Proof of Theorem 263i

The theorem to be proved is

$[x \le y \quad \rightarrow \quad 2 \uparrow x \le 2 \uparrow y] \quad \rightarrow \quad [x \le \mathrm{S}y \quad \rightarrow \quad 2 \uparrow x \le 2 \uparrow \mathrm{S}y]$

Suppose the theorem does not hold. Then, with the variables held fixed,

(H) $[[\neg\,(x) \le (y) \quad \vee \quad (2 \uparrow x) \le (2 \uparrow y)] \quad \& \quad [(x) \le (\mathrm{S}y)] \quad \& \quad [\neg\,(2 \uparrow x) \le (2 \uparrow (\mathrm{S}y))]]$

Special cases of the hypothesis and previous results:

0: $\neg\, x \le y \quad \vee \quad 2 \uparrow x \le 2 \uparrow y$ from H:x:y

1: $x \le \mathrm{S}y$ from H:x:y

2: $\neg\, 2 \uparrow x \le 2 \uparrow (\mathrm{S}y)$ from H:x:y

3: $\neg\, x \le \mathrm{S}y \quad \vee \quad x \le y \quad \vee \quad \mathrm{S}y = x$ from 262;x;y

4: $2 \uparrow x \le 2 \uparrow x$ from 60;$2 \uparrow x$

5: $2 \cdot (2 \uparrow y) = 2 \uparrow (\mathrm{S}y)$ from 126;2;y

6: $(2 \uparrow y) + (2 \uparrow y) = 2 \cdot (2 \uparrow y)$ from 118;$2 \uparrow y$

7: $2 \uparrow y \le (2 \uparrow y) + (2 \uparrow y)$ from 71;$2 \uparrow y$;$2 \uparrow y$

8: $\neg\, 2 \uparrow x \le 2 \uparrow y \quad \vee \quad \neg\, 2 \uparrow y \le 2 \cdot (2 \uparrow y) \quad \vee \quad 2 \uparrow x \le 2 \cdot (2 \uparrow y)$ from 73;$2 \uparrow x$;$2 \uparrow y$;$2 \cdot (2 \uparrow y)$

Equality substitutions:

9: $\neg\, \mathrm{S}y = x \quad \vee \quad 2 \uparrow x \le 2 \uparrow (\mathrm{S}y) \quad \vee \quad \neg\, 2 \uparrow x \le 2 \uparrow (x)$

10: $\neg\, 2 \cdot (2 \uparrow y) = 2 \uparrow (\mathrm{S}y) \quad \vee \quad \neg\, 2 \uparrow x \le 2 \cdot (2 \uparrow y) \quad \vee \quad 2 \uparrow x \le 2 \uparrow (\mathrm{S}y)$

11: $\neg\, (2 \uparrow y) + (2 \uparrow y) = 2 \cdot (2 \uparrow y) \quad \vee \quad \neg\, 2 \uparrow y \le (2 \uparrow y) + (2 \uparrow y) \quad \vee \quad 2 \uparrow y \le 2 \cdot (2 \uparrow y)$

Inferences:

12: $x \le y \quad \vee \quad \mathrm{S}y = x$ by

1: $x \le \mathrm{S}y$

3: $\neg\, x \le \mathrm{S}y \quad \vee \quad x \le y \quad \vee \quad \mathrm{S}y = x$

13: $\neg\, \mathrm{S}y = x \quad \vee \quad \neg\, 2 \uparrow x \le 2 \uparrow x$ by

2: $\neg\, 2 \uparrow x \le 2 \uparrow (\mathrm{S}y)$

9: $\neg\, \mathrm{S}y = x \quad \vee \quad 2 \uparrow x \le 2 \uparrow (\mathrm{S}y) \quad \vee \quad \neg\, 2 \uparrow x \le 2 \uparrow x$

14: $\neg\, 2\cdot(2\uparrow y) = 2\uparrow(\mathrm{S}y) \quad \vee \quad \neg\, 2\uparrow x \leq 2\cdot(2\uparrow y)$ by

2: $\neg\, 2\uparrow x \leq 2\uparrow(\mathrm{S}y)$

10: $\neg\, 2\cdot(2\uparrow y) = 2\uparrow(\mathrm{S}y) \quad \vee \quad \neg\, 2\uparrow x \leq 2\cdot(2\uparrow y) \quad \vee \quad 2\uparrow x \leq 2\uparrow(\mathrm{S}y)$

15: $\neg\, \mathrm{S}y = x$ by

4: $2\uparrow x \leq 2\uparrow x$

13: $\neg\, \mathrm{S}y = x \quad \vee \quad \neg\, 2\uparrow x \leq 2\uparrow x$

16: $\neg\, 2\uparrow x \leq 2\cdot(2\uparrow y)$ by

5: $2\cdot(2\uparrow y) = 2\uparrow(\mathrm{S}y)$

14: $\neg\, 2\cdot(2\uparrow y) = 2\uparrow(\mathrm{S}y) \quad \vee \quad \neg\, 2\uparrow x \leq 2\cdot(2\uparrow y)$

17: $\neg\, 2\uparrow y \leq (2\uparrow y)+(2\uparrow y) \quad \vee \quad 2\uparrow y \leq 2\cdot(2\uparrow y)$ by

6: $(2\uparrow y)+(2\uparrow y) = 2\cdot(2\uparrow y)$

11: $\neg\, (2\uparrow y)+(2\uparrow y) = 2\cdot(2\uparrow y) \quad \vee \quad \neg\, 2\uparrow y \leq (2\uparrow y)+(2\uparrow y) \quad \vee \quad 2\uparrow y \leq 2\cdot(2\uparrow y)$

18: $2\uparrow y \leq 2\cdot(2\uparrow y)$ by

7: $2\uparrow y \leq (2\uparrow y)+(2\uparrow y)$

17: $\neg\, 2\uparrow y \leq (2\uparrow y)+(2\uparrow y) \quad \vee \quad 2\uparrow y \leq 2\cdot(2\uparrow y)$

19: $x \leq y$ by

15: $\neg\, \mathrm{S}y = x$

12: $x \leq y \quad \vee \quad \mathrm{S}y = x$

20: $\neg\, 2\uparrow x \leq 2\uparrow y \quad \vee \quad \neg\, 2\uparrow y \leq 2\cdot(2\uparrow y)$ by

16: $\neg\, 2\uparrow x \leq 2\cdot(2\uparrow y)$

8: $\neg\, 2\uparrow x \leq 2\uparrow y \quad \vee \quad \neg\, 2\uparrow y \leq 2\cdot(2\uparrow y) \quad \vee \quad 2\uparrow x \leq 2\cdot(2\uparrow y)$

21: $\neg\, 2\uparrow x \leq 2\uparrow y$ by

18: $2\uparrow y \leq 2\cdot(2\uparrow y)$

20: $\neg\, 2\uparrow x \leq 2\uparrow y \quad \vee \quad \neg\, 2\uparrow y \leq 2\cdot(2\uparrow y)$

22: $2\uparrow x \leq 2\uparrow y$ by

19: $x \leq y$

0: $\neg\, x \leq y \quad \vee \quad 2\uparrow x \leq 2\uparrow y$

23: QEA by

21: $\neg\, 2\uparrow x \leq 2\uparrow y$

22: $2\uparrow x \leq 2\uparrow y$