Proof of Theorem 72i

The theorem to be proved is

$$x + (y + z) = (x + y) + z \quad \rightarrow \quad x + (y + Sz) = (x + y) + Sz$$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x + (y + z)) = ((x + y) + z)] \quad \& \quad [\neg\, (x + (y + (Sz))) = ((x + y) + (Sz))]]$

Special cases of the hypothesis and previous results:

0: $x + (y + z) = (x + y) + z$ from H:x:y:z

1: $\neg\, x + (y + (Sz)) = (x + y) + (Sz)$ from H:x:y:z

2: $S(y + z) = y + (Sz)$ from 12;y;z

3: $S(x + (y + z)) = x + (S(y + z))$ from 12;x;$y + z$

4: $S((x + y) + z) = (x + y) + (Sz)$ from 12;$x + y$;z

Equality substitutions:

5: $\neg\, x + (y + z) = (x + y) + z \;\vee\; \neg\, S(x + (y + z)) = x + (S(y + z)) \;\vee\; S((x + y) + z) = x + (S(y + z))$

6: $\neg\, S(y + z) = y + (Sz) \;\vee\; \neg\, x + (S(y + z)) = (x + y) + (Sz) \;\vee\; x + (y + (Sz)) = (x + y) + (Sz)$

7: $\neg\, S((x + y) + z) = (x + y) + (Sz) \;\vee\; \neg\, S((x + y) + z) = x + (S(y + z)) \;\vee\; (x + y) + (Sz) = x + (S(y + z))$

Inferences:

8: $\neg\, S(x + (y + z)) = x + (S(y + z)) \;\vee\; S((x + y) + z) = x + (S(y + z))$ by
 0: $x + (y + z) = (x + y) + z$
 5: $\neg\, x + (y + z) = (x + y) + z \;\vee\; \neg\, S(x + (y + z)) = x + (S(y + z)) \;\vee\; S((x + y) + z) = x + (S(y + z))$

9: $\neg\, S(y + z) = y + (Sz) \;\vee\; \neg\, x + (S(y + z)) = (x + y) + (Sz)$ by
 1: $\neg\, x + (y + (Sz)) = (x + y) + (Sz)$
 6: $\neg\, S(y + z) = y + (Sz) \;\vee\; \neg\, x + (S(y + z)) = (x + y) + (Sz) \;\vee\; x + (y + (Sz)) = (x + y) + (Sz)$

10: $\neg\, x + (\mathrm{S}(y+z)) = (x+y) + (\mathrm{S}z)$ by

2: $\mathrm{S}(y+z) = y + (\mathrm{S}z)$

9: $\neg\, \mathrm{S}(y+z) = y + (\mathrm{S}z) \quad \vee \quad \neg\, x + (\mathrm{S}(y+z)) = (x+y) + (\mathrm{S}z)$

11: $\mathrm{S}((x+y)+z) = x + (\mathrm{S}(y+z))$ by

3: $\mathrm{S}(x+(y+z)) = x + (\mathrm{S}(y+z))$

8: $\neg\, \mathrm{S}(x+(y+z)) = x + (\mathrm{S}(y+z)) \quad \vee \quad \mathrm{S}((x+y)+z) = x + (\mathrm{S}(y+z))$

12: $\neg\, \mathrm{S}((x+y)+z) = x + (\mathrm{S}(y+z)) \quad \vee \quad x + (\mathrm{S}(y+z)) = (x+y) + (\mathrm{S}z)$ by

4: $\mathrm{S}((x+y)+z) = (x+y) + (\mathrm{S}z)$

7: $\neg\, \mathrm{S}((x+y)+z) = (x+y) + (\mathrm{S}z) \quad \vee \quad \neg\, \mathrm{S}((x+y)+z) = x + (\mathrm{S}(y+z))$
$\vee \quad x + (\mathrm{S}(y+z)) = (x+y) + (\mathrm{S}z)$

13: $\neg\, \mathrm{S}((x+y)+z) = x + (\mathrm{S}(y+z))$ by

10: $\neg\, x + (\mathrm{S}(y+z)) = (x+y) + (\mathrm{S}z)$

12: $\neg\, \mathrm{S}((x+y)+z) = x + (\mathrm{S}(y+z)) \quad \vee \quad x + (\mathrm{S}(y+z)) = (x+y) + (\mathrm{S}z)$

14: QEA by

11: $\mathrm{S}((x+y)+z) = x + (\mathrm{S}(y+z))$

13: $\neg\, \mathrm{S}((x+y)+z) = x + (\mathrm{S}(y+z))$