Proof of Theorem 37

The theorem to be proved is

$\mathrm{C}(x, \mathrm{S}0, 0) = 0 \quad \leftrightarrow \quad x \neq 0$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(\mathrm{C}((x, \mathrm{S}0, 0))) = (0) \quad \vee \quad \neg\ (\mathrm{C}((x, \mathrm{S}0, 0))) = (0)] \quad \& \quad [(\mathrm{C}((x, \mathrm{S}0, 0))) = (0) \quad \vee \quad \neg\ (x) = (0)] \quad \& \quad [(x) = (0) \quad \vee \quad \neg\ (\mathrm{C}((x, \mathrm{S}0, 0))) = (0)] \quad \& \quad [(x) = (0) \quad \vee \quad \neg\ (x) = (0)]]$

Special cases of the hypothesis and previous results:

0: $\mathrm{C}((x, \mathrm{S}0, 0)) = 0 \quad \vee \quad \neg\ 0 = x$ from H:x

1: $0 = x \quad \vee \quad \neg\ \mathrm{C}((x, \mathrm{S}0, 0)) = 0$ from H:x

2: $0 = x \quad \vee \quad \mathrm{S}(\mathrm{P}x) = x$ from 22;x

3: $\mathrm{C}((0, \mathrm{S}0, 0)) = \mathrm{S}0$ from 33;S0;0;Px

4: $\mathrm{C}((\mathrm{S}(\mathrm{P}x), \mathrm{S}0, 0)) = 0$ from 33;S0;0;Px

5: $\neg\ \mathrm{S}0 = 0$ from 3;0

Equality substitutions:

6: $\neg\ 0 = x \quad \vee \quad \neg\ \mathrm{C}((x, \mathrm{S}0, 0)) = 0 \quad \vee \quad \mathrm{C}((x, \mathrm{S}x, x)) = x$

7: $\neg\ 0 = x \quad \vee \quad \neg\ \mathrm{C}((0, \mathrm{S}0, 0)) = \mathrm{S}0 \quad \vee \quad \mathrm{C}((x, \mathrm{S}x, x)) = \mathrm{S}x$

8: $\neg\ 0 = x \quad \vee \quad \mathrm{S}0 = 0 \quad \vee \quad \neg\ \mathrm{S}x = x$

9: $\neg\ \mathrm{S}(\mathrm{P}x) = x \quad \vee \quad \neg\ \mathrm{C}((\mathrm{S}(\mathrm{P}x), \mathrm{S}0, 0)) = 0 \quad \vee \quad \mathrm{C}((x, \mathrm{S}0, 0)) = 0$

10: $\neg\ \mathrm{C}((x, \mathrm{S}x, x)) = x \quad \vee \quad \neg\ \mathrm{C}((x, \mathrm{S}x, x)) = \mathrm{S}x \quad \vee \quad x = \mathrm{S}x$

Inferences:

11: $\neg\ 0 = x \quad \vee \quad \mathrm{C}((x, \mathrm{S}x, x)) = \mathrm{S}x$ by

3: $\mathrm{C}((0, \mathrm{S}0, 0)) = \mathrm{S}0$

7: $\neg\ 0 = x \quad \vee \quad \neg\ \mathrm{C}((0, \mathrm{S}0, 0)) = \mathrm{S}0 \quad \vee \quad \mathrm{C}((x, \mathrm{S}x, x)) = \mathrm{S}x$

12: $\neg\ \mathrm{S}(\mathrm{P}x) = x \quad \vee \quad \mathrm{C}((x, \mathrm{S}0, 0)) = 0$ by

4: $\mathrm{C}((\mathrm{S}(\mathrm{P}x), \mathrm{S}0, 0)) = 0$

9: $\neg\ \mathrm{S}(\mathrm{P}x) = x \quad \vee \quad \neg\ \mathrm{C}((\mathrm{S}(\mathrm{P}x), \mathrm{S}0, 0)) = 0 \quad \vee \quad \mathrm{C}((x, \mathrm{S}0, 0)) = 0$

