Proof of Theorem 272b

The theorem to be proved is

$x \leq 0 \quad \rightarrow \quad \mathrm{Q}x \leq \mathrm{Q}0$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[(x) \leq (0)] \quad \& \quad [\neg\, (\mathrm{Q}x) \leq (\mathrm{Q}0)]]$

Special cases of the hypothesis and previous results:

0: $x \leq 0$ from H:x

1: $\neg\, \mathrm{Q}x \leq \mathrm{Q}0$ from H:x

2: $\neg\, x \leq 0 \quad \vee \quad 0 = x$ from 57;x

3: $\mathrm{Q}0 \leq \mathrm{Q}0$ from 60;Q0

Equality substitutions:

4: $\neg\; x = 0 \quad \vee \quad \mathrm{Q}(x) \leq \mathrm{Q}0 \quad \vee \quad \neg\; \mathrm{Q}(0) \leq \mathrm{Q}0$

Inferences:

5: $0 = x$ by
0: $x \leq 0$
2: $\neg\, x \leq 0 \quad \vee \quad 0 = x$

6: $\neg\, 0 = x \quad \vee \quad \neg\, \mathrm{Q}0 \leq \mathrm{Q}0$ by
1: $\neg\, \mathrm{Q}x \leq \mathrm{Q}0$
4: $\neg\, 0 = x \quad \vee \quad \mathrm{Q}x \leq \mathrm{Q}0 \quad \vee \quad \neg\, \mathrm{Q}0 \leq \mathrm{Q}0$

7: $\neg\, 0 = x$ by
3: $\mathrm{Q}0 \leq \mathrm{Q}0$
6: $\neg\, 0 = x \quad \vee \quad \neg\, \mathrm{Q}0 \leq \mathrm{Q}0$

8: QEA by
5: $0 = x$
7: $\neg\, 0 = x$