Proof of Theorem 81

The theorem to be proved is

$Sx \leq y \quad \rightarrow \quad x < y$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(Sx) \leq (y)] \quad \& \quad [\neg (x) < (y)]]$

Special cases of the hypothesis and previous results:

0: $Sx \leq y$ from H:x:y

1: $\neg\, x < y$ from H:x:y

2: $x < y \;\vee\; \neg\, x \leq y \;\vee\; y = x$ from $56^{\leftarrow}$;x;y

3: $x \leq Sx$ from 63;x

4: $\neg\, x \leq Sx \;\vee\; \neg\, Sx \leq y \;\vee\; x \leq y$ from 73;x;Sx;y

5: $\neg\, Sx \leq x \;\vee\; (Sx) - x = 0$ from $55^{\rightarrow}$;Sx;x

6: $\neg\, (Sx) - x = 0$ from 21;x

Equality substitutions:

7: $\neg\, y = x \;\vee\; \neg\, Sx \leq y \;\vee\; Sx \leq x$

Inferences:

8: $\neg\, x \leq Sx \;\vee\; x \leq y$ by
0: $Sx \leq y$
4: $\neg\, x \leq Sx \;\vee\; \neg\, Sx \leq y \;\vee\; x \leq y$

9: $\neg\, y = x \;\vee\; Sx \leq x$ by
0: $Sx \leq y$
7: $\neg\, y = x \;\vee\; \neg\, Sx \leq y \;\vee\; Sx \leq x$

10: $\neg\, x \leq y \;\vee\; y = x$ by
1: $\neg\, x < y$
2: $x < y \;\vee\; \neg\, x \leq y \;\vee\; y = x$

11: $x \leq y$ by
3: $x \leq Sx$
8: $\neg\, x \leq Sx \;\vee\; x \leq y$

12: $\neg\, \mathrm{S}x \leq x$ by

6: $\neg\, (\mathrm{S}x) - x = 0$

5: $\neg\, \mathrm{S}x \leq x \quad \vee \quad (\mathrm{S}x) - x = 0$

13: $y = x$ by

11: $x \leq y$

10: $\neg\, x \leq y \quad \vee \quad y = x$

14: $\neg\, y = x$ by

12: $\neg\, \mathrm{S}x \leq x$

9: $\neg\, y = x \quad \vee \quad \mathrm{S}x \leq x$

15: QEA by

13: $y = x$

14: $\neg\, y = x$