Proof of Theorem 277

The theorem to be proved is

$x \preceq x \oplus y \quad \& \quad y \preceq x \oplus y$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[\neg\,(x) \preceq (x \oplus y) \quad \vee \quad \neg\,(y) \preceq (x \oplus y)]]$

Special cases of the hypothesis and previous results:

0: $\neg\, x \preceq x \oplus y \quad \vee \quad \neg\, y \preceq x \oplus y$ from H:x:y

1: $x \preceq x \oplus y \quad \vee \quad \neg\,\mathrm{Length}x \leq \mathrm{Length}(x \oplus y)$ from 264$^{\Leftarrow}$;x;$x \oplus y$

2: $y \preceq x \oplus y \quad \vee \quad \neg\,\mathrm{Length}y \leq \mathrm{Length}(x \oplus y)$ from 264$^{\Leftarrow}$;y;$x \oplus y$

3: $(\mathrm{Length}x) + (\mathrm{Length}y) = \mathrm{Length}(x \oplus y)$ from 260;x;y

4: $\mathrm{Length}x \leq (\mathrm{Length}x) + (\mathrm{Length}y)$ from 71;Length$x$;Length$y$

5: $(\mathrm{Length}y) + (\mathrm{Length}x) = (\mathrm{Length}x) + (\mathrm{Length}y)$ from 98;Length$x$;Length$y$

6: $\mathrm{Length}y \leq (\mathrm{Length}y) + (\mathrm{Length}x)$ from 71;Length$y$;Length$x$

Equality substitutions:

7: $\neg\,(\mathrm{Length}x) + (\mathrm{Length}y) = \mathrm{Length}(x \oplus y) \quad \vee \quad \neg\,\mathrm{Length}x \leq (\mathrm{Length}x) + (\mathrm{Length}y) \quad \vee \quad \mathrm{Length}x \leq \mathrm{Length}(x \oplus y)$

8: $\neg\,(\mathrm{Length}x) + (\mathrm{Length}y) = \mathrm{Length}(x \oplus y) \quad \vee \quad \neg\,(\mathrm{Length}y) + (\mathrm{Length}x) = (\mathrm{Length}x) + (\mathrm{Length}y) \quad \vee \quad (\mathrm{Length}y) + (\mathrm{Length}x) = \mathrm{Length}(x \oplus y)$

9: $\neg\,(\mathrm{Length}y) + (\mathrm{Length}x) = \mathrm{Length}(x \oplus y) \quad \vee \quad \neg\,\mathrm{Length}y \leq (\mathrm{Length}y) + (\mathrm{Length}x) \quad \vee \quad \mathrm{Length}y \leq \mathrm{Length}(x \oplus y)$

Inferences:

10: $\neg\,\mathrm{Length}x \leq (\mathrm{Length}x) + (\mathrm{Length}y) \quad \vee \quad \mathrm{Length}x \leq \mathrm{Length}(x \oplus y)$ by

3: $(\mathrm{Length}x) + (\mathrm{Length}y) = \mathrm{Length}(x \oplus y)$

7: $\neg\,(\mathrm{Length}x) + (\mathrm{Length}y) = \mathrm{Length}(x \oplus y) \quad \vee \quad \neg\,\mathrm{Length}x \leq (\mathrm{Length}x) + (\mathrm{Length}y) \quad \vee \quad \mathrm{Length}x \leq \mathrm{Length}(x \oplus y)$

11: $\neg\,(\mathrm{Length}y) + (\mathrm{Length}x) = (\mathrm{Length}x) + (\mathrm{Length}y) \quad \vee \quad (\mathrm{Length}y) + (\mathrm{Length}x) = \mathrm{Length}(x \oplus y)$ by

3: $(\text{Length}x) + (\text{Length}y) = \text{Length}(x \oplus y)$

8: $\neg\ (\text{Length}x) + (\text{Length}y) = \text{Length}(x \oplus y) \quad \vee \quad \neg\ (\text{Length}y) + (\text{Length}x) = (\text{Length}x) + (\text{Length}y) \quad \vee \quad (\text{Length}y) + (\text{Length}x) = \text{Length}(x \oplus y)$

12: $\text{Length}x \leq \text{Length}(x \oplus y)$ by

4: $\text{Length}x \leq (\text{Length}x) + (\text{Length}y)$

10: $\neg\ \text{Length}x \leq (\text{Length}x) + (\text{Length}y) \quad \vee \quad \text{Length}x \leq \text{Length}(x \oplus y)$

13: $(\text{Length}y) + (\text{Length}x) = \text{Length}(x \oplus y)$ by

5: $(\text{Length}y) + (\text{Length}x) = (\text{Length}x) + (\text{Length}y)$

11: $\neg\ (\text{Length}y) + (\text{Length}x) = (\text{Length}x) + (\text{Length}y) \quad \vee \quad (\text{Length}y) + (\text{Length}x) = \text{Length}(x \oplus y)$

14: $\neg\ (\text{Length}y) + (\text{Length}x) = \text{Length}(x \oplus y) \quad \vee \quad \text{Length}y \leq \text{Length}(x \oplus y)$ by

6: $\text{Length}y \leq (\text{Length}y) + (\text{Length}x)$

9: $\neg\ (\text{Length}y) + (\text{Length}x) = \text{Length}(x \oplus y) \quad \vee \quad \neg\ \text{Length}y \leq (\text{Length}y) + (\text{Length}x) \quad \vee \quad \text{Length}y \leq \text{Length}(x \oplus y)$

15: $x \preceq x \oplus y$ by

12: $\text{Length}x \leq \text{Length}(x \oplus y)$

1: $x \preceq x \oplus y \quad \vee \quad \neg\ \text{Length}x \leq \text{Length}(x \oplus y)$

16: $\text{Length}y \leq \text{Length}(x \oplus y)$ by

13: $(\text{Length}y) + (\text{Length}x) = \text{Length}(x \oplus y)$

14: $\neg\ (\text{Length}y) + (\text{Length}x) = \text{Length}(x \oplus y) \quad \vee \quad \text{Length}y \leq \text{Length}(x \oplus y)$

17: $\neg\ y \preceq x \oplus y$ by

15: $x \preceq x \oplus y$

0: $\neg\ x \preceq x \oplus y \quad \vee \quad \neg\ y \preceq x \oplus y$

18: $y \preceq x \oplus y$ by

16: $\text{Length}y \leq \text{Length}(x \oplus y)$

2: $y \preceq x \oplus y \quad \vee \quad \neg\ \text{Length}y \leq \text{Length}(x \oplus y)$

19: QEA by

17: $\neg\ y \preceq x \oplus y$

18: $y \preceq x \oplus y$