Proof of Theorem 32

The theorem to be proved is

$\mathrm{Eq}(x,y) = 0 \quad \leftrightarrow \quad x = y$ ⋆⋆

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(\mathrm{Eq}((x,y))) = (0) \;\vee\; \neg\,(\mathrm{Eq}((x,y))) = (0)] \;\;\&\;\; [(\mathrm{Eq}((x,y))) = (0) \;\vee\; (x) = (y)]$
$\&\;\; [\neg\,(x) = (y) \;\vee\; \neg\,(\mathrm{Eq}((x,y))) = (0)] \;\;\&\;\; [\neg\,(x) = (y) \;\vee\; (x) = (y)]]$

Special cases of the hypothesis and previous results:

0: $\mathrm{Eq}((x,y)) = 0 \;\vee\; y = x$ from H:x:y

1: $\neg\, y = x \;\vee\; \neg\,\mathrm{Eq}((x,y)) = 0$ from H:x:y

2: $(x-y)+(y-x) = \mathrm{Eq}((x,y))$ from 31;x;y

3: $\neg\,(x-y)+(y-x) = 0 \;\vee\; x-y = 0$ from 15;$x-y$;$y-x$

4: $\neg\,(x-y)+(y-x) = 0 \;\vee\; y-x = 0$ from 15;$x-y$;$y-x$

5: $\neg\, x-y = 0 \;\vee\; \neg\, y-x = 0 \;\vee\; y = x$ from 30;x;y

6: $x-x = 0$ from 19;x

7: $0+0 = 0$ from 12;0

Equality substitutions:

8: $\neg\,\mathrm{Eq}((x,y)) = 0 \;\vee\; \neg\,(x-y)+(y-x) = \mathrm{Eq}((x,y)) \;\vee\; (x-y)+(y-x) = 0$

9: $\neg\, y = x \;\vee\; (x-y)+(y-x) = 0 \;\vee\; \neg\,(x-x)+(x-x) = 0$

10: $\neg\,(x-y)+(y-x) = \mathrm{Eq}((x,y)) \;\vee\; \neg\,(x-y)+(y-x) = 0 \;\vee\; \mathrm{Eq}((x,y)) = 0$

11: $\neg\, x-x = 0 \;\vee\; (x-x)+(x-x) = 0 \;\vee\; \neg\,(0)+(0) = 0$

Inferences:

12: $\neg\,\mathrm{Eq}((x,y)) = 0 \;\vee\; (x-y)+(y-x) = 0$ by
2: $(x-y)+(y-x) = \mathrm{Eq}((x,y))$
8: $\neg\,\mathrm{Eq}((x,y)) = 0 \;\vee\; \neg\,(x-y)+(y-x) = \mathrm{Eq}((x,y)) \;\vee\; (x-y)+(y-x) = 0$

13: $\neg\,(x-y)+(y-x) = 0 \;\vee\; \mathrm{Eq}((x,y)) = 0$ by
2: $(x-y)+(y-x) = \mathrm{Eq}((x,y))$
10: $\neg\,(x-y)+(y-x) = \mathrm{Eq}((x,y)) \;\vee\; \neg\,(x-y)+(y-x) = 0 \;\vee\; \mathrm{Eq}((x,y)) = 0$

14: $(x-x)+(x-x)=0 \quad \vee \quad \neg\, 0+0=0$ by

6: $x-x=0$

11: $\neg\, x-x=0 \quad \vee \quad (x-x)+(x-x)=0 \quad \vee \quad \neg\, 0+0=0$

15: $(x-x)+(x-x)=0$ by

7: $0+0=0$

14: $(x-x)+(x-x)=0 \quad \vee \quad \neg\, 0+0=0$

16: $\neg\, y=x \quad \vee \quad (x-y)+(y-x)=0$ by

15: $(x-x)+(x-x)=0$

9: $\neg\, y=x \quad \vee \quad (x-y)+(y-x)=0 \quad \vee \quad \neg\,(x-x)+(x-x)=0$

CLAIM: $\neg\, \mathrm{Eq}((x,y))=0$ Suppose not. Then

17: $\mathrm{Eq}((x,y))=0$

18: $\neg\, y=x$ by

17: $\mathrm{Eq}((x,y))=0$

1: $\neg\, y=x \quad \vee \quad \neg\, \mathrm{Eq}((x,y))=0$

19: $(x-y)+(y-x)=0$ by

17: $\mathrm{Eq}((x,y))=0$

12: $\neg\, \mathrm{Eq}((x,y))=0 \quad \vee \quad (x-y)+(y-x)=0$

20: $\neg\, x-y=0 \quad \vee \quad \neg\, y-x=0$ by

18: $\neg\, y=x$

5: $\neg\, x-y=0 \quad \vee \quad \neg\, y-x=0 \quad \vee \quad y=x$

21: $x-y=0$ by

19: $(x-y)+(y-x)=0$

3: $\neg\,(x-y)+(y-x)=0 \quad \vee \quad x-y=0$

22: $y-x=0$ by

19: $(x-y)+(y-x)=0$

4: $\neg\,(x-y)+(y-x)=0 \quad \vee \quad y-x=0$

23: $\neg\, y-x=0$ by

21: $x-y=0$

20: $\neg\, x-y=0 \quad \vee \quad \neg\, y-x=0$

24: $\neg\, \mathrm{Eq}((x,y))=0$ The CLAIM is proved, and 17–23 will not be used after this:

22: $y-x=0$

23: $\neg\, y-x=0$

25: $y = x$ by

24: $\neg \mathrm{Eq}((x, y)) = 0$

0: $\mathrm{Eq}((x, y)) = 0 \quad \vee \quad y = x$

26: $\neg\ (x - y) + (y - x) = 0$ by

24: $\neg \mathrm{Eq}((x, y)) = 0$

13: $\neg\ (x - y) + (y - x) = 0 \quad \vee \quad \mathrm{Eq}((x, y)) = 0$

27: $(x - y) + (y - x) = 0$ by

25: $y = x$

16: $\neg\ y = x \quad \vee \quad (x - y) + (y - x) = 0$

28: QEA by

26: $\neg\ (x - y) + (y - x) = 0$

27: $(x - y) + (y - x) = 0$