Proof of Theorem 69i

The theorem to be proved is

$[x-(y+z)=(x-y)-z] \quad \rightarrow \quad [x-(y+\mathrm{S}z)=(x-y)-\mathrm{S}z]$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x-(y+z))=((x-y)-z)] \quad \& \quad [\neg\,(x-(y+(\mathrm{S}z)))=((x-y)-(\mathrm{S}z))]]$

Special cases of the hypothesis and previous results:

0: $x-(y+z)=(x-y)-z$ from H:x:y:z

1: $\neg\, x-(y+(\mathrm{S}z))=(x-y)-(\mathrm{S}z)$ from H:x:y:z

2: $\mathrm{S}(y+z)=y+(\mathrm{S}z)$ from 12;y;z

3: $\mathrm{P}(x-(y+z))=x-(\mathrm{S}(y+z))$ from 17;x;$y+z$

4: $\mathrm{P}((x-y)-z)=(x-y)-(\mathrm{S}z)$ from 17;$x-y$;z

Equality substitutions:

5: $\neg\, x-(y+z)=(x-y)-z \quad \vee \quad \neg\,\mathrm{P}(x-(y+z))=x-(\mathrm{S}(y+z)) \quad \vee \quad \mathrm{P}((x-y)-z)=x-(\mathrm{S}(y+z))$

6: $\neg\,\mathrm{S}(y+z)=y+(\mathrm{S}z) \quad \vee \quad \neg\, x-(\mathrm{S}(y+z))=(x-y)-(\mathrm{S}z) \quad \vee \quad x-(y+(\mathrm{S}z))=(x-y)-(\mathrm{S}z)$

7: $\neg\,\mathrm{P}((x-y)-z)=(x-y)-(\mathrm{S}z) \quad \vee \quad \neg\,\mathrm{P}((x-y)-z)=x-(\mathrm{S}(y+z)) \quad \vee \quad (x-y)-(\mathrm{S}z)=x-(\mathrm{S}(y+z))$

Inferences:

8: $\neg\,\mathrm{P}(x-(y+z))=x-(\mathrm{S}(y+z)) \quad \vee \quad \mathrm{P}((x-y)-z)=x-(\mathrm{S}(y+z))$ by

0: $x-(y+z)=(x-y)-z$

5: $\neg\, x-(y+z)=(x-y)-z \quad \vee \quad \neg\,\mathrm{P}(x-(y+z))=x-(\mathrm{S}(y+z)) \quad \vee \quad \mathrm{P}((x-y)-z)=x-(\mathrm{S}(y+z))$

9: $\neg\,\mathrm{S}(y+z)=y+(\mathrm{S}z) \quad \vee \quad \neg\, x-(\mathrm{S}(y+z))=(x-y)-(\mathrm{S}z)$ by

1: $\neg\, x-(y+(\mathrm{S}z))=(x-y)-(\mathrm{S}z)$

6: $\neg\,\mathrm{S}(y+z)=y+(\mathrm{S}z) \quad \vee \quad \neg\, x-(\mathrm{S}(y+z))=(x-y)-(\mathrm{S}z) \quad \vee \quad x-(y+(\mathrm{S}z))=(x-y)-(\mathrm{S}z)$

10: $\neg\, x - (\mathrm{S}(y+z)) = (x-y) - (\mathrm{S}z)$ by

2: $\mathrm{S}(y+z) = y + (\mathrm{S}z)$

9: $\neg\, \mathrm{S}(y+z) = y + (\mathrm{S}z) \quad \vee \quad \neg\, x - (\mathrm{S}(y+z)) = (x-y) - (\mathrm{S}z)$

11: $\mathrm{P}((x-y)-z) = x - (\mathrm{S}(y+z))$ by

3: $\mathrm{P}(x-(y+z)) = x - (\mathrm{S}(y+z))$

8: $\neg\, \mathrm{P}(x-(y+z)) = x - (\mathrm{S}(y+z)) \quad \vee \quad \mathrm{P}((x-y)-z) = x - (\mathrm{S}(y+z))$

12: $\neg\, \mathrm{P}((x-y)-z) = x - (\mathrm{S}(y+z)) \quad \vee \quad x - (\mathrm{S}(y+z)) = (x-y) - (\mathrm{S}z)$ by

4: $\mathrm{P}((x-y)-z) = (x-y) - (\mathrm{S}z)$

7: $\neg\, \mathrm{P}((x-y)-z) = (x-y) - (\mathrm{S}z) \quad \vee \quad \neg\, \mathrm{P}((x-y)-z) = x - (\mathrm{S}(y+z))$
$\vee \quad x - (\mathrm{S}(y+z)) = (x-y) - (\mathrm{S}z)$

13: $\neg\, \mathrm{P}((x-y)-z) = x - (\mathrm{S}(y+z))$ by

10: $\neg\, x - (\mathrm{S}(y+z)) = (x-y) - (\mathrm{S}z)$

12: $\neg\, \mathrm{P}((x-y)-z) = x - (\mathrm{S}(y+z)) \quad \vee \quad x - (\mathrm{S}(y+z)) = (x-y) - (\mathrm{S}z)$

14: QEA by

11: $\mathrm{P}((x-y)-z) = x - (\mathrm{S}(y+z))$

13: $\neg\, \mathrm{P}((x-y)-z) = x - (\mathrm{S}(y+z))$