulls back to become the sheaf  $\mathcal{L}_p$  on

$\mathbb{G}_{m,K}$ , where  $\rho$  is the (necessarily nontrivial) character of  $\mathbb{G}_{m,K}(K) := K^\times$  which is  $z \mapsto \chi(z^{1-q})$ ; indeed, the intrinsic norm from  $U(K) = K^\times$  to  $U(k) = \{\text{elements } x \text{ in } K^\times \text{ with } x^{1+q} = 1\}$  is the surjective homomorphism  $z \mapsto z^{1-q}$ . The sheaf  $\mathcal{L}_{\psi(b/\text{Tr})}$  on  $U[1/\text{Tr}]$  pulls back to become the sheaf  $\mathcal{L}_{\psi(b/(x+1/x))} = \mathcal{L}_{\psi(bx/(x^2+1))}$  on  $U[1/\text{Tr}] \otimes_k K = \mathbb{A}_K^1[1/x(x^2+1)]$ . Thus both the vanishing of  $H_c^2$  and the calculation of  $\chi_c$  have already been proven above: just repeat the proof of Theorem 1, but over the field  $K$ . Q.E.D.

*Remark.* In both Theorems 1 and 2, the  $H_c^1$  is in fact pure of weight one, because (cf. [3, 3.2.3]) the lisse rank one sheaf  $\mathcal{F}$  is nontrivially ramified at each of the missing points, including  $\infty$ .

QUESTION. One gets a natural generalization of the sum considered in Theorem 2 by allowing  $K/k$  to have arbitrary degree  $n \geq 2$ . The “split” version of this sum is the “inverted  $n$ -variable Kloosterman sum”

$$\sum_{\prod x_i=1, \sum x_i \neq 0} \chi_1(x_1)\chi_2(x_2) \cdots \chi_n(x_n)\psi(b/(x_1 + x_2 + \cdots + x_n)).$$

What can be said about such sums, and what are they good for?

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