13: $\neg\, 0 = x \quad \vee \quad \neg\, \mathrm{S}x = x$ by

5: $\neg\, \mathrm{S}0 = 0$

8: $\neg\, 0 = x \quad \vee \quad \mathrm{S}0 = 0 \quad \vee \quad \neg\, \mathrm{S}x = x$

CLAIM: $0 = x$ Suppose not. Then

14: $\neg\, 0 = x$

15: $\neg\, \mathrm{C}((x, \mathrm{S}0, 0)) = 0$ by

14: $\neg\, 0 = x$

1: $0 = x \quad \vee \quad \neg\, \mathrm{C}((x, \mathrm{S}0, 0)) = 0$

16: $\mathrm{S}(\mathrm{P}x) = x$ by

14: $\neg\, 0 = x$

2: $0 = x \quad \vee \quad \mathrm{S}(\mathrm{P}x) = x$

17: $\neg\, \mathrm{S}(\mathrm{P}x) = x$ by

15: $\neg\, \mathrm{C}((x, \mathrm{S}0, 0)) = 0$

12: $\neg\, \mathrm{S}(\mathrm{P}x) = x \quad \vee \quad \mathrm{C}((x, \mathrm{S}0, 0)) = 0$

18: $0 = x$ The CLAIM is proved, and 14–17 will not be used after this:

16: $\mathrm{S}(\mathrm{P}x) = x$

17: $\neg\, \mathrm{S}(\mathrm{P}x) = x$

19: $\mathrm{C}((x, \mathrm{S}0, 0)) = 0$ by

18: $0 = x$

0: $\mathrm{C}((x, \mathrm{S}0, 0)) = 0 \quad \vee \quad \neg\, 0 = x$

20: $\neg\, \mathrm{C}((x, \mathrm{S}0, 0)) = 0 \quad \vee \quad \mathrm{C}((x, \mathrm{S}x, x)) = x$ by

18: $0 = x$

6: $\neg\, 0 = x \quad \vee \quad \neg\, \mathrm{C}((x, \mathrm{S}0, 0)) = 0 \quad \vee \quad \mathrm{C}((x, \mathrm{S}x, x)) = x$

21: $\mathrm{C}((x, \mathrm{S}x, x)) = \mathrm{S}x$ by

18: $0 = x$

11: $\neg\, 0 = x \quad \vee \quad \mathrm{C}((x, \mathrm{S}x, x)) = \mathrm{S}x$

22: $\neg\, \mathrm{S}x = x$ by

18: $0 = x$

13: $\neg\, 0 = x \quad \vee \quad \neg\, \mathrm{S}x = x$

23: $\mathrm{C}((x, \mathrm{S}x, x)) = x$ by

19: $\mathrm{C}((x, \mathrm{S}0, 0)) = 0$

20: $\neg\, \mathrm{C}((x, \mathrm{S}0, 0)) = 0 \quad \vee \quad \mathrm{C}((x, \mathrm{S}x, x)) = x$

24: $\neg\ \mathrm{C}((x, \mathrm{S}x, x)) = x \quad \vee \quad \mathrm{S}x = x$ by

21: $\mathrm{C}((x, \mathrm{S}x, x)) = \mathrm{S}x$

10: $\neg\ \mathrm{C}((x, \mathrm{S}x, x)) = x \quad \vee \quad \neg\ \mathrm{C}((x, \mathrm{S}x, x)) = \mathrm{S}x \quad \vee \quad \mathrm{S}x = x$

25: $\neg\ \mathrm{C}((x, \mathrm{S}x, x)) = x$ by

22: $\neg\ \mathrm{S}x = x$

24: $\neg\ \mathrm{C}((x, \mathrm{S}x, x)) = x \quad \vee \quad \mathrm{S}x = x$

26: QEA by

23: $\mathrm{C}((x, \mathrm{S}x, x)) = x$

25: $\neg\ \mathrm{C}((x, \mathrm{S}x, x)) = x$