## Proof of Theorem 273

The theorem to be proved is

$x \prec y \quad \rightarrow \quad x < y$ ⋆

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x) \prec (y)] \quad \& \quad [\neg\, (x) < (y)]]$

### Special cases of the hypothesis and previous results:

0: $x \prec y$ from H:$x$:$y$

1: $\neg\, x < y$ from H:$x$:$y$

2: $\neg\, x \prec y \quad \vee \quad \text{Length}x < \text{Length}y$ from 265$^{\rightarrow}$;$x$;$y$

3: $\neg\, \text{Length}x < \text{Length}y \quad \vee \quad 2{\uparrow}(\text{Length}x) < 2{\uparrow}(\text{Length}y)$ from 145;Length$x$;Length$y$

4: $2 \uparrow (\text{Length}x) = \text{Q}x$ from 261;$x$

5: $2 \uparrow (\text{Length}y) = \text{Q}y$ from 261;$y$

6: $\neg\, y \leq x \quad \vee \quad \text{Q}y \leq \text{Q}x$ from 272;$y$;$x$

7: $\neg\, \text{Q}x < \text{Q}y \quad \vee \quad \neg\, \text{Q}y \leq \text{Q}x$ from 80;Q$x$;Q$y$

8: $y \leq x \quad \vee \quad x < y$ from 79;$y$;$x$

### Equality substitutions:

9: $\neg\, 2 \uparrow (\text{Length}x) = \text{Q}x \quad \vee \quad \neg\, 2 \uparrow (\text{Length}x) < 2 \uparrow (\text{Length}y) \quad \vee \quad \text{Q}x < 2 \uparrow (\text{Length}y)$

10: $\neg\, 2 \uparrow (\text{Length}y) = \text{Q}y \quad \vee \quad \neg\, \text{Q}x < 2 \uparrow (\text{Length}y) \quad \vee \quad \text{Q}x < \text{Q}y$

### Inferences:

11: $\text{Length}x < \text{Length}y$ by
0: $x \prec y$
2: $\neg\, x \prec y \quad \vee \quad \text{Length}x < \text{Length}y$

12: $y \leq x$ by
1: $\neg\, x < y$
8: $y \leq x \quad \vee \quad x < y$

13: $\neg\, 2\uparrow(\mathrm{Length}x) < 2\uparrow(\mathrm{Length}y) \quad \vee \quad \mathrm{Q}x < 2\uparrow(\mathrm{Length}y)$ by

4: $2\uparrow(\mathrm{Length}x) = \mathrm{Q}x$

9: $\neg\, 2\uparrow(\mathrm{Length}x) = \mathrm{Q}x \quad \vee \quad \neg\, 2\uparrow(\mathrm{Length}x) < 2\uparrow(\mathrm{Length}y) \quad \vee \quad \mathrm{Q}x < 2\uparrow(\mathrm{Length}y)$

14: $\neg\, \mathrm{Q}x < 2\uparrow(\mathrm{Length}y) \quad \vee \quad \mathrm{Q}x < \mathrm{Q}y$ by

5: $2\uparrow(\mathrm{Length}y) = \mathrm{Q}y$

10: $\neg\, 2\uparrow(\mathrm{Length}y) = \mathrm{Q}y \quad \vee \quad \neg\, \mathrm{Q}x < 2\uparrow(\mathrm{Length}y) \quad \vee \quad \mathrm{Q}x < \mathrm{Q}y$

15: $2\uparrow(\mathrm{Length}x) < 2\uparrow(\mathrm{Length}y)$ by

11: $\mathrm{Length}x < \mathrm{Length}y$

3: $\neg\, \mathrm{Length}x < \mathrm{Length}y \quad \vee \quad 2\uparrow(\mathrm{Length}x) < 2\uparrow(\mathrm{Length}y)$

16: $\mathrm{Q}y \leq \mathrm{Q}x$ by

12: $y \leq x$

6: $\neg\, y \leq x \quad \vee \quad \mathrm{Q}y \leq \mathrm{Q}x$

17: $\mathrm{Q}x < 2\uparrow(\mathrm{Length}y)$ by

15: $2\uparrow(\mathrm{Length}x) < 2\uparrow(\mathrm{Length}y)$

13: $\neg\, 2\uparrow(\mathrm{Length}x) < 2\uparrow(\mathrm{Length}y) \quad \vee \quad \mathrm{Q}x < 2\uparrow(\mathrm{Length}y)$

18: $\neg\, \mathrm{Q}x < \mathrm{Q}y$ by

16: $\mathrm{Q}y \leq \mathrm{Q}x$

7: $\neg\, \mathrm{Q}x < \mathrm{Q}y \quad \vee \quad \neg\, \mathrm{Q}y \leq \mathrm{Q}x$

19: $\mathrm{Q}x < \mathrm{Q}y$ by

17: $\mathrm{Q}x < 2\uparrow(\mathrm{Length}y)$

14: $\neg\, \mathrm{Q}x < 2\uparrow(\mathrm{Length}y) \quad \vee \quad \mathrm{Q}x < \mathrm{Q}y$

20: $QEA$ by

18: $\neg\, \mathrm{Q}x < \mathrm{Q}y$

19: $\mathrm{Q}x < \mathrm{Q}y$